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Research Article

A Multiple Hilbert-Type Integral Inequality with the Best Constant Factor

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By introducing the norm $\|x\|_\alpha$ $(x \in \mathbb{R})$ and two parameters α, λ, we give a multiple Hilbert-type integral inequality with a best possible constant factor. Also its equivalent form is considered.

1. Introduction

If $p > 1$, $1/p + 1/q = 1$, $f, g \geq 0$, satisfy $0 < \int_0^\infty f^p(t)dt < \infty$ and $0 < \int_0^\infty g^q(t)dt < \infty$, then the well-known Hardy-Hilbert's integral inequality is given by (see [1, 2])

$$\iint_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left[\int_0^\infty f^p(t)dt\right]^{1/p}\left[\int_0^\infty g^q(t)dt\right]^{1/q}, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Its equivalent form is

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy < \left(\frac{\pi}{\sin(\pi/p)}\right)^p \int_0^\infty f^p(x)dx, \tag{1.2}$$

where the constant factor $(\pi/\sin(\pi/p))^p$ is still the best possible.

Hardy-Hilbert integral inequality is important in analysis and applications. During the past few years, many researchers obtained various generalizations, variants, and extensions of inequality (1.1) (see [3–9] and the references cited therein).

Hardy et al. [1] gave a Hilbert-type integral inequality similar to (1.1) as

$$\iint_0^\infty \frac{\ln(x/y)}{x-y} f(x)g(y)\,dx\,dy < \left(\frac{\pi}{\sin(\pi/p)}\right)^2 \left(\int_0^\infty f^p(x)dx\right)^{1/p} \left(\int_0^\infty g^q(x)dx\right)^{1/q}, \quad (1.3)$$

where the constant factor $(\pi/\sin(\pi/p))^2$ is the best possible.

Recently, Yang gave a generalization of (1.3) as (see [9])

$$\begin{aligned}\iint_0^\infty &\frac{\ln(x/y)f(x)g(y)}{x^\lambda - y^\lambda}dx\,dy \\ &< \left(\frac{\pi}{\lambda\sin(\pi/p)}\right)^2 \left(\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx\right)^{1/p} \left(\int_0^\infty x^{(q-1)(1-\lambda)} g^q(x)dx\right)^{1/q},\end{aligned} \quad (1.4)$$

where the constant factor $(\pi/\lambda\sin(\pi/p))^2$ is the best possible. Its equivalent form is

$$\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{\ln(x/y)f(x)}{x^\lambda - y^\lambda}dx\right)^p dy < \left(\frac{\pi}{\lambda\sin(\pi/p)}\right)^{2p} \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx, \quad (1.5)$$

where the constant factor $(\pi/\lambda\sin(\pi/p))^{2p}$ is the best possible.

At present, because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hilbert-type integral inequalities have been studied. Hong [10] obtained the following. If

$$a > 0, \quad \sum_{i=1}^n \frac{1}{p_i} = 1, \quad p_i > 1, \quad r_i = \frac{1}{p_i}\prod_{i=1}^n p_i, \quad \lambda > \frac{1}{a}\left(n - 1 - \frac{1}{r_i}\right), \quad i = 1,2,\dots,n, \quad (1.6)$$

then

$$\begin{aligned}\int_\alpha^\infty \cdots \int_\alpha^\infty &\frac{1}{\left(\sum_{i=1}^n (x_i-\alpha)^a\right)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1\,dx_2\cdots dx_n \\ &\le \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(\lambda)} \prod_{i=1}^n \left[\Gamma\left(\frac{1}{a}\left(1-\frac{1}{r_i}\right)\right)\Gamma\left(\lambda - \frac{1}{a}\left(n-1-\frac{1}{r_i}\right)\right)\int_\alpha^\infty (t-\alpha)^{n-1-\alpha\lambda} f_i^{p_i}(t)dt\right]^{1/p_i}.\end{aligned} \quad (1.7)$$

Yang and Kuang, and others obtained some multiple Hilbert-type integral inequalities (see [5, 11, 12]).

The main objective of this paper is to build multiple Hilbert-type integral inequalities with best constant factor of (1.4) and (1.5).

For this reason, we introduce signs as

$$\begin{aligned}\mathbb{R}_n^+ &= \{x = (x_1,x_2,\dots,x_n) : x_1,x_2,\dots,x_n > 0\}, \\ \|x\|_\alpha &= \left(x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha\right)^{1/\alpha}, \quad \alpha > 0,\end{aligned} \quad (1.8)$$

and we agree with $\|x\|_\alpha < c$ representing $\{x \in \mathbb{R}_n^+ : \|x\|_\alpha < c\}$.

