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Research Article

About K-Positivity Properties of Time-Invariant Linear Systems Subject to Point Delays

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This paper discusses nonnegativity and positivity concepts and related properties for the state and output trajectory solutions of dynamic linear time-invariant systems described by functional differential equations subject to point time delays. The various nonnegativities and positivities are introduced hierarchically from the weakest one to the strongest one while separating the corresponding properties when applied to the state space or to the output space as well as for the zero-initial state or zero-input responses. The formulation is first developed by defining cones for the input, state and output spaces of the dynamic system, and then extended, in particular, to cones being the three first orthants each being of the corresponding dimension of the input, state, and output spaces.

1. Introduction

Positive systems have an important relevance since the input, state, and output signals in many physical or biological systems are necessarily positive [1–19]. Therefore, important attention has been paid to such systems in the last decades. For instance, an hydrological system composed of a set of lakes in which the input is the inflow into the upstream lake and the output is the outflow from the downstream lake is externally positive system since the output is always positive under a positive input [8]. Also, hyperstable single-input single-output systems are externally positive since the impulse response kernel is everywhere positive. This also implies that the associated transfer functions (provided they are time invariant) are positive real and their input/output instantaneous power and time-integral energy are positive. However, hyperstable systems of second and higher orders are not guaranteed to be externally positive since the impulse response kernel matrix is everywhere positive definite but not necessarily positive [19]. The properties of those systems like, for instance, stability, controllability/reachability or pole assignment through feedback become more difficult to analyze than in standard systems because those properties have to be simultaneously compatible with the nonnegativity/positivity concepts (see, e.g., [7–13, 18]). Nonnegativity/positivity properties apply for both continuous-time and discrete-time systems and are commonly formulated on the first orthant which is an important case in applications [7–15, 18, 20]. However, there are also studies of characterizations of the nonnegativity/positivity properties in more abstract spaces in terms of the solutions belonging to appropriate K-cones [3–6]. On the other hand, positive solutions of singular problems including nonlinearities have been studied in [1, 14]. In particular, positive solutions in singular boundary problems possessing second-order Caratheodory functions have been investigated in [1]. In [2], the property of total positivity is discussed in a context of constructing Knot intersection algorithms for a given space of functions. Also, eigenvalue regions for discrete and continuous-time positive linear systems have been obtained in [13] by using available information on the main diagonal entries of the system matrix while the absolute stability of discrete-time positive systems has been investigated in [17] when subject to unknown nonlinearities within a class of differential constraints with related positivity properties. Also, the properties of controllability and reachability as well as the stability of positive systems using 2D discrete state-space models and graph theoretic formalisms have been studied in the literature (see, e.g., [7, 9, 10, 12, 20, 21]). The reachability and controllability as well as the related pole-assignment problem have been also exhaustively investigated for continuous-time positive systems (see, e.g., [7, 13, 22–24]).

On the other hand, many dynamic systems like, for instance, transportation and signal transmission problems, war-peace models, or biological models (as the sunflower equation or prey-predator dynamics) possess either external delays; that is, either in the input or output, or internal ones, that is, in the state. The properties of the above second kind of systems are more difficult to investigate because of their infinite-dimensional nature [21, 25–36] although they are very important in some control applications like, for instance, the synthesis of sliding-mode controllers under delays [21, 25, 26]. The analytic problem becomes more difficult when delays are distributed or time varying [30, 31, 33, 36]. Positive systems with delays in both the continuous-time and discrete-time cases have been also investigated (see, e.g., [37–39]). Small delays are often introduced in the models as elements disturbing the delay-free dynamics, rather than in parameterized form, and their effect is analyzed as a dynamic perturbation of the differential system. Associate techniques simplify the analytical treatment but the obtained solutions are approximate. The use of disturbing signals on the nominal dynamics is also common in control theory problems involving the use of backstepping techniques or the synthesis of reduced-order controllers (see, e.g., [40, 41]). However, a direct inclusion of the delay effect on the dynamics leads, in general, to tighter calculus of the solution trajectories, [21, 25–36].

The main objective of this paper is to study the nonnegativity/positivity properties of time-invariant continuous-time dynamic systems under constant point delays. Since generalizations to any finite number of commensurate or incommensurate point delays from the case of only one single delay are mathematically trivial, a single delay is considered for the sake of simplicity. The formulation is first stated in K-cones defined for the input (which is admitted to be impulsive and to possess jump discontinuities), state and output spaces which are proper in general although some results are either proved or pointed out to be extendable for less restrictive cones. In a second stage, particular results are focused on the first orthant $\mathbb{R}_+^n$ of $\mathbb{R}^n$ since this is the typical characterization of nonnegativity/positivity in most of physical applications. The main new contribution of the paper is the study of a hierarchically established set of positivity concepts formulated in generic cones for a class of systems subject to point delays. The positivity properties induce a classification of the system at hand involving admissible pairs of nonnegative input and zero initial conditions. In that way, the systems are classified as nonnegative systems (admitting identically null components or input and outputs) and positive systems which possess at least one of its relevant components positive for all time. The above classification is refined as strong positive systems with all its relevant components being positive for the zero-input or zero-state cases and weak positive systems which are positive for either the zero-input or zero-state cases. Finally, strict (strict strong) positive systems have all their relevant components being positive for any admissible input/initial state pair (for the zero-input or zero-state admissible pairs). For these systems, all input/output components become excited (i.e., they reach positive values) for any admissible input-output pairs. The above concepts are referred to as external when they only apply to the output components for identically zero initial conditions.

Notation.

(1) $\mathbb{R}_+^n = \{z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n : z_i \geq 0\}$; $\mathbb{R}_-^n = \{z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n : z_i \leq 0\}$ are subsets of $\mathbb{R}^n$ ($\mathbb{R}$ being the real field) relevant to characterize nonnegativity and nonpositivity, respectively. $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$ are the set of integers, nonnegative integers and negative integers, respectively.

(2) The set of linear operators Γ from the linear real space $\mathbf{X}$ to the linear real space $\mathbf{Y}$ is denoted by $\mathbf{L}(\mathbf{X},\mathbf{Y})$ with $\mathbf{L}(\mathbf{X},\mathbf{X})$ being simply denoted as $\mathbf{L}(\mathbf{X})$. The set of $n \times m$ real matrices belongs trivially to $\mathbf{L}(\mathbb{R}^n, \mathbb{R}^m)$ and a matrix function $F : \mathbf{I} \cap \mathbb{R}_+ \to \mathbf{L}(\mathbb{R}^m, \mathbb{R}^n)$ is simply denoted by $F(t) \in \mathbb{R}^{n\times m}$, for all $t \in \mathbf{I}$, since $F : \mathbf{I} \to \mathbb{R}^{n\times m}$.

(3) The space of truncated real n-vector functions $L_{qe}^n(\mathbb{R}_+, \mathbb{R}^n)$ is defined for any $q \geq 1$ as follows: $f \in L_{qe}^n(\mathbb{R}_+, \mathbb{R}^n)$ if and only if $f_t \in L_q^n(\mathbb{R}_+, \mathbb{R}^n)$ for all finite $t \geq 0$ where $f_t : [0,\infty) \to \mathbb{R}^n$ is defined as $f_t(\tau) = f(\tau)$ for all $0 \leq \tau \leq t$ and $f_t(\tau) = 0$, otherwise and $L_q^n(\mathbb{R}_+, \mathbb{R}^n) = \{f : [0,\infty) \to \mathbb{R}^n : \exists \|f\|_q = (\int_0^\infty (f^T(\tau)f(\tau))^q d\tau)^{1/q} < \infty\}$ is the Banach space (being furthermore a Hilbert space if $q = 2$) of real q-integrable n-vector functions on $\mathbb{R}_+$, endowed with norm $\|f\|_q$, the associate inner product being defined accordingly. Furthermore, define

$$L_{tq}^n(\mathbb{R}_+, \mathbb{R}^n) = \left\{f : [0,\infty) \longrightarrow \mathbb{R}^n : \exists \|f\|_{tq} = \left(\int_0^\infty (f^T(\tau)f(\tau))^q d\tau\right)^{1/q} < \infty\right\},$$

for any given $t < \infty$

$$L_{qe}^n(\mathbb{R}_+, \mathbb{R}^n) = \left\{f : [0,\infty) \longrightarrow \mathbb{R}^n : \exists \|f\|_{tq} = \left(\int_0^\infty (f^T(\tau)f(\tau))^q d\tau\right)^{1/q} < \infty,\ \forall t < \infty\right\},$$

$$L_{t\infty}^n(\mathbb{R}_+,\mathbb{R}^n) = \left\{f : [0,\infty) \longrightarrow \mathbb{R}^n : \text{ess} \sup_{0\le\tau\le t\in\mathbb{R}_+} (\|f(\tau)\|_E) < \infty\right\} \text{ for a given } t<\infty,$$

$$L_{\infty e}^n(\mathbb{R}_+,\mathbb{R}^n) = \left\{f : [0,\infty) \longrightarrow \mathbb{R}^n : \text{ess} \sup_{0\le\tau\le t\in\mathbb{R}_+} (\|f(\tau)\|_E) < \infty,\ \forall t<\infty\right\},$$

$$L_{\infty}^n(\mathbb{R}_+,\mathbb{R}^n) = \left\{f : [0,\infty) \longrightarrow \mathbb{R}^n : \text{ess}\sup_{t\in\mathbb{R}_+} (\|f(t)\|_E) < \infty\right\} \tag{1.1}$$

with $\|f(t)\|_E$ denoting the Euclidean norm for any $t \in \mathbb{R}_+$. Note that from the standard definition of the essential supremum $\|f(t)\|_E \ge \text{ess}\,\text{Sup}_{t\in\mathbb{R}_+}(\|f(t)\|_E)$ for $t \in BD(f) \cup UBD(f)$, where $BD(f)$ and $UBD(f)$ are subsets of $\mathbb{R}_+$ of finite cardinal where $\|f(t)\|_E$ is bounded and unbounded (i.e., it is impulsive within $UBD(f)$), respectively. In other words, $f(t)$ is bounded on $BD(f)$ and impulsive on $UBD(f)$. Both BD and UBD have zero Lebesgue measures considered as subsets of $\mathbb{R}$ and may be empty implying that the essential supremum equalizes the supremum. Thus, $g : \mathbb{R}_+ \to \mathbb{R}^n$ defined by $g(t) = 0$, for all $t \in \mathbb{R}_+/(BD(f) \cup UBD(f))$ and $g(t) = f(t)(\neq 0)$, for all $t \in BD(f) \cup UBD(f)$, for all $f \in L_\infty^n(\mathbb{R}_+,\mathbb{R}^n)$ has a support of zero measure.

(4) $C^{n^{(q)}}(\mathbb{R}_+,\mathbb{R}^n)$ is the space of q-continuously differentiable real n-vector functions on $\mathbb{R}_+$ for any integer $q \ge 1$, $C^{n(0)}(\mathbb{R}_+,\mathbb{R}^n)$ is the set of continuous real n-vector functions on $\mathbb{R}_+$ and $C^{n\times n}(\mathbb{R}^{n\times n})$ and $C^{n\times n(q)}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ are, respectively, the sets of square real n-matrices and that of q-continuously differentiable square real n-matrix functions on $\mathbb{R}_+$. Real n-matrices and real n-matrix functions are also in the sets of linear operators on $\mathbb{R}^n$, $\mathbf{L}(\mathbb{R}^n)$. Similarly, the notations $C^{n\times m}(\mathbb{R}^{n\times m})$ and $C^{n\times m(q)}(\mathbb{R}_+,\mathbb{R}^{n\times m})$ apply "mutatis-mutandis" for rectangular real $n \times m$ matrices and matrix functions.

(5) The simplified notations L_{qe}^n, L_{tq}^n, L_q^n, L_∞^n, $L_{t\infty}^n$ and $C^{n(q)}$ are used for $L_{qe}^n(\mathbb{R}_+,\mathbb{R}^n)$, $L_{tq}^n(\mathbb{R}_+,\mathbb{R}^n)$, $L_q^n(\mathbb{R}_+,\mathbb{R}^n)$, $L_\infty^n(\mathbb{R}_+,\mathbb{R}^n)$, $L_{t\infty}^n(\mathbb{R}_+,\mathbb{R}^n)$ and $C^{n(q)}(\mathbb{R}_+,\mathbb{R}^n)$, respectively, since no confusion is expected. If $n = 1$, then the n superscript in the spaces of functions of functions are omitted.

(6) $U(t)$ is the Heaviside (unity step) real function defined by $U(t) = 1$ for $t \ge 0$ and $U(t) = 0$, otherwise; and I_n denotes the n-identity matrix.

(7) $\{0_n\}$ is the set consisting of the isolated point $0 \in \mathbb{R}^n$. Any subset $\overline{q}$ of ordered consecutive natural numbers is defined by $\overline{q} = \{1,2,\dots,q\}$.

(8) A set $K \subseteq \mathbb{R}^n$ of interior K^0 and boundary (frontier) K^{Fr} which is identical to all finite nonnegative linear combinations of elements in itself is said to be a cone. If K is convex then it is a convex polyhedral cone since it is finitely generated.

