

Research Article

On Janowski Starlike Functions

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For analytic functions $f(z)$ in the open unit disc \mathbb{U} with $f(0) = 0$ and $f'(0) = 1$, applying the fractional calculus for $f(z)$, a new fractional operator $D^\lambda f(z)$ is introduced. Further, a new subclass $\mathcal{S}_\lambda^*(A, B)$ consisting of $f(z)$ associated with Janowski function is defined. The objective of the present paper is to discuss some interesting properties of the class $\mathcal{S}_\lambda^*(A, B)$.

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1. Introduction and preliminaries

Let Ω be the class of analytic functions $w(z)$ in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$. For arbitrary fixed real numbers A and B which satisfy $-1 \leq B < A \leq 1$, we say that $p(z)$ belongs to the class $\mathcal{P}(A, B)$ if

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (1.1)$$

is analytic in \mathbb{U} and $p(z)$ is given by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}) \quad (1.2)$$

for some $w(z) \in \Omega$. This class, $\mathcal{P}(A, B)$, was first introduced by Janowski [1]. Therefore, we call $f(z)$ in the class $\mathcal{P}(A, B)$ Janowski functions. Further, let \mathcal{A} be class of functions

$f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.3}$$

which are analytic in \mathbb{U} . We recall here the following definitions of the fractional calculus (fractional integrals and fractional derivatives) given by Owa [2, 3] (also by Srivastava and Owa [4]).

Definition 1.1. The fractional integral of order λ is defined, for $f(z) \in \mathcal{A}$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0), \tag{1.4}$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. The fractional derivative of order λ is defined, for $f(z) \in \mathcal{A}$, by

$$D_z^\lambda f(z) = \frac{d}{dz} (D_z^{\lambda-1} f(z)) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1), \tag{1.5}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.3. Under the hypothesis of Definition 1.2, the fractional derivative of order $(n+\lambda)$ is defined, for $f(z) \in \mathcal{A}$, by

$$D_z^{\lambda+n} f(z) = \frac{d^n}{dz^n} (D_z^\lambda f(z)) \quad (0 \leq \lambda < 1, n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}). \tag{1.6}$$

By means of the above definitions for the fractional calculus, we see that

$$D_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda} \quad (\lambda > 0, k > 0),$$

$$D_z^\lambda z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \quad (0 \leq \lambda < 1, k > 0),$$

$$D_z^{n+\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)} z^{k-n-\lambda} \quad (0 \leq \lambda < 1, k > 0, n \in \mathbb{N}_0, k-n \neq -1, -2, -3, \dots). \tag{1.7}$$

Therefore, we conclude that for any real λ ,

$$D_z^\lambda z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \quad (k > 0, k-\lambda \neq -1, -2, -3, \dots). \tag{1.8}$$

With the definitions of the fractional calculus, we introduce the fractional operator $D^\lambda f(z)$, for $f(z) \in \mathcal{A}$, by

$$D^\lambda f(z) = \Gamma(2 - \lambda)z^\lambda D_z^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n \quad (\lambda \neq 2, 3, 4, \dots). \quad (1.9)$$

If $\lambda = 1$, then

$$D^1 f(z) = Df(z) = zf'(z) \quad (1.10)$$

and if $\lambda \neq 2, 3, 4, \dots$ and $\alpha \neq 2, 3, 4, \dots$, then

$$D^\alpha (D^\lambda f(z)) = D^\lambda (D^\alpha f(z)) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(2-\alpha)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\alpha)} a_n z^n, \quad (1.11)$$

$$D(D^\lambda f(z)) = z(D^\lambda f(z))' = \Gamma(2-\lambda)z^\lambda (\lambda D_z^\lambda f(z) + zD_z^{\lambda+1} f(z)).$$

Let $\mathcal{F}_\lambda^*(A, B)$ be the subclass of \mathcal{A} consisting of functions $f(z)$ satisfying

$$\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} = p(z) \quad (\lambda \neq 2, 3, 4, \dots) \quad (1.12)$$

for some $p(z) \in \mathcal{P}(A, B)$. Note that (1.12) is equivalent to

$$\lambda + \frac{zD_z^{\lambda+1} f(z)}{D_z^\lambda f(z)} = p(z) \quad (\lambda \neq 2, 3, 4, \dots). \quad (1.13)$$