2. Lemmas

First we give some multiple integral formulas.

Lemma 2.1 (see [13]). *If* $p_i > 0$, $i = 1,2,\dots,n$, $f(\tau)$ *is a measurable function, then*

$$\begin{aligned}&\int_{t_1,t_2,\dots,t_n>0;\,t_1+t_2+\cdots+t_n\le 1} f(t_1+t_2+\cdots+t_n)\,t_1^{p_1-1}t_2^{p_2-1}\cdots t_n^{p_n-1}\,dt_1\,dt_2\cdots dt_n \\ &\quad= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^1 f(\tau)\tau^{p_1+p_2+\cdots+p_n-1}\,d\tau.\end{aligned} \tag{2.1}$$

Lemma 2.2. *If* $r > 0$, $p_i > 0$, $i = 1,2,\dots,n$, $f(\tau)$ *is a measurable function, then*

$$\begin{aligned}&\int_{t_1,t_2,\dots,t_n>0;\,t_1+t_2+\cdots+t_n\le r} f(t_1+t_2+\cdots+t_n)\,t_1^{p_1-1}t_2^{p_2-1}\cdots t_n^{p_n-1}\,dt_1\,dt_2\cdots dt_n \\ &\quad= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^r f(\tau)\tau^{p_1+p_2+\cdots+p_n-1}\,d\tau,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\int_{t_1,t_2,\dots,t_n>0} f(t_1+t_2+\cdots+t_n)\,t_1^{p_1-1}t_2^{p_2-1}\cdots t_n^{p_n-1}\,dt_1\,dt_2\cdots dt_n \\ &\quad= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^\infty f(\tau)\tau^{p_1+p_2+\cdots+p_n-1}\,d\tau.\end{aligned} \tag{2.3}$$

Proof. Setting $t_i/r = u_i$ $(i = 1,2,\dots,n)$ on the left-hand side of (2.2) we obtain (2.2) from Lemma 2.1. □

From (2.1) and (2.3), we have the following lemma.

Lemma 2.3.

$$\begin{aligned}&\int_{t_1,t_2,\dots,t_n>0;\,t_1+t_2+\cdots+t_n\ge 1} f(t_1+t_2+\cdots+t_n)\,t_1^{p_1-1}t_2^{p_2-1}\cdots t_n^{p_n-1}\,dt_1dt_2\cdots dt_n \\ &\quad= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_1^\infty f(\tau)\tau^{p_1+p_2+\cdots+p_n-1}\,d\tau.\end{aligned} \tag{2.4}$$

Setting $t_i = (x_i/a_i)^{\alpha_i}$ $(i = 1,2,\dots,n)$ in (2.1), (2.2), (2.3), (2.4) we have the following lemma.

Lemma 2.4. *If $p_i > 0$, $a_i > 0$, $\alpha_i > 0$, $i = 1,2,\dots,n$, $f(\tau)$ is a measurable function, then*

$$\int_{x_1,x_2,\dots,x_n>0;(x_1/a_1)^{\alpha_1}+(x_2/a_2)^{\alpha_2}+\cdots+(x_1/a_n)^{\alpha_n}\le 1} f\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\left(\frac{x_2}{a_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_1}{a_n}\right)^{\alpha_n}\right) \times x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}dx_1\,dx_2\cdots dx_n$$
$$= \frac{a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}\Gamma(p_1/\alpha_1)\Gamma(p_2/\alpha_2)\cdots\Gamma(p_n/\alpha_n)}{\alpha_1\alpha_2\cdots\alpha_n\Gamma(p_1/\alpha_1+p_2/\alpha_2+\cdots+p_n/\alpha_n)}\int_0^1 f(\tau)\tau^{p_1/\alpha_1+p_2/\alpha_2+\cdots+p_n/\alpha_n-1}d\tau,$$

