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Research Article

Oscillatory Property of Solutions for $p(t)$-Laplacian Equations

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We consider the oscillatory property of the following $p(t)$-Laplacian equations $-(|u'|^{p(t)-2}u')' = 1/t^{\theta(t)}g(t,u)$, $t > 0$. Since there is no Picone-type identity for $p(t)$-Laplacian equations, it is an unsolved problem that whether the Sturmian comparison theorems for $p(x)$-Laplacian equations are valid or not. We obtain sufficient conditions of the oscillatory of solutions for $p(t)$-Laplacian equations.

1. Introduction

In recent years, the study of differential equations and variational problems with nonstandard $p(x)$-growth conditions have been an interesting topic (see [1–6]). The study of such problems arise from nonlinear elasticity theory, electrorheological fluids (see [3, 6]). On the asymptotic behavior of solutions of $p(x)$-Laplacian equations on unbounded domain, we refer to [5].

In this paper, we consider the oscillation problem

$$-\triangle_{p(t)} u := -\left(|u'|^{p(t)-2}u'\right)' = \frac{1}{t^{\theta(t)}}g(t,u), \quad t > 0, \tag{1.1}$$

where $p : \mathbb{R} \to (1,\infty)$ is a function, and $-\triangle_{p(t)}$ is called $p(t)$-Laplacian.

By an oscillatory solution we mean one having an infinite number of zeros on $0 < t < \infty$. Otherwise, the solution is said to be nonoscillatory. Hence, a nonoscillatory solution eventually keeps either positive or negative. It is called a positive (or negative) solution.

If $p(t) \equiv p$ is a constant, then $-\triangle_{p(t)}$ is the well-known p-Laplacian, and (1.1) is the usual p-Laplacian equation. But if $p(t)$ is a function, the $-\triangle_{p(t)}$ is more complicated than $-\triangle_p$, since it represents a nonhomogeneity and possesses more nonlinearity; for example, if Ω is bounded, the Rayleigh quotient

$$\lambda_{p(t)} = \inf_{u\in W_0^{1,p(t)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (1/p(t))|\nabla u|^{p(t)}dt}{\int_\Omega (1/p(t))|u|^{p(t)}dt}, \tag{1.2}$$

is zero in general, and only under some special conditions $\lambda_{p(t)} > 0$ (see [2]), but the fact that $\lambda_p > 0$ is very important in the study of p-Laplacian problems.

It is well known that, there exists Picone-type identity for p-Laplacian equations, and then it is easy to obtain Sturmian comparison theorems for p-Laplacian equations, which is very important in the study of the oscillation of the solutions of p-Laplacian equations. There are many papers about the oscillation problem of p-Laplacian equations (see [7–10]). On the typical p-Laplacian problem

$$-\triangle_p u = \frac{\lambda}{t^p}|u|^{p-2}u, \quad t > 0, \tag{1.3}$$

when $\lambda > ((p-1)/p)^p$, then all the solutions oscillation, but when $\lambda \le ((p-1)/p)^p$, then all the solutions are nonoscillation (see [10]). But there is no Picone-type identity for $p(t)$-Laplacian equations, it is an unsolved problem that whether the Sturmian comparison theorems for $p(x)$-Laplacian equations are valid or not. The results on the oscillation problem of $p(t)$-Laplacian equations are rare.

We say a function $f : \mathbb{R} \to \mathbb{R}$ possesses property (H) if it is continuous and satisfies $\lim_{t\to\infty} f(t) = f_\infty$, and $t^{|f(t)-f_\infty|} \le M^*$ for $t > 0$.

Throughout the paper, we always assume that

(A_1) $\theta \in C(\mathbb{R}^+,\mathbb{R})$, $p \in C^1(\mathbb{R},(1,\infty))$ and satisfies

$$1 < \inf_{x\in\mathbb{R}} p(x) \le \sup_{x\in\mathbb{R}} p(x) < +\infty; \tag{1.4}$$

(A_2) g is continuous on $\mathbb{R}^+ \times \mathbb{R}$, $g(t,\cdot)$ is increasing for any fixed $t > 0$, $g(t,u)u > 0$ for any $u \ne 0$ and satisfies

$$0 < \varliminf_{t\to+\infty} g(t,u)u \le \varlimsup_{t\to+\infty} g(t,u)u < +\infty, \quad \forall u \in \mathbb{R}\setminus\{0\}. \tag{1.5}$$

The main results of this paper are as follows.