Finally, for $h(z) \in \mathcal{A}$ and $s(z) \in \mathcal{A}$, we say that $h(z)$ is subordinate to $s(z)$, denoted by $h(z) \prec s(z)$, if there exists some function $w(z) \in \Omega$ such that

$$h(z) = s(w(z)) \quad (z \in \mathbb{U}). \quad (1.14)$$

In particular, if $s(z)$ is univalent in \mathbb{U} , then the subordination $h(z) \prec s(z)$ is equivalent to $h(0) = s(0)$ and $h(\mathbb{U}) \subset s(\mathbb{U})$ (see [5]).

2. Main results

To discuss our problems, we need the following lemma due to Jack [6] or Miller and Mocanu [7].

LEMMA 2.1. *Let $w(z)$ be a nonconstant analytic in \mathbb{U} with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_1 , then one has*

$$z_1 w'(z_1) = k w(z_1), \quad (2.1)$$

where k is real and $k \geq 1$.

Next, we have the following lemma.

LEMMA 2.2. Let $f(z) \in \mathcal{A}$ and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}. \tag{2.2}$$

Then, the following fractional differential equation:

$$D_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z) \quad (\lambda \neq 2, 3, 4, \dots) \tag{2.3}$$

has the solution

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} b_n z^n. \tag{2.4}$$

Proof. It is easy to see that

$$D_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z) = \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} b_n z^{n-\lambda} \right), \tag{2.5}$$

$$D_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda} \right),$$

which gives

$$a_n = \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} b_n. \tag{2.6}$$

This completes the proof of the lemma. □

Next, we derive the following theorem.

THEOREM 2.3. If $f(z) \in \mathcal{A}$ satisfies the condition

$$\left(\frac{(D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right) < \begin{cases} \frac{(A-B)z}{1+Bz} = F_1(z), & B \neq 0, \\ Az = F_2(z), & B = 0, \end{cases} \tag{2.7}$$

for some λ ($\lambda \neq 2, 3, 4, \dots$), then $f(z) \in \mathcal{S}_\lambda^*(A, B)$. This result is sharp because the extremal function is the solution of the fractional differential equation

$$D_z^\lambda f(z) = \begin{cases} \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} (1+Bz)^{(A-B)/B}, & B \neq 0, \\ \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} e^{Az}, & B = 0. \end{cases} \tag{2.8}$$

Proof. We define the function $w(z)$ by

$$\frac{D^\lambda f(z)}{z} = \begin{cases} (1+Bw(z))^{(A-B)/B}, & B \neq 0, \\ e^{Aw(z)}, & B = 0. \end{cases} \quad (2.9)$$

When $(1+Bw(z))^{(A-B)/B}$ and $e^{Aw(z)}$ have the value 1 at $z = 0$ (i.e., we consider the corresponding Riemann branch), then $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$, and

$$\left(z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right) = \begin{cases} \frac{(A-B)zw'(z)}{1+Bw(z)}, & B \neq 0, \\ Azw'(z), & B = 0. \end{cases} \quad (2.10)$$

Now, it is easy to realize that the subordination (2.7) is equivalent to $|w(z)| < 1$ for all $z \in \mathbb{U}$. Indeed, assume the contrary. Then, there exists a point $z_1 \in \mathbb{D}$ such that $|w(z_1)| = 1$. Then, by Lemma 2.1, $z_1 w'(z_1) = kw(z_1)$ for some real $k \geq 1$; for such $z_1 \in \mathbb{U}$, then we have

$$\left(z_1 \frac{(D^\lambda f(z_1))'}{D^\lambda f(z_1)} - 1 \right) = \begin{cases} \frac{(A-B)kw(z_1)}{1+Bw(z_1)} = F_1(w(z_1)) \notin F_1(\mathbb{U}), & B \neq 0, \\ Akw(z_1) = F_2(w(z_1)) \notin F_2(\mathbb{U}), & B = 0, \end{cases} \quad (2.11)$$