(9) The notation $f : \mathrm{Dom}(f) \to K \subseteq \mathbb{R}^n$ (K being a cone) is abbreviated as $f \in K$. Then, if $\mathrm{Dom}(f) \subseteq \mathbb{R}_+$, $\mathrm{Dom}(g) \subseteq \mathbb{R}_+$, then $f \in K$, $g \in K'$, $(f,g) \in K \times K'$ mean $f(t) \in K$, $g(\tau) \in K'$, $(f(t),g(\tau)) \in K \times K'$, for all $t \in \mathrm{Dom}(f)$, for all $\tau \in \mathrm{Dom}(g)$ if $K \subseteq \mathbb{R}^n$ and $K' \subseteq \mathbb{R}^{n'}$ are cones. Simple notations concerning

cones useful for analysis of state/output trajectories of dynamic systems are

$$\{a_n\} \in K \Longleftrightarrow a \in K \subseteq \mathbb{R}^n; \qquad \{a_n\} = f \in K \Longleftrightarrow \exists t \in \mathrm{Dom}(f) : f(t) = a,$$
$$\{a_n\} \neq f \in K \Longleftrightarrow \neg\exists t \in \mathrm{Dom}(f) : f(t) = a \Longleftrightarrow f(t) \neq a, \quad \forall t \in \mathrm{Dom}(f) \tag{1.2}$$

for any $f : \mathrm{Dom}(f) \to K \subseteq \mathbb{R}^n$.
The simplified notation $X/\{0_n\} := \{0_n \neq x \in X\}$ will be used

2. Dynamic system with point delays

Consider the linear time-invariant system (S) with finite point constant delay $h \geq 0$ described in state-space form by
(S)

$$\dot{x}(t) = Ax(t) + A_0 x(t-h) + Bu(t), \tag{2.1}$$
$$y(t) = Cx(t) + Du(t), \tag{2.2}$$

$x(t) \in \mathbf{X} \subseteq \mathbb{R}^n$, $u(t) \in \mathbf{U} \subseteq \mathbb{R}^m$ and $y(t) \in \mathbf{Y} \subseteq \mathbb{R}^p$ are, respectively, the state, input, and output real vector functions in the respective vector spaces **X**, **U**, and **Y** for all $t \geq 0$. A, A_0, B, C, and D are real matrices of dynamics, delayed dynamics, input, output, and input-output interconnections, respectively, of appropriate orders and then linear operators in $\mathbf{L}(\mathbb{R}^{\mathbf{n}}) \equiv L(\mathbb{R}^{\mathbf{n}}, \mathbb{R}^{\mathbf{n}}), \mathbf{L}(\mathbb{R}^{\mathbf{n}}), \mathbf{L}(\mathbb{R}^m, \mathbb{R}^n), \mathbf{L}(\mathbb{R}^n, \mathbb{R}^p)$, and $\mathbf{L}(\mathbb{R}^p)$, respectively. The system (2.1) is assumed to be subject to any function of initial conditions $\varphi \in \mathrm{IC}([-h,0], \mathbb{R}^n)$ which is of the form $\varphi(t) = \varphi^{(1)}(t) + \varphi^{(2)}(t) + \varphi^{(3)}(t)$, where

(1) $\varphi^{(1)} : [-h,0] \to \mathbb{R}^n_+$ is a piecewise continuous real n-vector function,

(2) $\varphi^{(2)} : [-h,0] \to \mathbb{R}^n_+$ has bounded discontinuities on a subset of zero measure of $[-h,0]$; that is, it consists of a finite set of bounded discontinuities so that it is of support of zero measure,

(3) $\varphi^{(3)} : [-h,0] \to \mathbb{R}^n_+$ is either null or impulsive of the form $\varphi^{(3)}(t) = \sum_{i=1}^{N_3} \varphi_i \delta(t - t_i)$ with $t_i \in [-h,0)$ being an ordered set of real numbers, $\varphi_i \in \mathbb{R}^n_+$ with $i \in \overline{N}_3$ (N_3 being finite) and $\delta : [-h,0] \to \mathbb{R}^n_+$ is a Dirac distribution centred at $t = 0$.

Then, $\mathrm{IC}([-h,0], \mathbb{R}^n)$ is an admissible set of initial conditions. If $u \in L^m_{qe}(\mathbb{R}_+, \mathbf{U})$ for any integer $q \geq 1$ then a unique solution $x \in C^{n(1)}(\mathbb{R}_+, \mathbb{R}^n)$ is proved to exist for any $\varphi \in \mathrm{IC}([-h,0], \mathbb{R}^n)$ and any input space $\mathbf{U} \subseteq \mathbb{R}^m$. The following result holds.

Theorem 2.1. *The state trajectory solution of (2.1) is in $C^{n^{(1)}} \cap L^n_{\infty e}$ and unique on $\mathbb{R}_+$ for any $\varphi \in \mathrm{IC}([-h,0], \mathbb{R}^n)$ and any $u \in L^m_{qe}$ for any real constant $q \geq 1$. Such a solution is defined explicitly by any of the two identical expressions below for all $t \in \mathbb{R}_+$:*

$$x(t) = e^{At}\left[x_0 + \int_{-h}^{0} e^{-A(\tau+h)} A_0 \varphi(\tau)\,d\tau + \int_{0}^{t-h} e^{-A(\tau+h)} A_0 x(\tau)\,d\tau + \int_{0}^{t} e^{-A\tau} Bu(\tau)\,d\tau\right] \tag{2.3}$$

$$= \Psi(t,0)x_0 + \int_{-h}^{0} \Psi(t,\tau) A_0 \varphi(\tau)\,d\tau + \int_{0}^{t} \Psi(t,\tau) Bu(\tau)\,d\tau, \tag{2.4}$$

where $x(0) = \varphi(0) = x_0$, $e^{At} \in \mathbb{R}^{n\times n}$ is an $n \times n$ real matrix function (and also an operator in $\mathbf{L}(\mathbb{R}^n)$, for all $t \in \mathbb{R}$), which is a C_0-semigroup of infinitesimal generator A and

$\Psi : \mathbb{R}\times\mathbb{R} \to \mathbf{L}(\mathbb{R}^n)$ *is a strong evolution operator which satisfies*

$$\dot{\Psi}(t,\tau) = \frac{d\Psi(t,\tau)}{dt} = A\Psi(t,\tau) + A_0\Psi(t-h,\tau) \tag{2.5}$$

for all $t \geq \tau \geq 0$ *with* $\psi(t,t) = I_n$ *for* $t \geq 0$ *and* $\psi(t,\tau) = 0$ *for* $\tau > t$, *which is uniquely pointwisely defined for all* $t \geq \tau \geq 0$ *by*

$$\Psi(t,\tau) = e^{A(t-\tau)}\left[I_n + \int_{\tau+h}^{t} e^{-A\sigma}A_0\Psi(\sigma-h,\tau)\,d\sigma\right]. \tag{2.6}$$

Proof. Since $\varphi \in \mathrm{IC}([-h,0],\mathbb{R}^n)$ is a function of initial conditions, define the segment of state-trajectory solution $x_{[t]} : [t-h,t] \to \mathbb{R}^n$ on $[-h,0]$ as $x_{[0]} \equiv x(t) = \varphi(t)$ for $t \in [-h,0]$ with $x(0) = \varphi(0) = x_0$. Equation (2.3) is identical via such a definition to

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\big(A_0x(\tau-h) + Bu(\tau)\big)\,d\tau \tag{2.7}$$

after joining the second and third right-hand side terms into one and converting the integral within the interval $[-h,t-h]$ into one on $[0,t]$ with the change of integration variable $\tau \to \tau+h$. Taking time-derivatives with respect to "t," then one gets directly using (2.7) again:

$$\begin{aligned}\dot{x}(t) &= A\left[e^{At}x_0 + \int_0^t e^{A(t-\tau)}\big(A_0x(\tau-h) + Bu(\tau)\big)\,d\tau\right] + A_0x(t-h) + Bu(t)\\ &= Ax(t) + A_0x(t-h) + Bu(t)\end{aligned} \tag{2.8}$$

which is identical to (2.1). Thus (2.7), and then (2.3), satisfy (2.1) for the given initial conditions. Note that all the entries $\alpha_{ij} : \mathbb{R}_+ \to \mathbb{R};\ i,j \in \overline{n}$ of $e^{At} = (\alpha_{ij})(t)$ are in L_{qe} for any finite $p \geq 1$ since they are of exponential order. The following cases can occur.

(a) $u \in L_{qe}^m$ for some finite $q > 1$. Since e^{At} is of exponential order, $\alpha_{ij} \in L_{se}$ for $s = q/(q-1);\ i,j \in \overline{n}$ and also from (2.6) $\Psi_{ij} : t\times[0,t] \to L_{se}\cap L_{\infty e};\ i,j \in \overline{n}$ where $\Psi(t,\tau) = (\Psi_{ij}(t,\tau))$ is also of exponential order. Since $1/q + 1/s = 1$, Hölder's inequality might be applied to get $(\Psi(t,\tau)Bu(\tau)) \in L_e^n$ implying $(\int_0^{t^+}\Psi(t,\tau)Bu(\tau)d\tau) \in L_{\infty e}^n$ for any finite $t \geq 0$ since the integrand is bounded and the integral is performed on a finite interval.

Also, $(\Psi(t,0)x_0 + \int_{-h}^{0}\Psi(t,\tau)A_0\varphi(\tau)d\tau) \in L_{\infty e}^n$, since

$$\int_0^{t^+}\Psi(t,\tau)Bu(\tau)\,d\tau = \int_0^{t^+}\Psi(t,\tau)B\overline{u}(\tau)\,d\tau + \gamma_u(t)\left(\sum_{i=1}^{N^+(t)}\Psi(t,t_{ui})Bu(t_{ui})\right) \tag{2.9}$$

with the indicator function $\gamma_u(t) = 0$ if $u(t)$ is not impulsive in $[0,t]$ and $\gamma(t) = 1$, otherwise, with $N^-(t)$, $N^+(t) \geq N^-(t)$ being finite positive integers and t_{ui} ($i \in \overline{N}^-(t)$, $i \in \overline{N}^+(t)$) are ordered sets of real numbers in $(0,t)$ and $(0,t]$, respectively, with $\overline{u}(t) = u(t)$ for all $t \neq t_{ui}$; and

$$\int_{-h}^{0} \Psi(t,\tau)A_0\varphi(\tau)d\tau = \int_{-h}^{0} \Psi(t,\tau)A_0\left(\varphi^{(1)}(\tau)+\varphi^{(2)}(\tau)\right)d\tau + \gamma_\varphi(t)\left(\sum_{i=1}^{N_3} K_i\Psi(t,t_i)A_0\right) \tag{2.10}$$

with the indicator function $\gamma_\varphi(t) = 0$ if $\varphi(t)$ is not impulsive in $[-h,0)$ and $\gamma_\varphi(t) = 1$, otherwise. Then, $x \in C^{n^{(1)}} \cap L^n_{\infty e}$ from (2.4). Finally, since (2.1) is a linear time-invariant differential system, it satisfies a locally Lipschitz condition over any subinterval of $\mathbb{R}_+$ so that uniqueness of the state trajectory follows on such an interval. By iterative construction of the whole trajectory by joining trajectory segments with $x(t) \equiv \varphi(t)$ $t \in [-h,0]$ the state-trajectory uniqueness on $\mathbb{R}_+$ follows.

(b) $u \in L^m_{\infty e}$ (i.e., $q = \infty$). Then, from (2.6) to $\Psi_{ij} : t \times [0,t] \to L_{1e} \cap L_{\infty e}$ for any finite $t \geq 0$ so that $x \in C^{n^{(1)}} \cap L^n_{\infty e}$. The remaining of the proof follows as in (a).

(c) $u \in L^m_{1e}$ (i.e., $q = 1$). Then, $\Psi_{ij} : t \times [0,t] \to L_{\infty e}$ and $s = \infty$ so that $x \in C^{n^{(1)}} \cap L^n_{\infty e}$. The remaining of the proof follows as in (a). □

Since $L^m_{1e} \cap L^m_{\infty e} \subset L^m_{qe}$ for any $q \geq 1$, the following result follows from Theorem 2.1.

Corollary 2.2. *The state trajectory solution of (2.1) is in $C^{n^{(1)}} \cap L^n_{\infty e}$ and unique on $\mathbb{R}_+$ for any $\varphi \in \mathrm{IC}([-h,0],\mathbb{R}^n)$ and any $u \in L^m_{1e} \cap L^m_{\infty e}$.*

Note that Theorem 2.1 gives the solution in a closed form based either in a C_0-semigroup $e^{A(\cdot)}$ of generated by the infinitesimal generator A or in a strong evolution operator $\Psi(\cdot,\cdot)$. The first one is familiarly known in control theory as the state-transition matrix which is a fundamental matrix of the delay-free differential system $\dot{z}(t) = Az(t)$. The internal delayed state contributes to the solution as a forcing term which is superposed to the external input for all time. The second version of the solution is obtained through a strong evolution operator. In this case, the delayed dynamics only contribute to the solution through the interval-type initial conditions. The expression (2.6) reflects the fact that the strong evolution operator depends on both the delay-free and delayed dynamics and then removes the direct influence of the delayed dynamics in the solution (2.4) for all $t > 0$ while the state-transition matrix in (2.3) is independent of the delayed dynamics so that such dynamics act as a forcing term for all time. The fact that the delay system is infinite dimensional is reflected in the fact that the strong evolution operator possess infinitely many eigenvalues in the second solution expression (2.4). The fact that the state transition matrix is not sufficient to describe the unforced response, requiring the incorporation of the state evolution for all preceding times to build such a solution, dictates that the solution is of infinite memory type and the infinite dimensional when using the first expression (2.3) of the solution. A different approach has been presented in [42] to build the solution of time-delay systems with point delays based on the Lambert matrix function approach. This form of the solution has the form of an infinite series of modes with associated coefficients which again reflects its infinite-dimensional nature.

The initial conditions do not appear explicitly in the solution and the series coefficients depend on the initial conditions and the preshape functions. The strong evolution operator can be calculated explicitly via (2.6) in the approach of this paper and through the Lambert matrix functions and associate coefficients in the approach of [28]. Since the solution is unique under the given weak conditions, the three expressions of the solution lead in fact to the same solution for all time.

3. Cone characterization via set topology

A cone $K \subseteq \mathbb{R}^n$ is said to be proper if it is closed, 0-pointed (i.e., $K \cap (-K) = \{0_n\}$), solid (i.e., K^0 is nonempty) and convex. K is convex cone if and only if $K + K \subseteq K$ (the sum being referred to Minkowski sum of sets) and $\lambda K \subseteq K$, for all $\lambda \in \mathbb{R}_+$ (see, e.g., [3]). An alternative characterization is that K is a convex cone if it is a nonempty set and $\lambda x + \mu y \in K$, for all $x, y \in K$; for all $\lambda, \mu \in \mathbb{R}_+$.