$$\int_{x_1,x_2,\dots,x_n>0;(x_1/a_1)^{\alpha_1}+(x_2/a_2)^{\alpha_2}+\cdots+(x_1/a_n)^{\alpha_n}\le r} f\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\left(\frac{x_2}{a_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_1}{a_n}\right)^{\alpha_n}\right) \times x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}dx_1\,dx_2\cdots dx_n$$
$$= \frac{a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}\Gamma(p_1/\alpha_1)\Gamma(p_2/\alpha_2)\cdots\Gamma(p_n/\alpha_n)}{\alpha_1\alpha_2\cdots\alpha_n\Gamma(p_1/\alpha_1+p_2/\alpha_2+\cdots+p_n/\alpha_n)}\int_0^r f(\tau)\tau^{p_1/\alpha_1+p_2/\alpha_2+\cdots+p_n/\alpha_n-1}d\tau,$$

$$\int_{x_1,x_2,\dots,x_n>0} f\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\left(\frac{x_2}{a_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_1}{a_n}\right)^{\alpha_n}\right)x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}dx_1\,dx_2\cdots dx_n$$
$$= \frac{a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}\Gamma(p_1/\alpha_1)\Gamma(p_2/\alpha_2)\cdots\Gamma(p_n/\alpha_n)}{\alpha_1\alpha_2\cdots\alpha_n\Gamma(p_1/\alpha_1+p_2/\alpha_2+\cdots+p_n/\alpha_n)}\int_0^\infty f(\tau)\tau^{p_1/\alpha_1+p_2/\alpha_2+\cdots+p_n/\alpha_n-1}d\tau,$$

$$\int_{x_1,x_2,\dots,x_n>0;(x_1/a_1)^{\alpha_1}+(x_2/a_2)^{\alpha_2}+\cdots+(x_1/a_n)^{\alpha_n}\ge 1} f\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\left(\frac{x_2}{a_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_1}{a_n}\right)^{\alpha_n}\right) \times x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}dx_1\,dx_2\cdots dx_n$$
$$= \frac{a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}\Gamma(p_1/\alpha_1)\Gamma(p_2/\alpha_2)\cdots\Gamma(p_n/\alpha_n)}{\alpha_1\alpha_2\cdots\alpha_n\Gamma(p_1/\alpha_1+p_2/\alpha_2+\cdots+p_n/\alpha_n)}\int_1^\infty f(\tau)\tau^{p_1/\alpha_1+p_2/\alpha_2+\cdots+p_n/\alpha_n-1}d\tau. \tag{2.5}$$

In particular, if $p > 0$, $\alpha > 0$, $f(\tau)$ is a measurable function, then

$$\int_{x_1,x_2,\dots,x_n>0;x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha\le 1} f(x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha)dx_1\,dx_2\cdots dx_n = \frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_0^1 f(\tau)\tau^{n/\alpha-1}d\tau, \tag{2.6}$$

$$\int_{x_1,x_2,\dots,x_n>0;x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha\ge 1} f(x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha)dx_1\,dx_2\cdots dx_n = \frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_1^\infty f(\tau)\tau^{n/\alpha-1}d\tau. \tag{2.7}$$

The following result holds.

Lemma 2.5. *If $p>1$, $n\in\mathbb{Z}_+$, $\alpha>0$, $\lambda>0$, define the weight function $w_{\alpha,\lambda}(x,p)$ as*

$$w_{\alpha,\lambda}(x,p)=\int_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda}\left[\frac{\|x\|_\alpha}{\|y\|_\alpha}\right]^{n-\lambda/p}dy. \tag{2.8}$$

Then

$$w_{\alpha,\lambda}(x,p)=\|x\|_\alpha^{n-\lambda}\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2. \tag{2.9}$$

Proof. By (2.6) and (2.7), we have

$$\begin{aligned}w_{\alpha,\lambda}(x,p)&=\|x\|_\alpha^{n-\lambda/p}\int_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda}\|y\|_\alpha^{\lambda/p-n}dy\\&=\|x\|_\alpha^{n-\lambda/p}\int_{y_1,y_2,\dots,y_n>0}\frac{\ln\left(\left(y_1^\alpha+y_2^\alpha+\cdots+y_n^\alpha\right)^{1/\alpha}/\|x\|_\alpha\right)}{\left(y_1^\alpha+y_2^\alpha+\cdots+y_n^\alpha\right)^{\lambda/\alpha}-\|x\|_\alpha^\lambda}\\&\qquad\times\left(y_1^\alpha+y_2^\alpha+\cdots+y_n^\alpha\right)^{(1/\alpha)(\lambda/p-n)}dy_1\,dy_2\cdots dy_n\\&=\|x\|_\alpha^{n-\lambda/p}\frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_0^\infty\frac{\ln\left(t^{1/\alpha}/\|x\|_\alpha\right)}{t^{\lambda/\alpha}-\|x\|_\alpha^\lambda}t^{(1/\alpha)(\lambda/p-n)}t^{n/\alpha-1}dt.\end{aligned} \tag{2.10}$$