Theorem 1.1. *Assume that $\varlimsup_{t\to+\infty}\theta(t) < \varliminf_{t\to+\infty} p(t)$, suppose that (1.1) has a positive solution u, then u is increasing for t sufficiently large, and u tends to $+\infty$ as $t \to +\infty$.*

Theorem 1.2. *Assume that p possesses property (H) and $g(t,u) = |u|^{q(t)-2}u$, where θ satisfies*

$$\varlimsup_{t\to+\infty} \theta(t) < \varliminf_{t\to+\infty} q(t), \tag{1.6}$$

where q satisfies

$$1 < \varlimsup_{t\to+\infty} q(t) < \varliminf_{t\to+\infty} p(t), \tag{1.7}$$

or $\lim_{t\to+\infty} q(t) = \lim_{t\to+\infty} p(t)$ *and* $q(t)$ *possesses property* (H)*, then all the solutions of (1.1) are oscillatory.*

2. Proofs of main results

In the following, we denote $-(\varphi(t,u'))' = -(|u'|^{p(t)-2}u')'$, and use C_i and c_i to denote positive constants.

Proof of Theorem 1.1. Let $u(t)$ be a positive solution of (1.1) , then there exists a $T > 0$ such that $u(t) > 0$ for $t \geq T$. Hence, by (A_2), we have

$$(\varphi(t,u'))' = -\frac{1}{t^{\theta(t)}}g(t,u) < 0 \quad \text{for } t > T. \tag{2.1}$$

We first show that $u' > 0$ for $t > T$. If it is false, we suppose that there exists a $t_1 \geq T$ such that $u'(t_1) \leq 0$. Since $ug(t,u) > 0$ when $u \neq 0$, by (2.1), we have

$$\varphi(t,u'(t)) < \varphi(t_1,u'(t_1)) \leq 0 \quad \text{for } t > t_1. \tag{2.2}$$

Hence we can find a $t_2 > t_1$ such that $u'(t_2) < 0$. Integrating both sides of (2.1) from t_2 to t, we get $\varphi(t,u'(t)) \leq \varphi(t_2,u'(t_2)) < 0$ for $t > t_2$, and therefore

$$u'(t) \leq -\left|u'(t_2)\right|^{(p(t_2)-1)/(p(t)-1)} \leq -\min_{t\geq t_2}\left|u'(t_2)\right|^{(p(t_2)-1)/(p(t)-1)} := -a < 0. \tag{2.3}$$

Integrate this inequality to obtain $u(t) \leq -a(t-t_2)+u(t_2) \to -\infty$, as $t \to +\infty$. It is a contradiction. Thus, $u(t)$ is increasing for $t \geq T$.

We next suppose that there exists a $K > 0$ such that $u(t) \leq K$ for $t \geq T$. Since $u(t)$ is increasing, then $u(t) \geq u(T)$ for $t \geq T$. From (2.1), we have

$$0 < \varphi(t,u'(t)) = \varphi(T,u'(T)) - \int_T^t \frac{1}{t^{\theta(t)}}g(t,u)dt. \tag{2.4}$$

Since u is a bounded positive solution, then it is easy to see that

$$\begin{aligned} 0 = \lim_{t\to+\infty}\varphi(t,u'(t)) &= \varphi(T,u'(T)) - \lim_{t\to+\infty}\int_T^t \frac{1}{t^{\theta(t)}}g(t,u)dt, \\ \varphi(t,u'(t)) &= \int_t^{+\infty} \frac{1}{t^{\theta(t)}}g(t,u)dt. \end{aligned} \tag{2.5}$$