K is a cone if and only if $(-K)$ is a cone and K is a proper cone if and only if $(-K)$ is a proper cone. A 0-pointed cone is in an abbreviated notation simply said to be pointed. As a counterpart to proper cone, K will be said to be improper if it is nonproper.

A convex solid cone K is said to be boundary-linked if $K \cap (-K) = Z_K \cup \{0_n\}$ where $Z_K = Z'_K \cap K^{\mathrm{Fr}}$ with $Z'_K = \{0 \neq z \in K^{\mathrm{Fr}}\} \subset K^{\mathrm{Fr}}$ (which can be empty). An example of boundary-linked cone in $\mathbb{R}^n$ is the union of the first and fourth orthants $K_p := \mathbb{R}_+ \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}_+,\ y \in \mathbb{R}\}$ with $K_p \cap (-K_p) = \{(0, y) : y \in \mathbb{R}\}$ (i.e., the ordinate axis).

Note that if $K = \mathbb{R}^n_+$ (the first orthant) then $Z'_K = \{0 \neq z \in K : z_i = 0 \text{ some } i \in \overline{n}\} \subset K^{\mathrm{Fr}}$. Note also that $Z'_K = \varnothing \Rightarrow Z_K = \varnothing$. Note also that $x \in Z_k \Leftrightarrow (-x) \in (-Z_k)$, where $(-Z_K) = (-Z'_K) \cap K^{\mathrm{Fr}}$ and $Z_K = \varnothing \Leftrightarrow (-Z_K) = \varnothing$ since K and $(-K)$ are cones. Note that $\{0_n\} \not\subset Z_K$, and $(-Z'_K) = \{0 \neq z \in (-K) : z \in K^{\mathrm{Fr}}\} \subset (-K)^{\mathrm{Fr}}$ and $Z_K = \varnothing \Leftrightarrow (-Z_K) = \varnothing$ since K and $(-K)$ are cones. Finally, note that $\{0_n\} \not\subset Z_K$ and $\{0_n\} \in K^{\mathrm{Fr}} \not\subset Z_K$ if K is convex since $\lambda K \subseteq K$, for all $\lambda \in \mathbb{R}_+$. Note also that $K^{\mathrm{Fr}} \supset Z_K \cup \{0_n\} \neq K^{\mathrm{Fr}}$ if $Z_K \neq \varnothing$. Note also that cones are unbounded as easily deduced as follows.

The following assertions hold for a given cone $K \subseteq \mathbb{R}^n$.

Assertion 3.1. *If K is boundary-linked and $n > 1$ then K is improper.*

Proof. If $n > 1$ then $K \cap (-K) = Z_K \cup \{0_n\} \neq \{0_n\}$, since $\{0_n\} \not\subset Z_K$, so that K is not pointed and then improper. □

Note that if $n = 1$ then trivially $Z_K = \varnothing$ since $\{0_1\} \not\subset Z_K$ so that $K \cap (-K) = \{0_1\}$ and K is pointed.

Assertion 3.2. *If $\{0_n\} \subset K^{\mathrm{Fr}}$, then K^0 is not a cone.*

Proof. Consider any $z \in K^0$ and $\lambda = 0 (\in R_+)$. Then, $\lambda z = \{0_n\} \not\subset K^0$ since $\{0_n\} \subset K^{\mathrm{Fr}}$. Thus, the property $\lambda K^0 \subseteq K^0$ for all $\lambda \in R_+$ fails and K^0 is not a cone. □

Assertion 3.3. *If K is proper, then K^0 is not a cone.*

Proof. K proper $\Rightarrow K \cap (-K) = \{0_n\}$ (since K is pointed) $\Rightarrow \{0_n\} \subset K^{\mathrm{Fr}}$ and the proof follows from Assertion 3.2. □

Assertion 3.4. *If K is boundary-linked, then K^0 is not a cone.*

Proof. K boundary-linked $\Rightarrow K \cap (-K) \supset \{0_n\}$ and the proof follows from Assertion 3.2. $\square$

ASSERTION 3.5. *If K is convex and $Z_K \cup \{0_n\} \subset K^0$, then K is open and $K^0 = K$ is a convex cone.*

Proof. Take any $z_0 \in K$. Since K is a convex cone, $K + K \subseteq K$. Proceeding recursively, $z = kz_0 \in K$ for any positive integer k and K is unbounded so that $z \in K^0$ and then $2z \in K^0$. Thus, $K^0 + K^0 \subseteq K^0$. Since, furthermore $Z_K \cup \{0_n\} \subset K^0$, K is open so that K^0 is a convex cone. $\square$

ASSERTION 3.6. *If K is closed convex and $\{0_n\} \in K^{\mathrm{Fr}}$, then K^0 is not a cone.*

Proof. Take $z \in K^0$ then $\{0_n\} \not\subset K^0$ for $0 = \lambda \in R_+$ so that K^0 is not the union of all finite nonnegative linear combinations of all the elements in K^0 so that it is not a cone. $\square$

Note that if K is an open cone, then $K^0 = K$ is trivially a cone.

ASSERTION 3.7. *If K is boundary-linked, then $Z_k = -Z_k$.*

Proof. Define the set $\overline{K}^{\mathrm{Fr}} = K^{\mathrm{Fr}}/(Z_K \cup \{0_n\})$ so that $K = K^0 \cup Z_K \cup \{0_n\} \cup \overline{K}$. Note also that $x \in Z_k \Leftrightarrow (-x) \in (-Z_k)$, $x \in K^0 \Leftrightarrow (-x) \in (-K^0)$ and $x \in \overline{K} \Leftrightarrow (-x) \in (-\overline{K})$ since K and $(-K)$ are both cones; and $\{0_n\} \subset K \cap (-K)$ since K is boundary linked. As a result, $(-K) = (-K^0) \cup (-Z_K) \cup \{0_n\} \cup (-\overline{K})$. From the distributive property of the intersection of sets with respect to their union in the Cantor's algebra, simple calculations yield $K \cap (-K) = (Z_K \cap (-Z_K)) \cup \{0_n\} = Z_k \cup \{0_n\}$ since K is boundary linked. Since $\{0_n\} \not\subset (Z_K \cap (-Z_K))$ then $Z_K = Z_K \cap (-Z_K)$. The proof is complete after proving that $Z_K = Z_K \cap (-Z_K) \Leftrightarrow Z_K = -Z_K$. Since $Z_K = -Z_K \Rightarrow Z_K = Z_K \cap (-Z_K)$, it is sufficient to prove $Z_K = Z_K \cap (-Z_K) \Rightarrow Z_K = -Z_K$. Proceed by contradiction by assuming that there exists a set $\emptyset \neq Z_{0K} \not\subset Z_K$ such that $(-Z_K) = Z_K \cup Z_{0K}$. Then, $\exists x \in Z_K \subset K^{\mathrm{Fr}}$ such that $K \ni (-x) \notin (-Z_K)$. Since $x \neq \{0_n\}$, $x \in K^0 \cup (K^{\mathrm{Fr}}/Z_K) \Rightarrow x \notin Z_K$ since $Z_K \not\subset K^0$ which establishes the contradiction so that $Z_K = -Z_K$. $\square$

ASSERTION 3.8. *If K is proper, then $(-K)$ is proper.*

Proof. $(-K)$ is convex if and only if K is convex, $K^0 \neq \emptyset \Leftrightarrow (-K^0) \neq \emptyset$ so that $(-K)$ is solid, $(-K) \cap K = K \cap (-K) = \{0\}$ so that $(-K)$ is pointed. Then, $(-K)$ is proper. $\square$

4. K-nonnegativity and positivity properties of the dynamic system (S)

Now, convex and solid cones $K_U \subseteq \mathbb{R}^m$, $K_Y \subseteq \mathbb{R}^p$, and $K \subseteq \mathbb{R}^n$, with associate sets

$$\begin{aligned} Z_{KU} &= Z'_{KU} \cap K_U^{\mathrm{Fr}} \quad \text{with } Z'_{KU} = \{0 \neq z \in K_U^{\mathrm{Fr}}\} \subset K_U^{\mathrm{Fr}}, \\ Z_K &= Z'_{KU} \cap K^{\mathrm{Fr}} \quad \text{with } Z'_K = \{0 \neq z \in K^{\mathrm{Fr}}\} \subset K^{\mathrm{Fr}}, \\ Z_{KY} &= Z'_{KY} \cap K_Y^{\mathrm{Fr}} \quad \text{with } Z'_{KY} = \{0 \neq z \in K_Y^{\mathrm{Fr}}\} \subset K_Y^{\mathrm{Fr}} \end{aligned} \tag{4.1}$$

are considered to characterize nonnegativity of the input, state, and output vectors, respectively, for the so-called admissible pairs of initial conditions and inputs defined precisely below.

Definition 4.1. An ordered pair $(u,\varphi) \in L_{qe}^m \times \mathrm{IC}([-h,0],\mathbb{R}^n)$, for some $q \geq 1$, is said to be admissible if $(u,\varphi) : \mathbb{R}_+ \times [-h,0] \to K_U \times K$ (i.e., $(u(t),\varphi(\tau)) \in K_U \times K$ for all $(t,\tau) \in \mathbb{R}_+ \times [-h,0]$).

Note that the trivial pair $(0,0) \in \{0_m\} \times \{0_n\} \subset K_U \times K$ which yields trivial state/output trajectory solutions $x(t) = 0$, $y(t) = 0$, for all $t \in \mathbb{R}_+$ is admissible. Note also from Theorem 2.1 and (2.1)-(2.2) that the state-trajectory and output trajectory solutions are unique on $\mathbb{R}_+$ for each admissible pair (u,φ) since $u \in L_{qe}^m \cap (\mathbb{R}_+ \times K_U)$ and $\varphi \in \mathrm{IC}([-h,0],\mathbb{R}^n) \cap ([-h,0] \times K)$. Finally, note that since $K_U \subseteq \mathbb{R}^m$ and $K \subseteq \mathbb{R}^n$, the above intersections of sets are not empty. Define sets $\overline{K}^{\mathrm{Fr}} = K^{\mathrm{Fr}}/(Z_K \cup \{0_n\})$ and $\overline{K}_Y^{\mathrm{Fr}} = K_Y^{\mathrm{Fr}}/(Z_K \cup \{0_n\})$. The following topological technical assumption facilitates the subsequent formalism.

Assumption 4.2. $K \subseteq \mathbb{R}^n$ is a convex solid cone fulfilling $Z_K \cup \{0_n\} \cup \overline{K}^{\mathrm{Fr}} \subset K^{\mathrm{Fr}} \subset K$.

Assumption 4.3. $K_Y \subseteq \mathbb{R}^p$ is a convex solid cone fulfilling $Z_{KY} \cup \{0_p\} \cup \overline{K}_Y^{\mathrm{Fr}} \subset K_Y^{\mathrm{Fr}} \subset K_Y$.

Note that if there are state (resp., output) trajectory solutions in $Z_K \cup \{0_n\}$ (resp., in $Z_{KY} \cup \{0_p\}$), then internally nonnegative (resp., externally nonnegative) trajectories are not positive since they exhibit zero components at some time instants. Assumptions 4.2-4.3 imply the following technical results.

Assertion 4.4. *If Assumptions 4.2-4.3 hold, then* $x \in K^0 \cup Z_K \Leftrightarrow x \neq \{0_n\}$ *for all* $x \in K$ *and* $y \in K_Y^0 \cup Z_{KY} \Leftrightarrow y \neq \{0_p\}$ *for all* $y \in K_Y$.

Assertion 4.5. *If Assumptions 4.2-4.3 hold and K is either boundary linked or proper then* $(K^0 \cup \overline{K}^{\mathrm{Fr}}) \cap ((-K^0) \cup (-\overline{K}^{\mathrm{Fr}})) = \varnothing$ *and* $(K^0 \cup \overline{K}^{\mathrm{Fr}}) \cap (-K) = \varnothing$. *If* K_Y *is either boundary linked or proper then* $(K_Y^0 \cup \overline{K}_Y^{\mathrm{Fr}}) \cap ((-K_Y^0) \cup (-\overline{K}_Y^{\mathrm{Fr}})) = \varnothing$ *and* $(K^0 \cup \overline{K}^{\mathrm{Fr}}) \cap (-K) = \varnothing$.

Proof. It is direct from $K^0 \cap (-K^0) = \varnothing$, $\overline{K}^{\mathrm{Fr}} \cap (-K^{\mathrm{Fr}}) = \varnothing$ and $(Z_K \cup \{0_n\}) \cap (\pm\overline{K}^{\mathrm{Fr}}) = \varnothing$ and similar results concerning K_Y. □

A set of definitions is now given to characterize different degrees of K-Nonnegativity according to the fact that there is some (positivity) or all (strict positivity) components of the state/output vectors strictly positive for all time or they are simply nonnegative for the given cones of the input, state, and output vectors. The nonnegativity properties are referred to as internal (resp., external) if they are fulfilled by the state vector (resp., output vector). Also, the positivity is strong (resp., weak) if it holds separately for the zero-state and zero input (resp., either for the zero state or zero input) state/output trajectory solutions.