Setting $(t^{1/\alpha}/\|x\|_\alpha)^\lambda=u$ we have

$$w_{\alpha,\lambda}(x,p)=\|x\|_\alpha^{n-\lambda}\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\frac{1}{\lambda^2}\int_0^\infty\frac{\ln u}{u-1}u^{1/p-1}du. \tag{2.11}$$

From [1, Theorem 342] we have $(1/\lambda^2)\int_0^\infty(\ln u/(u-1))u^{1/p-1}du=(\pi/\lambda\sin(\pi/p))^2$.
So we obtain

$$w_{\alpha,\lambda}(x,p)=\|x\|_\alpha^{n-\lambda}\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2. \tag{2.12}$$

Thus Lemma 2.5 is proved. □

Lemma 2.6. *If $\lambda>0$, $s>0$, then*

$$\int_1^\infty\frac{1}{x}\int_0^{1/x^\lambda}\frac{\ln u}{u-1}u^{s-1}du\,dx=\frac{2}{\lambda}\sum_{n=0}^\infty\frac{1}{(n+s)^3}. \tag{2.13}$$

Proof. Since

$$\frac{\ln u}{u-1}u^{s-1}=-\ln u\sum_{n=0}^\infty u^{n+s-1},\quad 0<u<1, \tag{2.14}$$

then

$$\begin{aligned}
\int_0^{1/x^\lambda} \frac{\ln u}{u-1} u^{s-1} du &= \sum_{n=0}^{\infty} \int_0^{1/x^\lambda} (-\ln u) u^{n+s-1} du \\
&= \sum_{n=0}^{\infty} \left[\frac{\lambda}{n+s} x^{-\lambda(n+s)} \ln x + \frac{1}{(n+s)^2} x^{-\lambda(n+s)} \right], \\
\int_1^{\infty} \frac{1}{x} \int_0^{1/x^\lambda} \frac{\ln u}{u-1} u^{s-1} du\, dx &= \int_1^{\infty} \left[\sum_{n=0}^{\infty} \left(\frac{\lambda}{n+s} x^{-\lambda(n+s)-1} + \frac{1}{(n+s)^2} x^{-\lambda(n+s)-1} \right) \right] dx \\
&= \sum_{n=0}^{\infty} \left[\int_1^{\infty} \frac{\lambda}{n+s} x^{-\lambda(n+s)-1} \ln x\, dx + \int_1^{\infty} \frac{1}{(n+s)^2} x^{-\lambda(n+s)-1} dx \right] \\
&= \frac{2}{\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+s)^3}.
\end{aligned} \tag{2.15}$$

□

We next give a key lemma in this paper.

LEMMA 2.7. *If $p > 1$, $1/p + 1/q = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $\lambda > 0$, $0 < \varepsilon < q\lambda/2p$, then*

$$\begin{aligned}
A := &\int_{\|x\|_\alpha \geq 1} \int_{\|y\|_\alpha \geq 1} \frac{\ln(\|x\|_\alpha / \|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda} \|x\|_\alpha^{-((n-\lambda)(p-1)+n+\varepsilon)/p} \|y\|_\alpha^{-((n-\lambda)(q-1)+n+\varepsilon)/q} dx\, dy \\
&\geq \left[\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)} \right]^2 \left[\frac{\pi}{\lambda \sin(\pi/p)} \right]^2 \frac{1}{\varepsilon}(1+o(1)), \quad \varepsilon \longrightarrow 0^+.
\end{aligned} \tag{2.16}$$