Denote $\theta_* = \{\underline{\lim}_{t\to+\infty} p(t) + \max\{1,\overline{\lim}_{t\to+\infty}\theta(t)\}\}/2$, when t is large enough, we have $u'(t) \geq \varphi^{-1}(t,\int_t^{+\infty}(1/t^{\theta_*})c\,dt)$, then

$$u(t) - u(T) \geq \int_T^t \varphi^{-1}\left(t,\int_t^{+\infty}\frac{1}{t^{\theta_*}}c\,dt\right)dt \longrightarrow +\infty. \tag{2.6}$$

It is a contradiction, thereby completing the proof. □

Proof of Theorem 1.2. If it is false, then we may assume that (1.1) has a positive solution u. From Theorem 1.1, we can see that u is increasing, then

$$0 \le \lim_{t\to+\infty} \varphi(t,u'(t)) = \varphi(T,u'(T)) - \lim_{t\to+\infty}\int_T^t \frac{1}{t^{\theta(t)}} g(t,u)dt. \tag{2.7}$$

If $\lim_{t\to+\infty}\varphi(t,u'(t)) > 0$, then there exists a positive constant a such that

$$\varphi(t,u'(t)) = \varphi(T,u'(T)) - \int_T^t \frac{1}{t^{\theta(t)}} g(t,u)dt = a + \int_t^{+\infty} \frac{1}{t^{\theta(t)}} g(t,u)dt, \tag{2.8}$$

then there exists a positive constant k such that $u(t) \ge kt$ for $t \ge T$. From (1.6), when t is large enough, we have

$$\varphi(T,u'(T)) \ge \varphi(t,u'(t)) = a + \int_t^{+\infty} \frac{1}{t^{\theta(t)}} (kt)^{q(t)-1} dt = +\infty. \tag{2.9}$$

It is a contradiction. Then we have

$$\lim_{t\to+\infty} \varphi(t,u'(t)) = 0, \tag{2.10}$$

$$\varphi(t,u'(t)) = \int_t^{+\infty} \frac{1}{t^{\theta(t)}} g(t,u)dt. \tag{2.11}$$

There are two cases.

(i) Equation (1.7) is satisfied. From (1.6) and (1.7), there exists a $T_1 > T$ which is large enough such that

$$\begin{gathered} \theta^+ := \sup_{t\ge T_1} \theta(t) < q^- := \inf_{t\ge T_1} q(t), \\ q^+ := \sup_{t\ge T_1} q(t) < p^- := \inf_{t\ge T_1} p(t). \end{gathered} \tag{2.12}$$

If $\theta^+ \le 1$, since u is increasing, then

$$\varphi(t,u'(t)) = \int_t^{+\infty} \frac{1}{t^{\theta(t)}} g(t,u)dt \ge \int_t^{+\infty} \frac{1}{t^{\theta^+}} c_1 dt = +\infty, \quad \forall t \ge T_1. \tag{2.13}$$

It is a contradiction to (2.10). Thus $1 < \theta^+ < p^-$. Since u is increasing, then

$$\varphi(t,u'(t)) = \int_t^{+\infty} \frac{1}{t^{\theta(t)}} g(t,u)dt \ge \int_t^{+\infty} \frac{1}{t^{\theta^+}} c_1 dt = \frac{c_1}{\theta^+ - 1} \frac{1}{t^{\theta^+ - 1}}, \quad \forall t \ge T_1, \tag{2.14}$$

$$u'(t) \ge \varphi^{-1}\left(t, \frac{c_1}{\theta^+ - 1} \frac{1}{t^{\theta^+ - 1}}\right), \quad \forall t \ge T_1. \tag{2.15}$$

Thus, there exist $T_2 > T_1$ and positive constants C_1 and c_2 such that

$$u'(t) \geq c_2\left(\frac{1}{t^{\theta^+-1}}\right)^{1/(p^--1)}, \quad u(t) \geq C_1 t^{-((\theta^+-1)/(p^--1))+1} = C_1 t^{(p^--\theta^+)/(p^--1)}, \quad \forall t > T_2. \tag{2.16}$$

From (2.11), when $t > T_2$, we have

$$\varphi(t,u'(t)) \geq \int_t^{+\infty} \frac{1}{t^{\theta^+}}\left(C_1 t^{(p^--\theta^+)/(p^--1)}\right)^{(q^--1)} dt = \int_t^{+\infty} \frac{(C_1)^{(q^--1)}}{t^{\theta^+-((p^--\theta^+)/(p^--1))(q^--1)}} dt. \tag{2.17}$$