In the previous standard literature on the subject, the nonnegativity/positivity properties are commonly referred to as external if they keep for the input/output descriptions; that is, the system is externally nonnegative/positive if any output trajectory is everywhere nonnegative/positive for all nonnegative/positive input. Similarly, the system is said to be internally nonnegative/positive (or, via an abbreviate notation, as nonnegative/positive) if both state and output trajectories are everywhere nonnegative/positive for any nonnegative/positive input [3, 7–20]. However, throughout this paper, the nonnegativity/positivity properties are referred to as internal (external) if they refer to the state (output) trajectory under nonnegative/positive input while no specification internal/external is given if both state and output trajectories exhibit the corresponding property. This novelty on previous literature is adopted since the nonnegativity/positivity properties for the state/output trajectories state-output trajectories each under specific conditions on the system parameterizations. Another novelty is the introduction of weak/strong nonnegativity/positivity to distinguish if the corresponding nonnegativity/positivity property holds for either the zero-initial state or zero-input responses rather than for general responses. The following sets of definitions apply to convex and solid cones K and K_Y which satisfy Assumptions 4.2-4.3 for all admissible pairs (u,φ) (see Definition 4.1).

Definition 4.6 (nonnegativity). (i) (S) is (K_U,K)-internally nonnegative ((K_U,K)-INN) if $x \in K$ for any admissible pair $(u,\varphi) \in K_U \times K$.

(ii) (S) is (K_U,K,K_Y)-externally nonnegative ((K_U,K,K_Y)-ENN) if $y \in K_Y$ for any admissible pair $(u,\varphi) \in K_U \times K$.

(iii) (S) is (K_U,K,K_Y)-nonnegative ((K_U,K,K_Y)-NN) if it is (K_U,K)-INN and (K_U,K,K_Y)-ENN.

The various definitions of positivity below apply to nonnegative systems when *at least one* state or output (or both state and output) component is strictly positive for all time provided that neither the input nor the function of initial conditions are identically zero. All the positivity definitions are referred to the appropriate cones.

Definition 4.7 (positivity). (i) (S) is (K_U,K)-internally positive ((K_U,K)-IP) if it is (K_U,K)-INN and $x \neq \{0_n\}$ for any admissible pair $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$.

(ii) (S) is (K_U,K,K_Y)-externally positive ((K_U,K,K_Y)-EP) if it is (K_U,K,K_Y)-ENN and $y \neq \{0_p\}$ for any admissible pair $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$.

(iii) (S) is (K_U,K)-positive ((K_U,K)-P) if it is (K_U,K)-P and (K_U,K,K_Y)-EP.

The various definitions of strict positivity apply to nonnegative systems when *all* the state or output (or both state and output) components are strictly positive for all time provided that neither the input nor the function of initial conditions are identically zero.

Definition 4.8 (strict positivity). (i) (S) is (K_U,K)-internally strictly positive [(K_U,K)-ISP] if it is (K_U,K)-INN and $x \in K^0 \cup \overline{K}^{\mathrm{Fr}}$ for any admissible pair $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$.

(ii) (S) is (K_U,K,K_Y)-externally strictly positive ((K_U,K,K_Y)-ESP) if it is (K_U,K,K_Y)-ENN and $y \in K_Y^0 \cup \overline{K}_Y^{\mathrm{Fr}}$ for any admissible pair $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$.

(iii) (S) is (K_U,K,K_Y)-strictly positive ((K_U,K,K_Y)-SIEP) if it is both (K_U,K_Y)-ISP and (K_U,K,K_Y)-ESP.

The various definitions of strong positivity below apply to nonnegative systems when *at least one* of the state or output (or both state and output) components are strictly positive for all time even if either the input or the function of initial conditions are identically zero. The strong positivity is said to be strict if the positivity property holds for all the components of the state or output (or state and output).

Definition 4.9 (strong positivity). (i) (S) is (K_U,K)-strongly internally positive ((K_U, K)-SIP) if it is (K_U,K)-INN and $x \neq \{0_n\}$ for any admissible pair $(u,\varphi) \in (K_U \times K/\{0_n\}) \cup (K_U/\{0_m\} \times K)$.

(ii) (S) is (K_U,K,K_Y)-strongly externally positive ((K_U,K,K_Y)-SEP) if it is (K_U,K, K_Y)-ENN and $y \neq \{0_p\}$ for any admissible pair $(u,\varphi) \in (K_U \times K/\{0_n\}) \cup (K_U/ \{0_m\} \times K)$.

(iii) (S) is (K_U,K,K_Y)-strongly positive ((K_U,K,K_Y)-SP) if it is (K_U,K)-SIP and (K_U, K,K_Y)-SEP.

Definition 4.10 (strong strict positivity). (i) (S) is (K_U,K)-strongly internally strictly positive ((K_U,K)-SISP) if it is (K_U,K)-INN and $x \in K^0 \cup \overline{K}^{\mathrm{Fr}}$ for any admissible pair $(u,\varphi) \in (K_U \times K/\{0_n\}) \cup (K_U/\{0_m\} \times K)$.

(ii) (S) is (K_U,K,K_Y)-strongly externally strictly positive ((K_U,K,K_Y)-SESP) if it is (K_U,K,K_Y)-ENN and $y \in K_Y^0 \cup \overline{K}_Y^{\mathrm{Fr}}$ for any admissible pair $(u,\varphi) \in (K_U \times K/\{0_n\}) \cup (K_U/\{0_m\} \times K)$.

(iii) (S) is (K_U,K,K_Y)-strongly strictly positive ((K_U,K,K_Y)-SSP) if it is (K_U,K)-SISP and (K_U,K,K_Y)-SESP.

The various definitions of weak positivity below apply to nonnegative systems when *at least one* of the state or output (or both state and output) components are strictly positive for all time for all admissible pairs of Definition 4.1 excluding either those being of the form $(0,\varphi)$ (zero-input weak positivity) or those being of the form $(u,0)$ (zero-initial state weak positivity) even if either the input or the function of initial conditions are identically zero. The weak positivity is said to be strict if the positivity property holds for all the components of the state or output (or state and output).

Definition 4.11 (weak positivity). (i) (S) is (K_U,K)-weakly internally positive ((K_U, K)-WIP) if it is (K_U,K)-INN and $x \neq \{0_n\}$ either for any admissible pair $(u,\varphi) \in K_U \times K/\{0_n\}$ (zero-input weakly internally positive) or for any admissible pair $(u,\varphi) \in K_U/ \{0_m\} \times K$ (zero-initial state weakly internally positive).

(ii) (S) is (K_U,K,K_Y)-weakly externally positive ((K_U,K,K_Y)-WEP) if it is (K_U,K, K_Y)-ENN and $y \neq \{0_p\}$ for any admissible pair $(u,\varphi) \in K_U \times K/\{0_n\}$ (zero-input weakly externally positive) or for any admissible pair $(u,\varphi) \in K_U/\{0_m\} \times K$ (zero-initial state weakly externally positive).

(iii) (S) is (K_U,K,K_Y)-weakly positive ((K_U,K,K_Y)-WP) if it is (K_U,K)-WIP and (K_U, K,K_Y)-WEP.

Definition 4.12 (weak strict positivity). (i) (S) is (K_U,K)-weakly internally strictly positive ((K_U,K)-WISP) if it is (K_U,K)-INN and $x \in K^0 \cup \overline{K}^{\mathrm{Fr}}$ either for any admissible pair $(u,\varphi) \in K_U \times K/\{0_n\}$ (zero-input weakly internally strictly positive) or for any admissible pair $(u,\varphi) \in K_U/\{0_m\} \times K$ (zero-initial state weakly internally strictly positive).

(ii) (S) is (K_U,K,K_Y)-weakly externally strictly positive ((K_U,K,K_Y)-WESP) if it is (K_U,K,K_Y)-ENN and $y \in K_Y^0 \cup \overline{K}_Y^{\text{Fr}}$ either for any admissible pair $(u,\varphi) \in K_U \times K/\{0_n\}$ (zero-input weakly externally strictly positive) or for any admissible pair $(u,\varphi) \in K_U/\{0_m\} \times K$ (zero-initial state weakly externally strictly positive).

(iii) (S) is (K_U,K,K_Y)-weakly strictly positive ((K_U,K,K_Y)-WSP) if it is (K_U,K)-WISP and (K_U,K,K_Y)-WESP.

Note that since $\{0_n\} \in K$ and $\{0_m\} \in K_U$ from Assumption 4.2, weak positivity implies positivity for either the forced state/output solution trajectory or the homogeneous the forced state/output solution trajectory. Also, weak internal (external) positivity implies internal (external) positivity since $x \in K^0 \cup Z_K \cup \overline{K}^{\text{Fr}}$ $(y \in K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\text{Fr}})$. The subsequent results are concerned with the facts that internal (external) strict positivity imply that the state/output trajectories are not in $Z_k(Z_{KY})$, Strong positivity implies weak positivity and weak positivity imply positivity so that mutual implications between some of the above definitions are proved. Weak strict positivity is linked to the basic properties of *excitability* and *transparency* delay-free positive systems in the first orthant [8]. Note also that weak positivity implies that the system is nonnegative but not necessarily either strong or strictly positive and not necessarily excitable. Concerning with positivity in the first orthant, an alternative concept of weak positivity was introduced in [7] being of interest in singular delay-free dynamical systems. Such systems are characterized by the matrix of dynamics being Metzler and all the remaining matrices parameterizing the system being of nonnegative entries. Although the parametrizations satisfy the conditions for positivity in the standard (nonsingular) case, trajectories can reach negative values at some time instants so that they do not lie in the class of positive systems even if the additional matrix E characterizing the singular nature possesses nonnegative entries [7]. In this context the weak-positivity concept of [7] is different from the current one since in the current approach the system is always nonnegative and it is positive (one relevant component is positive) for the zero input or zero state responses.

A (K_U,K)-IP system (S) is said to be excitable if for any admissible pair $(u,0) \in K_U/\{0_m\} \times \{0_n\}$ all the state variables are K-positive (i.e., $x \in K^0 \cup \overline{K}^{\text{Fr}}$) for any input $u \in K_U^0 \cup Z_{KU} \cup \overline{K}_U^{\text{Fr}}$. A (K_U,K,K_Y)-EP system (S) is said to be transparent if and only if for any admissible pair $(0,\varphi) \in \{0_m\} \times K/\{0_n\}$, all the output components are K_Y-positive; that is, $y \in K_Y^0 \cup \overline{K}_Y^{\text{Fr}}$. Then, from Definition 4.12(i), the following result holds.

Assertion 4.13. *If (S) is (K_U,K)-zero-initial state WISP then it is (K_U,K)-excitable. If (S) is (K_U,K,K_Y)-zero-input WESP then it is (K,K_Y)-transparent.*

The converses in Asser 4.13 are not true in general since generic admissible pairs (u,φ) in $K_U/\{0_m\} \times K$ and $K_U \times K/\{0_n\}$ are not involved in the definitions of excitability and transparency.

Theorem 4.14. *The following properties hold.*

(i) *If Assumption 4.2 holds, then (S) is (K_U,K)-ISP if and only if it is (K_U,K)-IP and $x \notin Z_K$ for any admissible pair $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$.*

(ii) *If Assumption 4.3 holds, then (S) is (K_U,K,K_Y)-ESP if and only if it is (K_U,K,K_Y)-EP and $y \notin Z_{KY}$ for any admissible pair $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$.*

(iii) *If Assumptions 4.2-4.3 hold, then (S) is* (K_U,K,K_Y)*-P if and only if it is* (K_U,K)*-IP and* (K_U,K,K_Y)*-EP and* $x \in K^0$ *and* $y \in K_Y^0$ *for any admissible pair* $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$.

Proof. (i) ("*If part*"): (S) is (K_U,K)-IP$\Rightarrow$(S) is (K_U,K)-INN from Definition 4.7(i) and $x \neq \{0_n\} \Leftrightarrow x \in K^0 \cup Z_K$ (from Assertion 4.4) for any admissible pair $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\}) \Rightarrow$(S) is (K_U,K)-ISP from Definition 4.8(i).

("*Only if part*"): (S) is (K_U,K)-ISP then $x \in K^0$ for any admissible pair $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$ then $\{0_n\} \neq x$ so that (S) is (K_U,K)-IP from Definition 4.7(i).

(ii) The proof is similar to that of (i) by using Definitions 4.7(ii) and 4.8(ii).

(iii) It follows from Definitions 4.7 and 4.1. □

Theorem 4.15. *The following properties hold.*

(i) *Under Assumption 4.2, if (S) is* (K_U,K)*-SIP, then it is* (K_U,K)*-IP and* (K_U,K)*-WIP.*

(ii) *Under Assumption 4.2, if (S) is* (K_U,K)*-SISP, then it is* (K_U,K)*-ISP and* (K_U,K)*-WISP.*

(iii) *Under Assumption 4.3, if (S) is* (K_U,K,K_Y)*-SEP, then it is* (K_U,K)*-EP and* (K_U,K,K_Y)*-WEP.*

(iv) *Under Assumption 4.3, if (S) is* (K_U,K,K_Y)*-SESP, then it is* (K_U,K)*-ESP and* (K_U,K,K_Y)*-WESP.*

(v) *Under Assumptions 4.2-4.3, if (S) is* (K_U,K,K_Y)*-SP, then it is* (K_U,K)*-P and* (K_U,K,K_Y)*-WP.*

(vi) *Under Assumptions 4.2-4.3, if (S) is* (K_U,K,K_Y)*-SSP, then it is* (K_U,K)*-SP and* (K_U,K,K_Y)*-WSP.*

(vii) *Under Assumption 4.2, if (S) is* (K_U,K)*-zero-initial state-WISP and K is boundary linked, then (S) is* (K_U,K)*-IP.*

If (S) is (K_U,K)*-zero-initial state WISP and K is proper, then (S) is* (K_U,K)*-ISP.*

If (S) is (K_U,K)*-zero-initial state WIP and K is proper, then (S) is* (K_U,K)*-IP.*

(viii) *Under Assumption 4.2, if (S) is* (K_U,K)*-zero-input WISP and K is boundary linked, then (S) is* (K_U,K)*-IP.*

If (S) is (K_U,K)*-zero-input WISP and K is proper, then (S) is* (K_U,K)*-ISP.*

If (S) is (K_U,K)*-zero-input-WIP and K is proper, then (S) is* (K_U,K)*-IP.*

(ix) *Under Assumption 4.3, if (S) is* (K_U,K,K_Y)*-zero-initial state-WESP and* K_Y *is boundary linked, then (S) is* (K_U,K,K_Y)*-EP.*

If (S) is (K_U,K,K_Y)*-zero-initial state-WESP and* K_Y *is proper, then (S) is* (K_U,K,K_Y)*-ESP.*

If (S) is (K_U,K,K_Y)*-zero-initial state WIP and* K_Y *is proper, then (S) is* (K_U,K,K_Y)*-IP.*

(x) *Under Assumption 4.3, (S) is* (K_U,K,K_Y)*-zero-input-WESP and* K_Y *is boundary linked, then (S) is* (K_U,K,K_Y)*-EP.*

If (S) is (K_U,K,K_Y)*-zero-input-WESP and* K_Y *is proper, then (S) is* (K_U,K,K_Y)*-ESP.*

If (S) is (K_U,K,K_Y)*-zero-input-WEP and* K_Y *is proper, then (S) is* (K_U,K,K_Y)*-EP.*

(xi) *Under Assumptions 4.2-4.3, if (S) is (K_U, K)-zero-initial state WSP and K and K_Y are boundary linked, then (S) is (K_U, K)-P.*

If (S) is (K_U, K)-zero-initial state WSP and K and K_Y are proper, then (S) is (K_U, K)-SP.