Proof. We have

$$\begin{aligned}
A = &\int_{\|x\|_\alpha \geq 1} \|x\|_\alpha^{\lambda/q-n-\varepsilon/p} dx \times \int_{y_1^\alpha + y_2^\alpha + \cdots + y_n^\alpha \geq 1} \frac{\ln\left((y_1^\alpha + y_2^\alpha + \cdots + y_n^\alpha)^{1/\alpha} / \|x\|_\alpha\right)}{(y_1^\alpha + y_2^\alpha + \cdots + y_n^\alpha)^{\lambda/\alpha} - \|x\|_\alpha^\lambda} \\
&\times (y_1^\alpha + y_2^\alpha + \cdots + y_n^\alpha)^{(1/\alpha)(\lambda/p-n-\varepsilon/q)} dy_1 dy_2 \cdots dy_n \\
= &\int_{\|x\|_\alpha \geq 1} \|x\|_\alpha^{\lambda/q-n-\varepsilon/p} dx \frac{\Gamma^n(1/\alpha)}{\alpha^n \Gamma(n/\alpha)} \int_1^{\infty} \frac{\ln(t^{1/\alpha}/\|x\|_\alpha)}{t^{\lambda/\alpha} - \|x\|_\alpha^\lambda} t^{(1/\alpha)(\lambda/p-n-\varepsilon/q)} t^{n/\alpha-1} dt.
\end{aligned} \tag{2.17}$$

Setting $(t^{1/\alpha}/\|x\|_\alpha)^\lambda = u$, we have

$$\begin{aligned}
A &= \int_{\|x\|_\alpha \geq 1} \|x\|_\alpha^{-n-\varepsilon} dx \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)} \frac{1}{\lambda^2} \int_{1/\|x\|_\alpha^\lambda}^{\infty} \frac{\ln u}{u-1} u^{1/p-1-\varepsilon/\lambda q} du \\
&= \int_{\|x\|_\alpha \geq 1} \|x\|_\alpha^{-n-\varepsilon} dx \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)} \frac{1}{\lambda^2} \int_0^{\infty} \frac{\ln u}{u-1} u^{1/p-1-\varepsilon/\lambda q} du \\
&\quad - \int_{\|x\|_\alpha \geq 1} \|x\|_\alpha^{-n-\varepsilon} dx \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)} \frac{1}{\lambda^2} \int_0^{1/\|x\|_\alpha^\lambda} \frac{\ln u}{u-1} u^{1/p-1-\varepsilon/\lambda q} du.
\end{aligned} \tag{2.18}$$

Notice

$$
\begin{aligned}
\int_{\|x\|_\alpha\geq 1}\|x\|_\alpha^{-n-\varepsilon}dx &= \int_{x_1,x_2,\dots,x_n>0;\,x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha\geq 1}\left(x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha\right)^{-(n+\varepsilon)/\alpha}dx_1\,dx_2\cdots dx_n\\
&= \frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_1^\infty u^{-(n+\varepsilon)/\alpha}u^{n/\alpha-1}du\\
&= \frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_1^\infty u^{-\varepsilon/\alpha-1}du = \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\cdot\frac{1}{\varepsilon},\\
&\frac{1}{\lambda^2}\int_0^\infty \frac{\ln u}{u-1}u^{1/p-1-\varepsilon/\lambda q}du = \left(\frac{\pi}{\lambda\sin(\pi/p)}\right)^2+o(1).
\end{aligned}
\tag{2.19}
$$

Further, from (2.7) and Lemma 2.6 we have

$$
\begin{aligned}
0 &\leq \int_{\|x\|_\alpha\geq 1}\|x\|_\alpha^{-n-\varepsilon}dx\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\frac{1}{\lambda^2}\int_0^{1/\|x\|_\alpha^\lambda}\frac{\ln u}{u-1}u^{1/p-1-\varepsilon/\lambda q}du\\
&\leq \int_{\|x\|_\alpha\geq 1}\|x\|_\alpha^{-n}dx\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\frac{1}{\lambda^2}\int_0^{1/\|x\|_\alpha^\lambda}\frac{\ln u}{u-1}u^{1/2p-1}du\\
&= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\frac{1}{\lambda^2}\int_1^\infty t^{-n/\alpha}\left[\int_0^{1/t^{\lambda/\alpha}}\frac{\ln u}{u-1}u^{1/2p-1}du\right]t^{n/\alpha-1}dt\\
&= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\frac{1}{\lambda^2}\frac{2\alpha}{\lambda}\sum_{n=0}^\infty\frac{1}{(n+1/2p)^3} = \frac{2\Gamma^n(1/\alpha)}{\alpha^{n-2}\lambda^3\Gamma(n/\alpha)}\sum_{n=0}^\infty\frac{1}{(n+1/2p)^3}.
\end{aligned}
\tag{2.20}
$$

Then

$$
A\geq\left[\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\right]^2\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2\frac{1}{\varepsilon}\left(1+o(1)\right).
\tag{2.21}
$$

□

3. Main results

Our main result is given in the following theorem.