but this contradicts the condition (2.7) of this theorem and so the assumption is wrong, that is, $|w(z)| < 1$ for every $z \in \mathbb{U}$. The sharpness of this result follows from the fact that

$$D_z^\lambda f(z) = \begin{cases} \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} (1+Bz)^{(A-B)/B}, & B \neq 0, \\ \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} e^{Az}, & B = 0, \end{cases} \implies$$

$$\frac{D^\lambda f(z)}{z} = \begin{cases} (1+Bz)^{(A-B)/B}, & B \neq 0, \\ e^{Az}, & B = 0, \end{cases} \implies$$

$$\left(z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right) = \begin{cases} \frac{(A-B)z}{1+Bz}, & B \neq 0, \\ Az, & B = 0, \end{cases} \implies$$

$$z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} = \begin{cases} \frac{1+Az}{1+Bz}, & B \neq 0, \\ 1+Az, & B = 0. \end{cases} \quad (2.12)$$

□

COROLLARY 2.4. *If $f(z) \in \mathcal{S}_\lambda^*(A, B)$, then*

$$\begin{aligned} & \left| (\Gamma(2-\lambda)z^{\lambda-1}D_z^\lambda f(z))^{B/(A-B)} - 1 \right| < 1, \quad B \neq 0, \\ & \left| \log(\Gamma(2-\lambda)z^{\lambda-1}D_z^\lambda f(z))^{1/A} \right| < 1, \quad B = 0. \end{aligned} \tag{2.13}$$

Proof. This corollary is a simple consequence of Theorem 2.3, and these inequalities are known as the Marx-Strohhacker inequalities for the class $\mathcal{S}_\lambda^*(A, B)$. \square

Next, our result is contained in the following theorem.

THEOREM 2.5. *If $f(z) \in \mathcal{S}_\lambda^*(A, B)$, then*

$$\begin{aligned} \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}(1-Br)^{(A-B)/B} \leq |D_z^\lambda f(z)| &\leq \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}(1+Br)^{(A-B)/B}, \quad B \neq 0, \\ \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}e^{-Ar} \leq |D_z^\lambda f(z)| &\leq \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}e^{Ar}, \quad B = 0. \end{aligned} \tag{2.14}$$

These results are sharp because extremal function is the solution of the fractional differential equation

$$D_z^\lambda f(z) = \begin{cases} \frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}(1+Bz)^{(A-B)/B}, & B \neq 0, \\ \frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}e^{Az}, & B = 0. \end{cases} \tag{2.15}$$

Proof. Janowski [1] proved that if $p(z) \in \mathcal{P}(A, B)$, then

$$\begin{aligned} \left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| &< \frac{(A-B)r}{1-B^2r^2}, \quad B \neq 0, \\ |p(z) - 1| &< Ar, \quad B = 0. \end{aligned} \tag{2.16}$$

Using the definition of the class $\mathcal{S}_\lambda^*(A, B)$, the inequality (2.16) can be rewritten in the form

$$\begin{aligned} \left| z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} - \frac{1-ABr^2}{1-B^2r^2} \right| &< \frac{(A-B)r}{1-B^2r^2}, \quad B \neq 0, \\ \left| z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right| &< Ar, \quad B = 0. \end{aligned} \tag{2.17}$$

From (2.17), with simple calculations, we get

$$\begin{aligned} \frac{1-(A-B)r-ABr^2}{1-B^2r^2} \leq \operatorname{Re} \left(z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} \right) &\leq \frac{1+(A-B)r-ABr^2}{1-B^2r^2}, \quad B \neq 0, \\ 1-Ar \leq \operatorname{Re} \left(z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} \right) &\leq 1+Ar, \quad B = 0. \end{aligned} \tag{2.18}$$