If (S) is (K_U, K)-zero-initial state WP and K and K_Y are proper, then (S) is (K_U, K)-P.

(xii) *Under Assumptions 4.2-4.3, if (S) is (K_U, K)-zero-input WSP and K and K_Y are boundary linked, then (S) is (K_U, K)-P.*

If (S) is (K_U, K)-zero-input WSP and K and K_Y are proper, then (S) is (K_U, K)-SP.

If (S) is (K_U, K)-zero-input WP and K and K_Y are proper, then (S) is (K_U, K)-P.

Proof. (i)-(ii): (S) is (K_U, K)-SIP$\Rightarrow x \neq \{0_n\} \Leftrightarrow x \in K^0 \cup Z_k$, from Definition 4.9(i) and Assertion 4.4, for any admissible pairs $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$ and either

$$(0,\varphi) \in (\{0_m\} \times K/\{0_n\}) \quad \text{or} \quad (u,0) \in (K_U/\{0_m\} \times \{0_n\}) \tag{4.2}$$

since

$$\begin{aligned}
&(K_U \times K/\{0_n\}) \cup (K_U/\{0_m\} \times K) \supset (K_U/\{0_m\} \times K/\{0_n\}) \\
&(K_U \times K/\{0_n\}) \cup (K_U/\{0_m\} \times K) \supset (K_U \times K/\{0_n\}) \\
&(K_U \times K/\{0_n\}) \cup (K_U/\{0_m\} \times K) \supset (K_U/\{0_m\} \times K)
\end{aligned} \tag{4.3}$$

$\Rightarrow (K_U, K)$-IP and (K_U, K)-WIP from Definitions 4.7(i) and 4.11(i) and (i) are proved.

The proof of (ii) is similar by using (S) is (K_U, K)-SISP for any $(u,\varphi) \in (K_U/\{0_m\} \times K/\{0_n\})$ and Definitions 4.10(i), 4.8(i), and 4.12(i).

(iii)-(vi): the proofs are very similar to those of (i)-(ii) the corresponding definitions (Definitions 4.8–4.12).

(vii) By assumption, any state-trajectory solution of (S) satisfies $x_{u0} \in K^0 \cup \overline{K}^{\text{Fr}}$ for an admissible pair $(u,0) \in K_U/\{0_m\} \times \{0_n\}$ since (S) is (K_U, K)-zero initial state WISP (see Definition 4.12(i)). Also, since (K_U, K)-zero initial state WISP implies that (S) is (K_U, K)-INN then $x_{0\varphi} \in K$ for any admissible pair $(0,\varphi) \in \{0_m\} \times K/\{0_n\} \subset K_U \times K$. From Theorem 2.1, (2.4), $x_{u\varphi} = x_{0\varphi} + x_{u0} \in K$ since (u,φ) is an admissible pair because both $(u,0)$ and $(0,\varphi)$ are admissible pairs. It is now proved by contradiction that $x_{u\varphi} \neq \{0_n\}$. Assume that $x_{u\varphi} = \{0_n\}$ then $x_{0\varphi} = -x_{u0} \in (-K^0) \cup (-\overline{K}^{\text{Fr}}) \Rightarrow x_{0\varphi} \notin K$ (from Assertion 4.5) a contradiction has been established since $x_{0\varphi} \in K$ so that $x_{u\varphi} \neq \{0_n\}$. Then, $x_{u\varphi} \in K^0 \cup Z_K \cup \overline{K}^{\text{Fr}}$ and (S) is (K_U, K)-IP. The first part of (vii) has been proved. If K is proper then, for any admissible pair $(u,0) \in K_U/\{0_m\} \times \{0_n\}$, the same contradiction $x_{0\varphi} \in K$ and $x_{u\varphi} = \{0_n\}$ implies $x_{0\varphi} \notin K$ follows for any admissible pairs $(u,0)$ and $(0,\varphi)$ so that $x_{u\varphi} \neq \{0_n\}$ implies $x_{u\varphi} \in K^0 \cup Z_K \cup \overline{K}^{\text{Fr}}$ and the second part of (vii) is proved. Finally, if (S) is (K_U, K)-zero initial state WISP then $\{0_n\} \neq x_{u\varphi} \in K^0 \cup Z_K$ still follows from the proof of the second property so that (S) is (K_U, K)-IP.

(viii)–(xii): The proofs follow under similar reasoning guidelines as those used to prove (vii). □

More explicit conditions about the various concepts of positivity are known given for the dynamic system (S) based on the properties of the various matrices parameterizing the description (2.1)-(2.2). First, note that since K_U, K, and K_Y are cones then the set of matrices $\Pi(K) \equiv \Pi(K,K)$, $\Pi(K_U,K)$, and $\Pi(K_U,K_Y)$ defined according to $\Pi(K_1,K_2) = \{M \in \mathbb{R}^{n_2\times n_1} : MK_1 \subseteq K_2\}$ where $K_{1,2} \subseteq \mathbb{R}^{n_1,n_2}$ are also cones. Thus, for matrices in cones of matrices, the following positivity concepts will be used provided that Assumption 4.2 holds.

Definition 4.16. (i) A n-matrix M is K-nonnegative (K-NN) if $M \in \Pi(K)$.

(ii) An n-matrix M is K-positive (K-P) if $M \in \Pi(K)$ and $M(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$.

(iii) An n-matrix M is K-strictly positive (K-SP) if $M \in \Pi(K)$ and $M(K/\{0_n\}) \subseteq K^0 \cup \overline{K}^{\mathrm{Fr}}$.

Since $K^0 \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}} \subset K$, if M is K-SP then it is K-P and K-INN. If M is K-P then it is K-NN. In the same way, for cones $K_i \subseteq \mathbb{R}^{n_i}$ $(i = 1,2)$, Definition 4.16 is extended as follows.

Definition 4.17. (i) A matrix $M \in \mathbb{R}^{n_1\times n_2}M$ is (K_1,K_2)-nonnegative ((K_1,K_2)-NN) if $M \in \Pi(K_1,K_2)$.

(ii) An n-matrix $M \in \mathbb{R}^{n_1\times n_2}$ is K-positive ((K_1,K_2)-P) if $M \in \Pi(K_1,K_2)$ and $M(K_1/\{0_{n_1}\}) \subseteq K_2^0 \cup Z_{K_2} \cup \overline{K}_2^{\mathrm{Fr}}$.

(iii) An n-matrix $M \in \mathbb{R}^{n_1\times n_2}$ is K-strictly positive ((K_1,K_2)-SP) if $M \in \Pi(K_1,K_2)$ and $M(K_1/\{0_{n_1}\}) \subseteq K_2^0 \cup \overline{K}_2^{\mathrm{Fr}}$.

The following results about nonnegativity and positivity of (S) are proved.

Theorem 4.18 (K-nonnegativity). *Let K_U, K, and K_Y be proper cones. Then, the following properties hold.*

(i) *(S) is (K_U,K)-INN if and only if $\Psi(t,\tau) \in \Pi(K)$ for all $t \in \mathbb{R}_+$, $\tau \in [-h,0]$, $A_0 \in \Pi(K)$ and $B \in \Pi(K_U,K)$.*

(ii) *(S) is (K_U,K,K_Y)-ENN if and only if $C\Psi(t,\tau) \in \Pi(K,K_Y)$ for all $t \in \mathbb{R}_+$, $\tau \in [-h,0]$, $CA_0 \in \Pi(K,K_Y)$, and $CB \in \Pi(K_U,K_Y)$.*

(iii) *(S) is (K_U,K,K_Y)-NN if and only if it is (K_U,K)-INN and $CK + DK_U \subseteq K_Y$.*

(iv) *(S) is (K_U,K,K_Y)-NN if and only if it is (K_U,K)-INN, $C \in \Pi(K,K_Y)$, and $D \in \Pi(K_U,$
$K_Y)$.*

(v) *(S) is (K_U,K,K_Y)-NN if and only if $\Psi(t,\tau) \in \Pi(K)$ for all $t \in \mathbb{R}_+$, $\tau \in [-h,0]$, $A_0 \in \Pi(K)$ and $B \in \Pi(K_U,K)$, $C \in \Pi(K,K_Y)$, and $D \in \Pi(K_U,K_Y)$.*

Proof. (i) ("*If part*"): $\Psi(t,\tau) \in \Pi(K)$ for $\tau \in [-h,0]$, $t \in \mathbb{R}_+ \Rightarrow s_0(t) = (\Psi(t,0)x_0) \in K$ for any $\varphi \in K$ and $t \in \mathbb{R}_+$.

$A_0 \in \Pi(K) \Rightarrow (A_0\varphi(\tau)) \in K$ since $\Psi(t,\tau) \in \Pi(K)$ for $\tau \in [-h,0]$, $t \in \mathbb{R}_+$.

Then, from the two above properties together with the definitions of the generalized Lebesgue integrals including integrals of Dirac distributions, one directly gets

$$\begin{aligned} s_\varphi(t) &= \int_{-h}^{0} \Psi(t,\tau)A_0\varphi(\tau)d\tau \\ &= \int_{-h}^{0} \Psi(t,\tau)A_0\big(\varphi^{(1)}(\tau)+\varphi^{(2)}(\tau)\big)d\tau + \sum_{i=1}^{N_3}\Psi(t,t_i)A_0K_i \\ &= \left(\lim_{\substack{\Delta\to 0\\ k\to\infty}}\left(\sum_{i=1}^{k}\Psi(t,i\Delta)A_0\varphi(i\Delta)\right)+\sum_{i=1}^{N_3}\Psi(t,t_i)A_0K_i\right)\in K \end{aligned} \tag{4.4}$$

since $K+K\subseteq K$ since K is proper and then convex so that $kK := K+\cdots+K(k)\subseteq K$ for any $k\in\mathbb{Z}_+$. In the same way, since $u\in K_U$

$$\begin{aligned} B\in\Pi(K_U,K) \Longrightarrow &\int_{0}^{t^\pm} \Psi(t,\tau)Bu(\tau)d\tau \\ &= \int_{0}^{t^\pm} \Psi(t,\tau)B\overline{u}(\tau)d\tau+\gamma_u(t)\left(\sum_{i=1}^{N^\pm(t)}\Psi(t,t_{ui})Bu(t_{ui})\right). \end{aligned} \tag{4.5}$$

Then, $x(t^\pm)=(s_0(t)+s_\varphi(t)+s_u(t^\pm))\in K$, for all $t\in\mathbb{R}_+$ from Theorem 2.1 and $3K\subseteq K$.

("*Only if part*"): if $\Psi(t,0)\notin\Pi(K)$ then $\exists$ (a nonzero) $x_0\in K$ such that $(\Psi(t,0)x_0)\notin K$ (otherwise, $\Psi(t,0)\in\Pi(K)$). Taking $\varphi:[-h,0)\to 0\in K\subseteq\mathbb{R}_+^n$; $u:\mathbb{R}_+\to 0\in K_U\subset\mathbb{R}_+^m$ so that (u,φ) is admissible being zero except for the value $(x_0,0)\neq 0$ at $t=0$. From Theorem 2.1, $x(t)=(\Psi(t,0)x_0)\notin K$ and then (S) is not K-INN. If either $A_0\notin\Pi(K)$ or $\Psi(t,t_i)\notin\Pi(K)$ for some $t_i\in[-h,0]$ then $(\Psi(t,t_i)A_0)\notin\Pi(K)$ for some $t_i\in[-h,0]$ (for all $t_i\in[-h,0]$ if $A_0\notin\Pi(K)$). Then, $x(t_i)=s_\varphi(t_i)=(\Psi(t,t_i)A_0K_i)\notin K$ (from Theorem 2.1) for the admissible pair (u,φ) being identically zero on $([-h,t_i)\cup[t_i,0])\times(\mathbb{R}_+^m\times\mathbb{R}_+^n)$ (i.e., everywhere except at $t=t_i$) since, otherwise, $(\Psi(t,t_i)A_0)\in\Pi(K)$. Finally, if $B\notin\Pi(K_U,K)$ then $x(t)=s_u(t)\notin K$ (from Theorem 2.1) for $\varphi:[-h,0]\to 0\in K$, some $u:[0,\infty)\to K_U$ and some $t\in\mathbb{R}_+$ (otherwise, $B\in\Pi(K_U,K)$). As a result, (S) is (K_U,K)-INN and (i) is proved.

(ii) is proved in a similar way as (i) by using Theorem 2.1 and (2.2).

(iii) ("*If part*"): assume that (S) is (K_U,K)-INN then $x\in K$ for all admissible $\mathbb{R}_+^m$. If, furthermore, $CK+DK_u\subseteq K_Y$ then $y\in K_Y$ for all admissible $\mathbb{R}_+^m$ so that (S) is (K_U,K,K_Y)-NN.