Denote $\theta_0 = \theta^+$, $\theta_1 = \theta^+ - ((p^- - \theta_0)/(p^- - 1))(q^- - 1)$. If $\theta_1 \leq 1$, then we have

$$\varphi(t,u'(t)) \geq \int_t^{+\infty} \frac{(C_1)^{(q^--1)}}{t^{\theta_1}} dt = +\infty. \tag{2.18}$$

It is a contradiction to (2.10). Thus $1 < \theta_1 < p^-$, and we have

$$u'(t) \geq \varphi^{-1}\left(t, \frac{(C_1)^{(q^--1)}}{\theta_1 - 1}\frac{1}{t^{\theta_1-1}}\right), \quad \forall t > T_2, \tag{2.19}$$

then, there exists $T_3 > T_2$ and positive constant c_3 and C_2 such that

$$u'(t) \geq c_3\left(\frac{1}{t^{\theta_1-1}}\right)^{1/(p^--1)}, \quad u(t) \geq C_2 t^{-((\theta_1-1)/(p^--1))+1} = C_2 t^{(p^--\theta_1)/(p^--1)}, \quad \forall t > T_3. \tag{2.20}$$

Thus

$$\varphi(t,u'(t)) = \int_t^{+\infty} \frac{1}{t^{\theta(t)}} g(t,u) dt \geq \int_t^{+\infty} \frac{(c_2)^{(q^--1)}}{t^{\theta^+-((p^--\theta_1)/(p^--1))(q^--1)}} dt. \tag{2.21}$$

Denote $\theta_2 = \theta^+ - ((p^- - \theta_1)/(p^- - 1))(q^- - 1)$. If $\theta_2 \leq 1$, then

$$\varphi(t,u'(t)) \geq \int_t^{+\infty} \frac{(c_3)^{(q^--1)}}{t^{\theta_2}} dt = +\infty. \tag{2.22}$$

It is a contradiction to (2.10). Thus $1 < \theta_2 < p^-$. So, we get a sequence $\theta_n > 1$ and satisfy $\theta_{n+1} = \theta^+ - ((p^- - \theta_n)/(p^- - 1))(q^- - 1)$, $n = 0, 1, 2, \ldots$. Then

$$\theta_{n+1} = \theta_0 + \sum_{k=0}^{n}\left(\frac{q^- - 1}{p^- - 1}\right)^k (\theta_1 - \theta_0), \quad n = 1, 2, \ldots. \tag{2.23}$$

Since (1.7) is valid, then $q^- < p^-$, thus

$$\lim_{n\to+\infty} \theta_{n+1} = \theta_0 - \frac{p^- - \theta_0}{p^- - q^-}(q^- - 1) \leq \theta_0 - (q^- - 1) < 1. \tag{2.24}$$

It is a contradiction to $\theta_n > 1$.

(ii) Equation (1.7) is not satisfied. Then $\lim_{t\to+\infty} q(t) = \lim_{t\to+\infty} p(t)$ and $q(t)$ possesses property (H). From (2.15), we can see that

$$u'(t) \geq \left(\frac{c_1}{\theta^+ - 1}\frac{1}{t^{\theta^+-1}}\right)^{1/(p(t)-1)}, \quad \forall t \geq T_1. \tag{2.25}$$

Since p possesses property (H), then, there exist $T_2 > T_1$ and positive constants C_1 and c_2 such that

$$u'(t) \geq c_2\left(\frac{1}{t^{\theta^+-1}}\right)^{1/(p_\infty-1)}, \quad u(t) \geq C_1 t^{-((\theta^+-1)/(p_\infty-1))+1} = C_1 t^{(p_\infty-\theta^+)/(p_\infty-1)}, \quad \forall t > T_2. \tag{2.26}$$