THEOREM 3.1. *If* $p>1$, $1/p+1/q=1$, $n\in\mathbb{Z}$, $\alpha>0$, $\lambda>0$, $f,g\geq 0$, *satisfy*

$$
\begin{aligned}
&0<\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx<\infty,\\
&0<\int_{\mathbb{R}_+^n}\|y\|_\alpha^{(n-\lambda)(q-1)}g^q(y)dy<\infty.
\end{aligned}
\tag{3.1}
$$

Then

$$J := \iint_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda} f(x)g(y)\,dx\,dy < \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2 \times\left[\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)\,dx\right]^{1/p}\left[\int_{\mathbb{R}_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g^q(y)\,dy\right]^{1/q}, \tag{3.2}$$

$$\int_{\mathbb{R}_+^n} \|y\|_\alpha^{\lambda-n}\left[\int_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda} f(x)\,dx\right]^p dy < \left[\left(\frac{\pi}{\lambda\sin(\pi/p)}\right)^2 \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\right]^p \int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)\,dx. \tag{3.3}$$

The constant factors $(\Gamma^n(1/\alpha)/\alpha^{n-1}\Gamma(n/\alpha))[\pi/\lambda\sin(\pi/p)]^2$, $[(\pi/\lambda\sin(\pi/p))^2(\Gamma^n(1/\alpha)/\alpha^{n-1}\Gamma(n/\alpha))]^p$ *are the best possible.*

Proof. By Hölder's inequality, one has

$$\begin{aligned}
J &= \iint_{\mathbb{R}_+^n} \left[\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda}\right]^{1/p}\left[\frac{\|x\|_\alpha}{\|y\|_\alpha}\right]^{(n-\lambda)/p+\lambda/pq} \|x\|_\alpha^{(1/q-1/p)(n-\lambda)} f(x) \\
&\quad \times\left[\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda}\right]^{1/q}\left[\frac{\|y\|_\alpha}{\|x\|_\alpha}\right]^{(n-\lambda)/q+\lambda/qp} \|y\|_\alpha^{(1/p-1/q)(n-\lambda)} g(y)\,dx\,dy \\
&\le \left\{\iint_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda}\left[\frac{\|x\|_\alpha}{\|y\|_\alpha}\right]^{n-\lambda+\lambda/q} \|x\|_\alpha^{(p/q-1)(n-\lambda)} f^p(x)\,dx\,dy\right\}^{1/p} \\
&\quad \times\left\{\iint_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda}\left[\frac{\|y\|_\alpha}{\|x\|_\alpha}\right]^{n-\lambda+\lambda/p} \|y\|_\alpha^{(q/p-1)(n-\lambda)} g^q(y)\,dx\,dy\right\}^{1/q} \\
&= \left\{\iint_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda}\left[\frac{\|x\|_\alpha}{\|y\|_\alpha}\right]^{n-\lambda/p} \|x\|_\alpha^{(p-2)(n-\lambda)} f^p(x)\,dx\,dy\right\}^{1/p} \\
&\quad \times\left\{\iint_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda}\left[\frac{\|y\|_\alpha}{\|x\|_\alpha}\right]^{n-\lambda/q} \|y\|_\alpha^{(q-2)(n-\lambda)} g^q(y)\,dx\,dy\right\}^{1/q} \\
&= \left[\int_{\mathbb{R}_+^n} w_{\alpha,\lambda}(x,p)\|x\|_\alpha^{(n-\lambda)(p-2)} f^p(x)\,dx\right]^{1/p}\left[\int_{\mathbb{R}_+^n} w_{\alpha,\lambda}(y,q)\|y\|_\alpha^{(n-\lambda)(q-2)} g^q(y)\,dy\right]^{1/q}.
\end{aligned} \tag{3.4}$$

According to the condition of taking equality in Hölder's inequality, if this inequality takes the form of an equality, then there exist constants C_1 and C_2, such that they are not all zero, and

$$\begin{aligned} &C_1 \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda}\left[\frac{\|x\|_\alpha}{\|y\|_\alpha}\right]^{n-\lambda/p}\|x\|_\alpha^{(p-2)(n-\lambda)} f^p(x) \\ &\quad = C_2 \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda}\left[\frac{\|y\|_\alpha}{\|x\|_\alpha}\right]^{n-\lambda/q}\|y\|_\alpha^{(q-2)(n-\lambda)} g^q(y), \quad \text{a.e. in } \mathbb{R}_+^n \times \mathbb{R}_+^n. \end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned} C_1\|x\|_\alpha^n\|x\|_\alpha^{(p-1)(n-\lambda)} f^p(x) &= C_2\|y\|_\alpha^n\|y\|_\alpha^{(q-1)(n-\lambda)} g^q(y) \\ &= C\ (\text{constant}), \quad \text{a.e. in } \mathbb{R}_+^n \times \mathbb{R}_+^n, \end{aligned} \tag{3.6}$$