("*Only if part*"): if (S) is not (K_U,K)-INN then it cannot be (K_U,K,K_Y)-NN from Definition 4.6(iii). If $CK+DK_U\not\subset K_Y$ then $\exists u\in K_U$ for some $x\in K$ (for some $x\in K$ so that the pair (φ,u) is admissible) such that $y\notin K_Y$ so that (S) is not (K_U,K,K_Y)-NN. Then, (S) being (K_U,K)-INN and $CK+DK_U\not\subset K_Y$ are necessary conditions for (S) being (K_U,K,K_Y)-NN.

(iv) It follows from property (iii) that since $CK+DK_U\subseteq K_Y \Leftrightarrow C\in\Pi(K,K_Y)$ and $D\in\Pi(K_U,K_Y)$ since $Cx\in K_Y$ for any $x\in K$, and $Du\in K_Y$ for any $u\in K_U$ so that $CK+DK_U\subseteq K_Y+K_Y\subseteq K_Y$ since K_Y is a convex cone.

(v) ("*If part*"): note that $(\Psi(t,0)x_0) \in K$, $(\Psi(t,\xi)A_0) \in \Pi(K)$, $(\int_{-h}^{0}\Psi(t,\xi)A_0\varphi(\xi)d\xi) \in K$, $(Bu) \in K$, $(\int_0^t \Psi(t,\tau)Bu(\tau)d\tau) \in K$ for all $\varphi \in K$, all $u \in K_U$, $\xi \in [-h,0]$, $\tau \in [0,t]$, $t \in \mathbb{R}_+$ then $x \in 3K \subseteq K$ (since K is a convex cone) from (2.4).

("*Only if part*"): proceed by contradiction by assuming, for instance, that (S) is (K_U,K,K_Y)-NN with $B \notin \Pi(K_U,K)$ and take $u : \mathbb{R}_+ \to K_U$ defined by $u(\tau) = K_\delta\delta(t-\tau)$ (so that $u(\tau) = 0$, for all $\tau \in [0,t)$) for some $K_\delta \in K_U$ such that $BK_\delta \notin K$. Such a $K_\delta \in K_U$ exists since, otherwise, $B \in \Pi(K_U,K)$. Then since $\int_0^{t^+} \Psi(t,\tau)BK_\delta\delta(t-\tau)d\tau = (BK_\delta) \notin K$ since $\Psi(t,t)$
$= I_n$. If $\varphi : [-h,0] \to 0 \in \mathbb{R}^n$ then $x \notin K$ (since $x(t^+) = (BK_\delta) \notin K$) so that (S) is not (K_U,K,K_Y)-NN from Theorem 2.1 for some admissible pair $(u,0)$. Then, $B \in \Pi(K_U,K)$ is a necessary condition for (S) to be (K_U,K,K_Y)-NN. The necessity of all the remaining given conditions is proved in a similar way by using nonzero admissible pairs $(u,0)$ or $(0,\varphi)$ to establish contradictions in terms of either $x \notin K$ or $y \notin K_Y$. □

Results on positivity and strict positivity (weak and strong) under necessary conditions in terms of nonnegativity follow.

Theorem 4.19 (K-internal positivity). *Let K_U, K, and K_Y be proper cones and let (S) be (K_U,K)-INN (see Theorem 4.18(i)). Then, the following properties hold.*

(i) *(S) is (K_U,K)-IP if and only if*

$$\begin{aligned}\Psi(t,\tau)(K/\{0_n\}) + A_0(K/\{0_n\}) + B(K_U/\{0_m\})\\ \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}} \quad \forall t \in \mathbb{R}_+ \text{ and all } \tau \in [-h,0].\end{aligned} \tag{4.6}$$

(ii) *(S) is (K_U,K)-IP if and only if $\Psi(t,\tau)(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$ for all $t \in \mathbb{R}_+$, $\tau \in [-h,0]$, $A_0(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$, and $B(K_U/\{0_m\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$.*

(iii) *(S) is (K_U,K)-ISP if and only if any of the equivalent properties (i)-(ii) hold with the replacement $Z_K \to \emptyset$.*

(iv) *(S) is (K_U,K)-WIP if and only if either $\Psi(t,\tau)(K/\{0_n\}) + A_0(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$ for all $t \in \mathbb{R}_+$, all $\tau \in [-h,0]$ (zero-input (K_U,K)-WIP), or $B(K_U/\{0_m\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$ (zero initial state (K_U,K)-WIP).*

(v) *(S) is K-WIP if either $\Psi(t,\tau)(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$ for all $t \in \mathbb{R}_+$, all $\tau \in [-h,0]$; and $A_0(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$ (zero-input K-WIP), or $B(K_U/\{0_m\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}$ (zero initial state (K_U,K)-WIP).*

(vi) *(S) is (K_U,K)-WISP if and only if any of Properties (iv)-(v) hold with the replacement $Z_K \to \emptyset$.*

(vii) *(S) is (K_U,K)-SIP if and only if*

$$\begin{aligned}&(\Psi(t,\tau)(K/\{0_n\}) + A_0(K/\{0_n\}) + BK_U)\\ &\quad\cup (\Psi(t,\tau)K + A_0K + B(K_U/\{0_m\})) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}\end{aligned} \tag{4.7}$$

for all $t \in \mathbb{R}_+$, $\tau \in [-h,0]$.

(viii) *(S) is (K_U,K)-SISP if and only if (vii) holds with the replacement $Z_K \to \emptyset$.*

Proof. (i) ("*If part*"): It follows directly since $x \in (K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}})$ for all admissible nonzero (φ, u). ("*Only if part*"): proceed by contradiction. If

$$\Psi(t,\tau)(K/\{0_n\}) + A_0(K/\{0_n\}) + B(K_U/\{0_m\}) \not\subset K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}, \\ \exists (u,\varphi) \in K_U/\{0_m\} \times K/\{0_n\} \tag{4.8}$$

is admissible such that $x \notin (K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}})$ so that (S) is not (K_U, K)-IP from Definition 4.7(i).

(ii) ("*If part*"): from Theorem 2.1 and the property $3K \subseteq K$ since K is convex to yield $\{0_n\} \neq x \in K$ for any admissible nonzero pair (u,φ) implying $x \in (K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}})$ so that (S) is (K_U, K)-IP from Definition 4.7(i).

("*Only if part*"): Similar to the proof of the "only if part" of (i).

(iii) It is similar to the proofs of (i)-(ii) via Definition 4.8(i). with the replacements $K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}} \to K^0 \cup \overline{K}^{\mathrm{Fr}}$ and $\{0_n\} \neq x \in K \to K \ni x \notin \{0_n\} \cup Z_K$ for any admissible nonzero pair $(u,\varphi) \in K_U \times K$.

(iv)-(v): the proofs are similar to those of (i)-(ii) from Theorem 2.1 and Definition 4.11(i), instead of Definition 4.7(i), since (S) is WIP if it is (K_U, K)-INN; that is, $x \in K$ for all admissible $(u,\varphi) \in K_U \times K$ and $x \in (K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}})$ for all admissible $(u,\varphi) \in K_U/\{0_m\} \times \{0_n\}$ or all admissible $(u,\varphi) \in K_U \times K/\{0_n\}$.

(vi)–(viii): they follow in a similar way as that of (iv)-(v) with the use of Definitions 4.12(i), 4.9(i), and 4.10(i) with Theorem 2.1 and the respective replacements:

$$\begin{aligned} x \in \left(K^0 \cup \overline{K}^{\mathrm{Fr}}\right) \quad & \text{for all admissible } (u,\varphi) \in K_U/\{0_m\} \times \{0_n\} \\ & \text{or for all admissible } (u,\varphi) \in K_U \times K/\{0_n\}, \\ x \in \left(K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}\right) \quad & \text{for all admissible } (u,\varphi) \in (K_U \times K/\{0_n\}) \cup (K_U/\{0_m\} \times \{0_n\}), \\ x \in \left(K^0 \cup \overline{K}^{\mathrm{Fr}}\right) \quad & \text{for all admissible } (u,\varphi) \in (K_U \times K/\{0_n\}) \cup (K_U/\{0_m\} \times \{0_n\}). \end{aligned} \tag{4.9}$$

□

Theorem 4.19 might be extended directly to corresponding external-type properties (i.e., related to the output of (S)) or to combined state-output properties as established now in the subsequent two results.

Theorem 4.20 (K-external positivity). *Let K_U, K, and K_Y be proper cones and let (S) be (K_U, K, K_Y)-ENN (see Theorem 4.18(ii)). Then, the following properties hold.*

(i) *(S) is (K_U, K, K_Y)-EP if and only if*

$$C(K/\{0_n\}) + D(K_U/\{0_m\}) \subseteq K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\mathrm{Fr}}. \tag{4.10}$$

(ii) *(S) is (K_U, K, K_Y)-EP if and only if $C(K/\{0_n\}) \subseteq K_Y^0 \cup Z_{K_Y} \cup \overline{K}_Y^{\mathrm{Fr}}$ and $D(K_U/\{0_m\}) \subseteq K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\mathrm{Fr}}$.*

(iii) *(S) is (K_U, K, K_Y)-ESP if and only if any of the equivalent properties (i)-(ii) hold with the replacement $Z_{KY} \to \varnothing$.*

(iv) *(S) is* (K_U, K, K_Y)*-WEP if and only if either*

$$C(K/\{0_n\}) \subseteq K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\mathrm{Fr}} \quad (\textit{zero-input } (K_U, K, K_Y)\textit{-WEP}), \tag{4.11}$$

or

$$D(K_U/\{0_m\}) \subseteq K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\mathrm{Fr}} \quad (\textit{zero initial State } (K_U, K, K_Y)\textit{-WEP}). \tag{4.12}$$

(v) *(S) is* (K_U, K, K_Y)*-WESP if and only if (iv) holds with the replacement* $Z_{KY} \to \varnothing$*.*

(vi) *(S) is* (K_U, K, K_Y)*-SEP if and only if* $(C(K/\{0_n\}) + DK_U) \cup (CK + D(K_U/\{0_m\})) \subseteq K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\mathrm{Fr}}$*.*

(vii) *(S) is* (K_U, K, K_Y)*-SESP if and only if (vi) holds with the replacement* $Z_{KY} \to \varnothing$*.*

The proof is similar to that of Theorem 4.19, and is thus omitted.

Theorem 4.21 (K-positivity). *Let* K_U*,* K*, and* K_Y *be proper cones and let (S) be* (K_U, K, K_Y)*-NN (see Theorem 4.18(iii)–(v)). Then, the following properties hold.*

(i) *(S) is* (K_U, K, K_Y)*-P if and only if*

$$\begin{gathered}\Psi(t,\tau)(K/\{0_n\}) + A_0(K/\{0_n\}) + B(K_U/\{0_m\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}} \\ \forall t \in \mathbb{R}_+, \textit{ all } \tau \in [-h, 0]; \\ C(K/\{0_n\}) + D(K_U/\{0_m\}) \subseteq K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\mathrm{Fr}}.\end{gathered} \tag{4.13}$$

(ii) *(S) is* (K_U, K, K_Y)*-P if and only if*

$$\begin{gathered}\Psi(t,\tau)(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}} \quad \forall t \in \mathbb{R}_+, \textit{ all } \tau \in [-h, 0]; \\ A_0(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}, \qquad B(K_U/\{0_m\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}, \\ C(K/\{0_n\}) \subseteq K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\mathrm{Fr}}, \qquad D(K_U/\{0_m\}) \subseteq K_Y^0 \cup Z_{KY} \cup \overline{K}_Y^{\mathrm{Fr}}.\end{gathered} \tag{4.14}$$

(iii) *(S) is* (K_U, K, K_Y)*-SP if and only if any of the equivalent properties (i)-(ii) hold with the replacements* $Z_K \to \varnothing$ *and* $Z_{KY} \to \varnothing$*.*

(iv) *(S) is* (K_U, K, K_Y)*-WP if and only if either*

$$\begin{gathered}\Psi(t,\tau)(K/\{0_n\}) + A_0(K/\{0_n\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}} \quad \forall t \in \mathbb{R}_+, \textit{ all } \tau \in [-h, 0], \\ C(K/\{0_n\}) \subseteq K_Y^0 \cup Z_{K_Y} \cup \overline{K}_Y^{\mathrm{Fr}} \quad [\textit{Zero-Input } (K_U, K, K_Y)\textit{-WP}],\end{gathered} \tag{4.15}$$

or

$$\begin{gathered}B(K_U/\{0_m\}) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}, \\ D(K_U/\{0_m\}) \subseteq K_Y^0 \cup Z_{K_Y} \cup \overline{K}_Y^{\mathrm{Fr}} \quad (\textit{zero initial state } (K_U, K, K_Y)\textit{-WP}).\end{gathered} \tag{4.16}$$

(v) *(S) is* (K_U, K, K_Y)*-WSP if and only if (iv) holds with the replacements* $Z_K \to \varnothing$ *and* $Z_{KY} \to \varnothing$*.*

(vi) *(S) is* (K_U, K, K_Y)*-SP if and only if*

$$\begin{aligned}&(\Psi(t,\tau)(K/\{0_n\}) + A_0(K/\{0_n\}) + BK_U)\\ &\quad \cup (\Psi(t,\tau)K + A_0K + B(K_U/\{0_m\})) \subseteq K^0 \cup Z_K \cup \overline{K}^{\mathrm{Fr}}\end{aligned} \tag{4.17}$$

for all $t \in \mathbb{R}_+$, *all* $\tau \in [-h, 0]$, *and*

$$(C(K/\{0_n\}) + DK_U) \cup (CK + D(K_U/\{0_m\})) \subseteq K_Y^0 \cup Z_{K_Y} \cup \overline{K}_Y^{\mathrm{Fr}}. \tag{4.18}$$

(vii) *(S) is* (K_U, K, K_Y)*-SSP if and only if (vi) holds with the replacements* $Z_K \to \varnothing$ *and* $Z_{KY} \to \varnothing$.