Since $\lim_{t\to+\infty} q(t) = \lim_{t\to+\infty} p(t)$ and $q(t)$ possesses property (H), then $q_\infty = p_\infty$. From (2.26), when $t > T_2$, we have

$$\varphi(t,u'(t)) = \int_t^{+\infty} \frac{1}{t^{\theta(t)}} g(t,u)dt \geq \int_t^{+\infty} \frac{(C_1)^{(q(t)-1)}}{t^{\theta^+-(p_\infty-\theta^+)}C}dt. \tag{2.27}$$

Denote $\theta_0 = \theta^+$, $\theta_1 = \theta^+ - (p_\infty - \theta_0)$. If $\theta_1 \leq 1$, then we have

$$\varphi(t,u'(t)) \geq \int_t^{+\infty} \frac{(C_1)^{(q(t)-1)}}{t^{\theta_1}}dt = +\infty. \tag{2.28}$$

It is a contradiction to (2.10). Thus $1 < \theta_1 < p_\infty$, and there exist $T_3 > T_2$ and positive constant c_3 and C_2 such that

$$u'(t) \geq c_3\left(\frac{1}{t^{\theta_1-1}}\right)^{1/(p_\infty-1)}, \quad u(t) \geq C_2 t^{-((\theta_1-1)/(p_\infty-1))+1} = C_2 t^{(p_\infty-\theta_1)/(p_\infty-1)}, \quad \forall t > T_3. \tag{2.29}$$

Repeating the above step, we can obtain a sequence $\{\theta_n\}$ such that

$$1 < \theta_{n+1} = \theta_n - (p_\infty - \theta^+) = \theta_0 - n(p_\infty - \theta^+). \tag{2.30}$$

It is a contradiction to (1.6). □

3. Applications

Let $\Omega = \{x \in \mathbb{R}^N \mid |x| > r_0\}$, p, q, and θ are radial. Let us consider

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \frac{1}{|x|^{\theta(x)}}|u|^{q(x)-2}u \text{ in } \Omega. \tag{3.1}$$

Write $t = |x|$. If u is a radial solution of (3.1), then (3.1) can be transformed into

$$-\left(t^{N-1}|u'|^{p(t)-2}u'\right)' = \frac{t^{N-1}}{t^{\theta(t)}}|u|^{q(t)-2}u, \quad t > r_0. \tag{3.2}$$

THEOREM 3.1. *Assume that* $p(t)$ *satisfies* $N < \inf p(x)$, *and* $\lim_{t\to+\infty} p(t) = p$, $p(t)$, $q(t)$, *and* $\theta(t)$ *satisfies the conditions of Theorem 1.2, then every radial solution of (3.1) is oscillatory.*

Proof. Denote $s = \int_0^t \tau^{(1-N)/(p(\tau)-1)} d\tau$, then $ds/dt = t^{(1-N)/(p(t)-1)}$, and $s \to +\infty$ if and only if $t \to +\infty$. It is easy to see that (3.2) can be transformed into

$$-\frac{d}{ds}\left(\left|\frac{d}{ds}u\right|^{p(s)-2}\frac{d}{ds}u\right) = t^{(N-1)/(p(t)-1)}\frac{t^{N-1}}{t^{\theta(t)}}g(t,u), \quad t > r_0. \tag{3.3}$$

It is easy to see that

$$\begin{aligned} 0 &< \underline{\lim}_{t\to+\infty}\left[\frac{t^{((N-1)/(p(t)-1))+N-1-\theta(t)}}{s^{-((p-1)/(p-N))(\theta(t)-((N-1)p/(p-1)))}}\right] \\ &\le \overline{\lim}_{t\to+\infty}\left[\frac{t^{((N-1)/(p(t)-1))+N-1-\theta(t)}}{s^{-((p-1)/(p-N))(\theta(t)-((N-1)p/(p-1)))}}\right] < +\infty. \end{aligned} \tag{3.4}$$

Since $\overline{\lim}_{t\to+\infty}\theta(t) < \underline{\lim}_{t\to+\infty}q(t)$, it is easy to see that

$$\frac{p-1}{p-N}\left(\overline{\lim}_{s\to+\infty}\theta(s) - \frac{(N-1)p}{p-1}\right) < \underline{\lim}_{s\to+\infty}q(s). \tag{3.5}$$

According to Theorem 1.2, then every radial solution of (3.1) is oscillatory. □

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