which contradicts (3.1). Hence we have

$$J < \left[\int_{\mathbb{R}_+^n} w_{\alpha,\lambda}(x,p)\|x\|_\alpha^{(n-\lambda)(p-2)} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^n} w_{\alpha,\lambda}(y,q)\|y\|_\alpha^{(n-\lambda)(q-2)} g^q(y)dy\right]^{1/q}. \tag{3.7}$$

By Lemma 2.5 and since $\pi/\sin(\pi/p) = \pi/\sin(\pi/q)$, we have

$$\begin{aligned} J &< \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2\left[\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right]^{1/p} \\ &\quad \times\left[\int_{\mathbb{R}_+^n}\|y\|_\alpha^{(n-\lambda)(q-1)} g^q(y)dy\right]^{1/q}. \end{aligned} \tag{3.8}$$

Hence (3.2) is valid.

For $0 < a < b < \infty$, let us define

$$\begin{aligned} g_{a,b}(y) &= \begin{cases} \|y\|_\alpha^{\lambda-n}\left[\int_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda} f(x)dx\right]^{p-1}, & a < \|y\|_\alpha < b, \\ 0, & 0 < \|y\|_\alpha \le a, \text{ or } \|y\|_\alpha \ge b, \end{cases} \\ \widetilde{g}(y) &= \|y\|_\alpha^{\lambda-n}\left[\int_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda} f(x)dx\right]^{p-1}, \quad y \in \mathbb{R}_+^n. \end{aligned} \tag{3.9}$$

By (3.1), for sufficiently small $a > 0$ and sufficiently large $b > 0$, we have

$$0 < \int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)} g_{a,b}^q(y)dy < \infty. \tag{3.10}$$

Hence by (3.2) we have

$$
\begin{aligned}
\int_{a<\|y\|_\alpha<b} & \|y\|_\alpha^{(n-\lambda)(q-1)}\widetilde{g}^q(y)\,dy \\
&= \int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{\lambda-n}\left[\int_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda}f(x)\,dx\right]^p dy \\
&= \int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{\lambda-n}\left[\int_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda}f(x)\,dx\right]^{p-1} \\
&\qquad\qquad \times\left(\int_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda}f(x)\,dx\right)dy \\
&= \iint_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda}f(x)g_{a,b}(y)\,dx\,dy \\
&< \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2\left[\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)\,dx\right]^{1/p} \\
&\quad \times\left[\int_{\mathbb{R}_+^n}\|y\|_\alpha^{(n-\lambda)(q-1)}g_{a,b}^q(y)\,dy\right]^{1/q} \\
&= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2\left[\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)\,dx\right]^{1/p} \\
&\quad \times\left[\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)}\widetilde{g}^q(y)\,dy\right]^{1/q}.
\end{aligned}
\tag{3.11}
$$

It follows that

$$
\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)}\widetilde{g}^q(y)\,dy < \left[\left(\frac{\pi}{\lambda\sin(\pi/p)}\right)^2\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\right]^p\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)\,dx. \tag{3.12}
$$

For $a\to 0^+$, $b\to+\infty$, by (3.1), we have

$$
\int_{\mathbb{R}_+^n}\|y\|_\alpha^{(n-\lambda)(q-1)}\widetilde{g}^q(y)\,dy \le \left[\left(\frac{\pi}{\lambda\sin(\pi/p)}\right)^2\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\right]^p\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)\,dx<\infty. \tag{3.13}
$$

Hence by (3.2) we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \|y\|_\alpha^{\lambda-n}\left[\int_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda} f(x)dx\right]^p dy \\
&\quad= \iint_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda} f(x)\widetilde{g}(y)dx\,dy \\
&\quad< \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2\left[\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right]^{1/p} \\
&\qquad\times\left[\int_{\mathbb{R}_+^n}\|y\|_\alpha^{(n-\lambda)(q-1)}\widetilde{g}^q(y)dy\right]^{1/q} \\
&\quad= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2\left[\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right]^{1/p} \\
&\qquad\times\left[\int_{\mathbb{R}_+^n}\|y\|_\alpha^{\lambda-n}\left[\int_{\mathbb{R}_+^n}\frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda} f(x)dx\right]^p dy\right]^{1/q}.
\end{aligned} \tag{3.14}$$

It follows that

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \|y\|_\alpha^{\lambda-n}\left[\int_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda-\|y\|_\alpha^\lambda} f(x)dx\right]^p dy \\
&\quad< \left[\left(\frac{\pi}{\lambda\sin(\pi/p)}\right)^2\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\right]^p\int_{\mathbb{R}_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx,
\end{aligned} \tag{3.15}$$

hence (3.3) is valid.