Since

$$\operatorname{Re}\left(z \frac{(D_z^\lambda f(z))'}{D_z^\lambda f(z)}\right) = r \frac{\partial}{\partial r} \log |D^\lambda f(z)|, \tag{2.19}$$

using (2.18) and (2.19), we obtain

$$\frac{1 - (A - B)r - AB r^2}{r(1 + Br)(1 - Br)} \leq \frac{\partial}{\partial r} \log |D^\lambda f(z)| \leq \frac{1 + (A - B)r - AB r^2}{r(1 + Br)(1 - Br)}, \quad B \neq 0, \tag{2.20}$$

$$\frac{1}{r} - A \leq \frac{\partial}{\partial r} \log |D^\lambda f(z)| \leq \frac{1}{r} + A, \quad B = 0.$$

Integrating both sides of (2.20) from 0 to r and after simple calculations, we complete the proof of the theorem. \square

COROLLARY 2.6. *Giving specific values to A and B , one obtains the distortion of the following class.*

- (i) $\mathcal{S}_\lambda^*(1, -1)$,
- (ii) $\mathcal{S}_\lambda^*(1 - 2\beta, -1)$, $0 \leq \beta < 1$,
- (iii) $\mathcal{S}_\lambda^*(1, -1 + 1/M)$, $M > 1/2$,
- (iv) $\mathcal{S}_\lambda^*(\beta, -\beta)$, $0 \leq \beta < 1$.

Finally, we discuss the coefficient inequalities for $f(z) \in \mathcal{S}_\lambda^*(A, B)$.

THEOREM 2.7. *If $f(z) \in \mathcal{S}_\lambda^*(A, B)$, then*

$$|a_n| \leq \begin{cases} \frac{|A - B|}{(n - 1)!} \frac{|\Gamma(n + 1 - \lambda)|}{\Gamma(n + 1) |\Gamma(2 - \lambda)|} \prod_{k=1}^{n-2} (k + |A - B|), & B \neq 0, \\ \frac{|A|}{(n - 1)!} \frac{|\Gamma(n + 1 - \lambda)|}{\Gamma(n + 1) |\Gamma(2 - \lambda)|} \prod_{k=1}^{n-2} (k + |A|), & B = 0. \end{cases} \tag{2.21}$$

Proof. Using the definition of the class, we can write, for $B \neq 0$,

$$\begin{aligned} z \frac{(D^\lambda f(z))'}{D^\lambda f(z)} = p(z) &\iff z(D^\lambda f(z))' = D^\lambda f(z)p(z) \\ &\implies z + 2a_2 \frac{\Gamma(3)\Gamma(2 - \lambda)}{\Gamma(3 - \lambda)} z^2 + 3a_3 \frac{\Gamma(4)\Gamma(2 - \lambda)}{\Gamma(4 - \lambda)} z^3 \\ &\quad + \dots + na_n \frac{\Gamma(n + 1)\Gamma(2 - \lambda)}{\Gamma(n + 1 - \lambda)} z^n + \dots \\ &= (1 + p_1 z + \dots + p_n z^n + \dots) \\ &\quad \cdot \left(z + a_2 \frac{\Gamma(3)\Gamma(2 - \lambda)}{\Gamma(3 - \lambda)} z^2 + a_3 \frac{\Gamma(4)\Gamma(2 - \lambda)}{\Gamma(4 - \lambda)} z^3 + \dots + a_n \frac{\Gamma(n + 1)\Gamma(2 - \lambda)}{\Gamma(n + 1 - \lambda)} z^n + \dots \right). \end{aligned} \tag{2.22}$$

Equating the coefficient of z^n in both sides of (2.22), we get

$$a_n = \frac{1}{(n-1)} \frac{\Gamma(n+1-\lambda)}{\Gamma(n+1)} \sum_{k=1}^{n-1} \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} a_k p_{n-k}, \quad a_1 \equiv 1. \tag{2.23}$$

On the other hand, if $p(z) \in \mathcal{P}(A, B)$, then $|p_n| \leq (A - B)$ (see [8]); so we obtain

$$|a_n| \leq \frac{|A - B|}{(n-1)} \frac{|\Gamma(n+1-\lambda)|}{\Gamma(n+1)} \sum_{k=1}^{n-1} \frac{\Gamma(k+1)}{|\Gamma(k+1-\lambda)|} |a_k|, \quad |a_1| \equiv 1. \tag{2.24}$$