The proof follows directly from Theorems 4.19 and 4.20.

Remark 4.22. All the above results are also applicable to (nonclosed) improper cones K fulfilling $\overline{K}^{\mathrm{Fr}} \not\subset K$ (so that K is trivially nonclosed although nonnecessarily open) which are pointed, solid, and convex by replacing in all the results $\overline{K}^{\mathrm{Fr}} \to \varnothing$ (resp., $\overline{K}_Y^{\mathrm{Fr}} \to \varnothing$) where $\overline{K}^{\mathrm{Fr}}$ appears since $x \in K \Rightarrow x \notin \overline{K}^{\mathrm{Fr}}$. The validity of the above nonnegative and positivity results to this case is obvious since points in $\overline{K}^{\mathrm{Fr}}$ or in K^0 are compatible with the various definitions of nonnegativity/positivity. Note that the replacements $\overline{K}^{\mathrm{Fr}} \to \varnothing$ are made by the irrelevance of $\overline{K}^{\mathrm{Fr}}$ (which is nonempty in general) in the statement of the corresponding positivity property.

5. Nonnegativity and positivity on the first orthant $\mathbb{R}_+^n$

The first orthant $\mathbb{R}_+^n$ $(n \geq 1)$ is clearly a pointed solid convex cone of interior $\mathbb{R}_+^{n^0}$ and boundary $\mathbb{R}_+^{n^{\mathrm{Fr}}}$, which is improper since $\mathbb{R}_+^n$ is open (if the infinity point is not included), such that $\mathbb{R}_+^{n^0} = \{z \in \mathbb{R}_+^n : z_i \neq 0, \text{ for all } i \in \overline{n}\}$, $\overline{\mathbb{R}}_+^{n^{\mathrm{Fr}}} = \mathbb{R}_+^{n^{\mathrm{Fr}}}/\{0_n\} \cup Z_{\mathbb{R}_+^n} = \varnothing$ (see Remark 4.22) and $Z_{\mathbb{R}_+^n} = \{0 \neq z \in \mathbb{R}^n : z_i = 0, \text{ some } i \in \overline{n}\}$. Note that $\mathbb{R}_+^n$ is also a polyhedral cone. Similarly, (first orthant) pointed solid convex cones might be defined for the state, input and output spaces of dimensions m and p. Alternatively, the set of (affinely) extended $\mathbb{R}_+^n$ closed (and then proper, i.e., a closed pointed solid convex cone) cone, $\mathrm{cl}(\mathbb{R}_+^n)$, might be considered in the formulation defined from the (affinely) extended set of nonnegative real numbers $\mathrm{cl}\,\mathbb{R}_+ = \mathbb{R}_+ \cup \{\infty\} = [0, \infty]$ (i.e., the compactification, or affine closure, of $\mathbb{R}_+$ defined by adding the affine infinity $+\infty$ to $\mathbb{R}_+$) while redefining $\overline{\mathbb{R}}_+^{n^{\mathrm{Fr}}} = \mathrm{cl}\,\mathbb{R}_+^n/(\{0_n\} \cup Z_{\mathbb{R}_+^n}) = \{\mathrm{cl}\,\mathbb{R}_+^{n^{\mathrm{Fr}}} \ni z \notin Z_{\mathrm{cl}\,\mathbb{R}_+^n} \cup \{0_n\}\}$ (see Remark 4.22). Similarly, (first orthant) proper cones $\mathrm{cl}\,\mathbb{R}_+^{n,m,p}$ are defined for the input and output spaces of interiors and boundaries

$$\begin{aligned}\mathbb{R}_+^{\ell^0} &= \{z \in \mathbb{R}_+^\ell : z_i \neq 0,\ \forall i \in \overline{\ell}\},\\ \overline{\mathbb{R}}_+^{\ell^{\mathrm{Fr}}} &= \mathrm{cl}\,\mathbb{R}_+^\ell/(\{0_\ell\} \cup Z_{\mathbb{R}_+^\ell}) = \{\mathrm{cl}\,\mathbb{R}_+^{\ell^{\mathrm{Fr}}} \ni z \notin Z_{\mathrm{cl}\,\mathbb{R}_+^\ell} \cup \{0_\ell\}\};\\ Z_{\mathrm{cl}\,\mathbb{R}_+^\ell} &= \{0 \neq z \in \mathrm{cl}\,\mathbb{R}^\ell : z_i \neq 0,\ \text{some } i \in \overline{\ell}\}\end{aligned} \tag{5.1}$$

for $\ell = m, p$. Both formulations are almost equivalent to practical effects except for unimportant details. The last one is adopted in order to refer the subsequent results to the more general ones obtained in the previous section. Definitions 4.16-4.17 might be extended "mutatis-mutandis" for matrices in the closed cone $\Pi(\mathrm{cl}\,\mathbb{R}_+^{n_1},\mathrm{cl}\,\mathbb{R}_+^{n_2}) \subseteq \mathrm{cl}\,\mathbb{R}^{n_1\times n_2}$. In the particular definitions from Definitions 4.9–4.17 related to the first orthant, the standard notation used in the above sections (i.e., $\mathrm{cl}\,\mathbb{R}_+^n$-NN, P, SP, etc.) is replaced with the simpler one NN, P, SP, and so forth. Respective alternative simplified notations for nonnegativity and positivity in the first orthant "≥ 0," "> 0," and "$\gg 0$" denote that the nonnegative, positive, and strictly positive matrices have, respectively, nonnegative entries, at least one positive entry or all their entries being positive since the state, input and output vectors of system (S) belong to cones $K = \mathrm{cl}\,\mathbb{R}_+^n$, $K_U = \mathrm{cl}\,\mathbb{R}_+^m$, and $K_Y = \mathrm{cl}\,\mathbb{R}_+^p$. Definition 4.16 for matrices in cones is extended for matrices in the closed cone $\Pi(\mathrm{cl}\,\mathbb{R}_+^n)$ as follows.

Definition 5.1. (i) A real square n-matrix M is nonnegative (NN, or via a simplified notation $M \geq 0$) if $M \in \Pi(\mathrm{cl}\,\mathbb{R}_+^n)$.

(ii) A real square n-matrix M is positive (P, or via the simplified notation $M > 0$) if $M \in \Pi(\mathrm{cl}\,\mathbb{R}_+^n)$ and $M(\mathrm{cl}\,\mathbb{R}_+^n/\{0_n\}) \subseteq \mathbb{R}_+^{n^0} \cup Z_{\mathbb{R}_+^n} \cup \overline{\mathbb{R}}_+^{n^{\mathrm{Fr}}}$.

(iii) A real square n-matrix M is strictly positive (SP or via the simplified notation $M \gg 0$) if $M \in \Pi(\mathrm{cl}\,\mathbb{R}_+^n)$ and $M((\mathrm{cl}\,\mathbb{R}_+^n/\{0_n\})) \subseteq K^0 \cup \overline{\mathbb{R}}_+^{n^{\mathrm{Fr}}}$.

Thus, all the remaining Definitions 4.1–4.12 of nonnegativity and positivity of (S) and Definition 4.17 for, in general, real rectangular matrices as well as Assumptions 4.2-4.3 also apply for the formalism in the first orthant so that the subsequent result follows directly.

Theorem 5.2. *Consider proper cones* $K = \mathrm{cl}\,\mathbb{R}_+^n$, $K_U = \mathrm{cl}\,\mathbb{R}_+^m$, $K_Y = \mathrm{cl}\,\mathbb{R}_+^p$, $\Pi(K) = \Pi(\mathrm{cl}\,\mathbb{R}_+^n)$, $\Pi(K_U,K) = \Pi(\mathrm{cl}\,\mathbb{R}_+^m,\mathrm{cl}\,\mathbb{R}_+^n)$ *and* $\Pi(K,K_Y) = \Pi(\mathrm{cl}\,\mathbb{R}_+^n,\mathrm{cl}\,\mathbb{R}_+^p)$. *Then, (S) is as follows.*

(i) *INN, ENN, and NN if the corresponding items of Theorem 4.18 hold.*

(ii) *IP, ISP, WIP, WISP, and SISP if the corresponding items of Theorem 4.19 hold.*

(iii) *EP, ESP, WEP, WESP, SEP, and SESP if the corresponding items of Theorem 4.20 hold.*

(iv) *P, SP, WP, WSP, SP, and SSP if the corresponding items of Theorem 4.21 hold.*

For any admissible pair (u,φ), global Lyapunov's stability (global Lyapunov's asymptotic stability) holds if all the eigenvalues of $\Psi(t,0)$ have modulus less than or equal to (less than) unity since the state trajectory is bounded for all admissible pairs (u,φ), $t \in \mathbb{R}_+$ (bounded for all $t \in \mathbb{R}_+$ and asymptotically converging to zero for (u,φ) being zero for t < 0 and $(0,x_0)$ bounded at $t = 0$). This follows directly from Theorem 2.1. If (S) is INN (see Theorem 5.2) then $\Psi(t,\tau) \in \Pi(\mathrm{cl}\,\mathbb{R}_+^n)$ and $\Psi(t,\tau)$ is $\mathrm{cl}\,\mathbb{R}_+^n$-irreducible for all $t(\geq \tau),\tau$ in $\mathrm{cl}\,\mathbb{R}_+$. Then, the following result holds.

Theorem 5.3. *The subsequent properties hold.*

(i) $A \in \Pi(\mathrm{cl}\,\mathbb{R}_+^n)$ *is* $\mathrm{cl}\,\mathbb{R}_+^n$*-irreducible if and only if* $(I_n + A)^{n-1} \gg 0$.

(ii) *If A is a Metzler matrix and $A_0 \geq 0$ then $\Psi(t,\tau) > 0$ for all $t \geq \tau \geq 0$.*

(iii) *If A is a Metzler matrix, $(I_n + A)^{n-1} \gg 0$ and $A_0 \geq 0$ then $\Psi(t,\tau) > 0$ is* $\mathrm{cl}\,\mathbb{R}_+^n$*-irreducible for all $t > \tau \geq 0$.*

(iv) *A is a Metzler matrix and $A_0 \geq 0$ and, furthermore, there exist real constants α, $\beta \geq \alpha$ such that $\alpha z \leq \Psi(t,0)z \leq \beta z$ for any prefixed $t > 0$ and some $z \gg 0$ (i.e., $z \in \mathbb{R}_+^{n^0} \cup \overline{\mathbb{R}}_+^n$), then (S) is IP if $B \in \Pi(\mathrm{cl}\,\mathbb{R}_+^m, \mathrm{cl}\,\mathbb{R}_+^n)$ and, furthermore,*

(1) *the (unforced) (S) is globally asymptotically Lyapunov's stable for any admissible pair $(0,\varphi)$ being uniformly bounded, except on a set of zero measure, if $\alpha,\beta \in (-1,1)$ and it is globally Lyapunov's stable for any admissible pair $(0,\varphi)$ being uniformly bounded, except on a set of zero measure, if $\alpha,\beta \in [-1,1]$. If A is a stability matrix and $\|A_0\|$ is sufficiently small compared to the stability abscissa of the matrix A, then (S) is globally asymptotically Lyapunov's stable,*

(2) *the forced (S) is $\mathbf{L}_p$-stable for any admissible pair (u,φ) being uniformly bounded, except on a subset of zero measure of its definition domain, if $\alpha,\beta \in (-1,1)$.*

(v) *A is a Metzler matrix, $(I_n + A)^{n-1} \gg 0$, $A_0 \geq 0$ and, furthermore, there exist real constants α, $\beta \geq \alpha$ with $\alpha,\beta \in [-1,1]$ such that $\alpha z < \Psi(t,0)z < \beta z$ for any prefixed $t > 0$ and some $\mathbb{R}^n \ni z > 0$ (i.e., $z \in \mathbb{R}_+^{n^0} \cup Z_{\mathbb{R}_+^n} \cup \overline{\mathbb{R}}_+^n$), or some $z \gg 0$ (i.e., $z \in \mathbb{R}_+^{n^0}$). Then the (unforced) (S) is IP if $B \in \Pi(\mathrm{cl}\,\mathbb{R}_+^m, \mathrm{cl}\,\mathbb{R}_+^n)$ and the stability properties (iv(1))-(iv(2)) hold.*

Proof. (i) is proved in [3].

(ii) If A is a Metzler matrix (i.e., all its off-diagonal entries are nonnegative) then the C_0-semigroup of infinitesimal generator A is $\mathrm{cl}\,\mathbb{R}_+^n$-positive; that is, $e^{At} > 0$ for all $t \in \mathbb{R}_+$ [7]. Then, if $A_0 \geq 0$ then the strong linear evolution operator $\Psi : [0,t] \times [0,\tau] \to \mathbf{L}(\mathbb{R}^\mathbf{n})$, for all $t,\tau(\leq t) \in \mathbb{R}_+$ is in $\Pi(\mathrm{cl}\,\mathbb{R}_+^n)$ for all $t,\tau(\leq t) \in \mathbb{R}_+$ which follows by direct calculus from (2.6) of Theorem 2.1 since $\Psi(t,\tau)$ is the sum of the two nonnegative matrices $e^{A(t-\tau)} > 0$ and $\int_{\tau+h}^t e^{A(t-\tau-\sigma)} A_0 \Psi(\sigma - h, \tau)d\sigma \geq 0$, for all $t,\tau(\leq t) \in \mathbb{R}_+$, the second one being nonnegative by recursion via (2.6) for all $t,\tau(\leq t) \in \mathbb{R}_+$ since $A_0 \geq 0$ and $\Psi(\sigma,\sigma) = I_n > 0$, for all $\sigma \in \mathbb{R}_+$.