Using the induction method form (2.24), we obtain,

$$\begin{aligned} |a_2| &\leq \frac{|A - B|}{1} \frac{|\Gamma(3-\lambda)|\Gamma(2)}{\Gamma(3)|\Gamma(2-\lambda)|}, \quad \text{for } n = 2, \\ |a_3| &\leq \frac{|A - B|}{2} \frac{|\Gamma(4-\lambda)|\Gamma(2)}{\Gamma(4)|\Gamma(2-\lambda)|} \left(1 + \frac{|A - B|}{1}\right), \quad \text{for } n = 3, \\ |a_4| &\leq \frac{|A - B|}{3} \frac{|\Gamma(5-\lambda)|\Gamma(2)}{\Gamma(5)|\Gamma(2-\lambda)|} \left(1 + \frac{|A - B|}{1}\right) \left(1 + \frac{|A - B|}{2}\right), \dots, \quad \text{for } n = 4, \\ |a_n| &\leq \frac{|A - B|}{(n-1)!} \frac{|\Gamma(n+1-\lambda)|}{\Gamma(n+1)|\Gamma(2-\lambda)|} \prod_{k=1}^{n-2} (k + |A - B|). \end{aligned} \tag{2.25}$$

□

Remark 2.8. One considers the extremal function $f(z)$ defined by

$$D_z^\lambda f(z) = \begin{cases} \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} (1 + Bz)^{(A-B)/B}, & B \neq 0, \\ \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} e^{Az}, & B = 0, \end{cases} \tag{2.26}$$

in Theorems 2.3 and 2.5.

If $B = 0$, then

$$\begin{aligned} D_z^\lambda f(z) &= \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} e^{Az} \\ &= \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{A^{n-1}}{(n-1)!} z^{n-\lambda} \right), \end{aligned} \tag{2.27}$$

$$D_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda} \right),$$

which gives

$$a_n = \frac{A^{n-1}\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)n!(n-1)!}. \tag{2.28}$$

If $B \neq 0$, then

$$\begin{aligned} D_z^\lambda f(z) &= \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} (1+Bz)^{(A-B)/B} \\ &= \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} \binom{A-B}{n-1} \frac{B^{n-1}}{n-1} z^{n-\lambda} \right), \end{aligned} \tag{2.29}$$

$$D_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda} \right),$$

which gives

$$\begin{aligned} a_n &= \binom{A-B}{n-1} B^{n-1} \frac{\Gamma(n+1-\lambda)}{n! \Gamma(2-\lambda)} \\ &= \frac{(A-B)(A-2B)(A-3B) \cdots (A-(n-1)B) \Gamma(n+1-\lambda)}{n! \Gamma(2-\lambda)} \\ &= \frac{(2-\lambda)_{n-1}}{(1)_n} \left(\prod_{j=1}^{n-1} (A-jB) \right), \end{aligned} \tag{2.30}$$

where $(a)_n$ denotes the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1 & (n = 0, a \neq 0), \\ a(a+1)(a+2) \cdots (a+n-1) & (n = 1, 2, 3, \dots), \end{cases} \tag{2.31}$$

so

$$\frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)} = (n-\lambda)(n-\lambda-1)(n-\lambda-2) \cdots (2-\lambda) = (2-\lambda)_{n-1}. \tag{2.32}$$

We note that, by giving specific values to A and B , we obtain the distortion and coefficient inequalities for the classes $\mathcal{S}_\lambda^*(1, -1)$, $\mathcal{S}_\lambda^*(1, 0)$, $\mathcal{S}_\lambda^*(\beta, -\beta)$ ($0 \leq \beta < 1$), $\mathcal{S}_\lambda^*(1, -1 + 1/M)$ ($M > 1/2$), and $\mathcal{S}_\lambda^*(1 - 2\beta, -1)$ ($0 \leq \beta < 1$).

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