Then, $\Psi(t,\tau)(\mathrm{cl}\,\mathbb{R}_+^n/\{0_n\}) \subseteq \mathbb{R}_+^{n^0} \cup Z_{\mathbb{R}_+^n} \cup \overline{\mathbb{R}}_+^{n^{\mathrm{Fr}}}$, for all $t,\tau(< t) \in \mathbb{R}_+$, or equivalently, $\Psi(t,\tau) > 0$, for all $t,\tau(\leq t) \in \mathbb{R}_+$ which proves (ii).

(iii) From (i)-(ii), $e^{At} > 0$ (since A is a Metzler matrix) from (ii), $\Psi(t,\tau) > 0$ for all $t \geq \tau \geq 0$ and A is irreducible (in the sense of $\mathrm{cl}\,\mathbb{R}_+^n$-irreducible) from (i) since $(I_n + A)^{n-1} \gg 0$. Now, note the following.

(a) A matrix Q is reducible, if and only if there exists a real n-permutation matrix P such that $P^TQP = \left[\begin{smallmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22}\end{smallmatrix}\right]$ with Q_{11} and Q_{22} being square submatrices of orders $n_1 < n$ and $n_2 < n$ with $n = n_1 + n_2$, [7].

(b) $e^{Qt} = \sum_{k=0}^{\infty}(Q^k t^k/k!)$ since e^{Qt} is the limit as $k \to \infty$ of everywhere convergent series $\sum_{i=0}^{k}(Q^i t^i/i!)$ for all $t \geq 0$.

(c) $P^T = P^{-1}$ (since P is a permutation matrix) implies $P^T Q^\ell P = [P^{-1}QP]^\ell$ for any $\ell \in \mathbb{Z}_+$ so that

$$P^T e^{Qt} P = P^{-1} e^{Qt} P = \begin{bmatrix} e^{Q_{11}t} & \widehat{Q}_{12}(t) \\ 0 & e^{Q_{22}t} \end{bmatrix} \quad \text{iff } Q \text{ is reducible.} \tag{5.2}$$

Since A is $\mathrm{cl}\,\mathbb{R}_+^n$-irreducible, there is no (nonsingular) transformation with associate n-matrix P which transforms A and $e^{At}(\mathbb{R}_+ \ni t > 0)$ into corresponding triangular similar matrices so that $e^{At} > 0$ is irreducible for all $\mathbb{R}_+ \ni t > 0$. Since $\Psi(t,\tau)$ is the sum of the matrix functions $e^{A(t-\tau)} > 0$, which are also irreducible for $t > \tau \geq 0$ and $\int_{\tau+h}^{t} e^{A(t-\tau-\sigma)} A_0 \Psi(\sigma - h,\tau) d\sigma \geq 0$, for all $t,\tau(\leq t) \in \mathbb{R}_+$ by using (2.6) of Theorem 2.1, then $\Psi(t,\tau) > 0$ and $\operatorname{cl}\mathbb{R}_+^n$-Irreducible for all $t > \tau \geq 0$, [3], and (iii) is proved.

(iv)-(v) property (iv) follows directly since if there exist real constants $\alpha,\beta \in [-1,1]$ such that $\alpha z \leq \Psi(t,0)z \leq \beta z$ for some $t > 0$ and some $z \gg 0$ then $\Psi(t,\tau) > 0$ for all $t > \tau \geq 0$ from (ii) since A is a Metzler matrix and $A_0 \geq 0$. Thus, (S) is IP if $B \in \Pi(\operatorname{cl}\mathbb{R}_+^m, \operatorname{cl}\mathbb{R}_+^n)$. Then, $\Psi(t,0) > 0$ for all $\mathbb{R}_+ \ni t > 0$ with (real) maximal eigenvalue being also the spectral radius in $(-1,1)$ if $\alpha,\beta \in (-1,1)$. Then the unforced (S) is globally asymptotically Lyapunov's stable while the forced (S) is $\mathbf{L}_q$-stable for any admissible pair (u,φ) being uniformly bounded except (possibly) on a set of zero measure with $u \in L_q^m \cap \mathbb{R}_+^m$, some $\mathbb{R}_+ \ni q \geq 1$. Now, consider a nonnegative real function $\overline{\psi} : \mathbb{R}_+ \to \mathbb{R}$ such that $\operatorname{Sup}_{t\geq\tau\geq 0}(\|\Psi(t,\tau)\|) \leq \overline{\psi}(t)$ for any matrix norm pointwise defined for the strong evolution operator Ψ since $\Psi \in C^{(1)}(\operatorname{cl}\mathbb{R}_+ \times \operatorname{cl}\mathbb{R}_+, \mathbf{L}(\operatorname{cl}\mathbb{R}_+^n))$. It follows from (2.6) that $\overline{\psi}(t) < \infty$ since $\Psi(0,0) = I_n$, being trivially bounded, implies via recursion that $\operatorname{Sup}_{t\geq\tau\geq 0}(\|\Psi(t,\tau)\|) \leq \overline{\psi}(t) < \infty$, for all $t \in \operatorname{cl}\mathbb{R}_+$ provided that $\|A_0\|$ is sufficiently small satisfying $1 > (k_A/\rho_A)(1 - e^{-\rho_A h})\|A_0\|$, where $k_A \geq 1$ and $\rho_A > 0$ are, respectively, a norm upper bound of $\operatorname{Sup}_{t\in\operatorname{cl}\mathbb{R}_+}(\|e^{At}\|) \leq k_A < \infty$ (for the same matrix norm as that used for $\Psi(t,\tau)$) and the minus stability abscissa of A (i.e., $(-\rho_A) < 0$ which is the absolute abscissa of the dominant (real) eigenvalue of the Metzler stability matrix A) [27–29, 34]. It has been proved that the unforced (S) is Lyapunov stable. On the other hand, from the above result in (2.6):

$$\overline{\psi}(t) \leq k_A \left(1 - \frac{k_A}{\rho_A}(1 - e^{-\rho_A h})\|A_0\|\right)^{-1} e^{-\rho_A t} < \infty, \quad \forall t \in \operatorname{cl}\mathbb{R}_+; \lim_{(t-\tau)\to\infty} (\|\Psi(t,\tau)\|) = 0 \tag{5.3}$$

so that the unforced (S) is globally asymptotically Lyapunov's stable and $\Psi \in C^{n\times n^{(1)}}(\operatorname{cl}\mathbb{R}_+, \mathbf{L}(\operatorname{cl}\mathbb{R}_{\mu+}^n))$ where $\mathbb{R}_{\mu+}^n = \{z \in \operatorname{cl}\mathbb{R}_+^n : \|z\| \leq \mu\}$, some finite $\mu \in \mathbb{R}_+$.

Remark in the proof. The last part of the above proof is also valid for the case $\Psi \in C^{(1)}(\operatorname{cl}\mathbb{R}_+ \times \operatorname{cl}\mathbb{R}_+, \mathbf{L}(\operatorname{cl}\mathbb{R}^n))$ so the condition for asymptotic stability in terms of A being a stability matrix and $\|A_0\|$ sufficiently small holds for any (S), (2.1)-(2.2), irrespective of its nonnegativity properties.

Finally, if $\alpha,\beta \in [-1,1]$ then the unforced (S) is guaranteed to be Lyapunov's stable. Property (v) follows in a similar way as property (iv) from (iii) with $\alpha,\beta \in [-1,1]$ since $\alpha z \leq \Psi(t,0)z \leq \beta z \Leftrightarrow \alpha z < \Psi(t,0)z < \beta z$ for some $z > 0$, or some $z \gg 0$, for any prefixed $t > 0$, since $\Psi(t,0) > 0$ is irreducible for all $t > 0$ with the spectral radius being a real maximal eigenvalue in $(-1,1)$ [3] (see also [7, 13]). □

Remark 5.4. Note that in order to test Theorem 5.3(v) for some $z > 0$, it is sufficient to check such a vector candidate among those not being in the set of eigenvalues of $\Psi(t,0)$, for any prefixed $t > 0$, since $\Psi(t,0)$ is $\operatorname{cl}\mathbb{R}_+^n$-irreducible if and only if $\Psi(t,0)$ has exactly one (up to scalar multiples) eigenvector z in the cone $\operatorname{cl}\mathbb{R}_+^n$ and this vector is in $\mathbb{R}_+^{n^0}$ so that $z \gg 0$. Also, $\Psi(t,0)$ is $\operatorname{cl}\mathbb{R}_+^n$-irreducible if and only if it has no eigenvector in the boundary of $\mathbb{R}_+^n$ so that any $z > 0$ cannot be an eigenvalue of $\Psi(t,0)$ for any $t \in \mathbb{R}_+$, [3].

Remark 5.5. Note that since (S) is linear and time invariant, it suffices to check the stability properties of Theorem 5.3(iv)-(v) for any prefixed $t > 0$ since the maximal eigenvalue of the strong evolution operator for any $t > \tau \geq 0$ is real of modulus less than unity for all $0 \neq t \in \mathbb{R}_+$. However, the irreducibility of the strong evolution operator does not hold for $\Psi(t,t) = I_n$ for any $t \in \mathbb{R}_+$ so that it has to be formulated for $\Psi(t,\tau)$ for any $t,\tau \in \mathbb{R}_+$ $t > \tau \geq 0$. Note that for $h = 0$, $\Psi(t,\tau) = e^{A(t-\tau)}$ so that, under Theorem 5.3(iv), its maximal eigenvalue is real positive less than unity for any $t,\tau \in \mathbb{R}_+$ $t > \tau \geq 0$ with the Metzler matrix A then being also a stability matrix so that its maximal eigenvalue is real negative. Thus, the delay-free unforced systems are globally Lyapunov's stable. The property of global asymptotic stability of the unforced delay-free system is then guaranteed since A is a Metzler stability matrix, $A_0 \geq 0$ with $\Psi(t,\tau)$ having real maximal eigenvalue less than unity for any $t,\tau \in \mathbb{R}_+$ with $t > \tau$ and some delay $h > 0$.

The next result links excitability and transparency with the parallel properties of delay-free positive systems.

Theorem 5.6. *The following properties hold.*

(i) *Assume $A_0 \geq 0$ and that the particular (S) under delay-free dynamics (i.e., $A_0 \equiv 0$) is $(\mathrm{cl}\,\mathbb{R}_+^m, \mathrm{cl}\,\mathbb{R}_+^n)$-excitable. Then (S) is $(\mathrm{cl}\,\mathbb{R}_+^m, \mathrm{cl}\,\mathbb{R}_+^n)$-excitable independent of the delay (i.e., for all delays $h \in [0,\infty)$), $\sum_{k=0}^{n-1} A^k B \gg 0$, $\sum_{k=0}^{n-1} (A + A_0)^k B \gg 0$ and $B > 0$.*

(ii) *Assume $A_0 > 0$ and that the particular (S) under delay-free dynamics (i.e., $A_0 \equiv 0$) is $(\mathrm{cl}\,\mathbb{R}_+^n, \mathrm{cl}\,\mathbb{R}_+^p)$-transparent. Then (S) is $(\mathrm{cl}\,\mathbb{R}_+^n, \mathrm{cl}\,\mathbb{R}_+^p)$-transparent independent of the delay and $\sum_{k=0}^{n-1} CA^k \gg 0$, $\sum_{k=0}^{n-1} C(A + A_0)^k \gg 0$ and $C > 0$.*

Proof. (i) From Theorem 2.1 ((2.4) and (2.6)), the state-trajectory solution for zero initial state is $x(t) = x_z(t) + \int_0^t \int_{t+h}^t e^{A(t-\tau-\sigma)} A_0 \Psi(\sigma - h,\tau) Bu(\tau) d\sigma\, d\tau \geq x_z(t) \gg 0$, $\mathbb{R}_+ \ni t > 0$, for all $h \in \mathbb{R}_+$ since $A_0 \geq 0$, $B > 0$ and $x_z(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \gg 0$ since (S) is $(\mathrm{cl}\,\mathbb{R}_+^m, \mathrm{cl}\,\mathbb{R}_+^n)$-excitable (for $A_0 = 0$) so that A is a Metzler matrix and $e^{At} > 0$ for $\mathbb{R}_+ \ni t > 0$ and $u(t) > 0$, $\mathbb{R}_+ \ni t > 0$. Since $x(t) \gg 0$ $\mathbb{R}_+ \ni t > 0$, for all $h \in \mathbb{R}_+$, (S) is $(\mathrm{cl}\,\mathbb{R}_+^m, \mathrm{cl}\,\mathbb{R}_+^n)$-excitable independent of the delay. If $h = 0$ (zero delay) (S) is $(\mathrm{cl}\,\mathbb{R}_+^m, \mathrm{cl}\,\mathbb{R}_+^n)$-excitable if and only if $\sum_{k=0}^{n-1} (A + A_0)^k B \gg 0$ since its delay-free dynamics are given by $\dot{x}(t) = (A_0 + A_1)x(t)$ [8]. By the same necessary and sufficient condition if h is infinity, or for $A_0 = 0$, $\sum_{k=0}^{n-1} A^k B \gg 0$. Those parametrical properties never hold if $B \geq 0$ with $B = 0$ so that $B > 0$. Property (i) is proved.

(ii) The proof is similar to that of (i) by substituting (2.4) into (2.2) for an admissible pair $(0,\varphi)$ with zero input and the use of the necessary and sufficient condition $\sum_{k=0}^{n-1} CA^k \gg 0$ of $(\mathrm{cl}\,\mathbb{R}_+^n, \mathrm{cl}\,\mathbb{R}_+^p)$-transparency of linear delay-free time invariant systems. □

A collateral interest of the problem focused on in this manuscript is its potential generalization to a wider class of problems. In particular, the results presented in the paper could be extended to singular dynamic systems as well as to hybrid systems composed of coupled continuous-time and digital states. They could be also potentially extended to more general descriptions involving ODE problems in the complex Euclidean space with the right-hand side being polynomials with, in general, nonconstant periodic coefficients [43–47].

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