If the constant factor $C_{n,\alpha}(\lambda,p) = (\pi/\lambda\sin(\pi/p))^2(\Gamma^n(1/\alpha)/\alpha^{n-1}\Gamma(n/\alpha))$ in (3.2) is not the best possible, then there exists a positive number k (with $k < C_{n,\alpha}(\lambda,p)$), such that (3.2) is still valid if one replaces $C_{n,\alpha}(\lambda,p)$ by k.

For $0<\varepsilon<q\lambda/2p$, by sitting

$$\begin{aligned}
f_\varepsilon(x) &= \begin{cases} \|x\|_\alpha^{-((n-\lambda)(p-1)+n+\varepsilon)/p}, & \|x\|_\alpha \ge 1, \\ 0, & \|x\|_\alpha < 1, \end{cases} \\
g_\varepsilon(y) &= \begin{cases} \|y\|_\alpha^{-((n-\lambda)(q-1)+n+\varepsilon)/q}, & \|y\|_\alpha \ge 1, \\ 0, & \|y\|_\alpha < 1 \end{cases}
\end{aligned} \tag{3.16}$$

we have

$$\begin{aligned}
&\iint_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda} f_\varepsilon(x) g_\varepsilon(y)\,dx\,dy \\
&\quad < k\left[\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f_\varepsilon^p(x)\,dx\right]^{1/p}\left[\int_{\mathbb{R}_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g_\varepsilon^q(y)\,dy\right]^{1/q}, \\
&\int_{\|x\|_\alpha \ge 1}\int_{\|y\|_\alpha \ge 1} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda} f_\varepsilon(x) g_\varepsilon(y)\,dx\,dy \\
&\quad < k\left[\int_{\|x\|_\alpha \ge 1} \|x\|_\alpha^{(n-\lambda)(p-1)} \|x\|_\alpha^{-(n-\lambda)(p-1)-n-\varepsilon}\,dx\right]^{1/p} \\
&\quad\quad \times\left[\int_{\|y\|_\alpha \ge 1} \|y\|_\alpha^{(n-\lambda)(q-1)} \|y\|_\alpha^{-(n-\lambda)(q-1)-n-\varepsilon}\,dy\right]^{1/q} \\
&\quad = k\int_{\|x\|_\alpha \ge 1} \|x\|_\alpha^{-n-\varepsilon}\,dx = k\cdot\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\cdot\frac{1}{\varepsilon}.
\end{aligned} \tag{3.17}$$

On the other hand, from Lemma 2.7 we have

$$\begin{aligned}
&\iint_{\mathbb{R}_+^n} \frac{\ln(\|x\|_\alpha/\|y\|_\alpha)}{\|x\|_\alpha^\lambda - \|y\|_\alpha^\lambda} f_\varepsilon(x) g_\varepsilon(y)\,dx\,dy \\
&\quad \ge \left[\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\right]^2\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2\frac{1}{\varepsilon}(1+o(1)), \quad \varepsilon \longrightarrow 0^+.
\end{aligned} \tag{3.18}$$

Hence we have

$$\begin{gathered}
\left[\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\right]^2\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2\frac{1}{\varepsilon}(1+o(1)) \le k\cdot\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\cdot\frac{1}{\varepsilon}, \\
\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2(1+o(1)) \le k.
\end{gathered} \tag{3.19}$$

By sitting $\varepsilon \to 0^+$ we have

$$C_{n,\alpha}(\lambda,p) = \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^2 \le k. \tag{3.20}$$

This contradicts the fact that $k < C_{n,\alpha}(\lambda,p)$, hence the constant factor in (3.2) is the best possible. Since inequality (3.2) is equivalent to (3.3), the constant factor in (3.3) is also the best possible. Thus the theorem is proved. □

Remark 3.2. By using (3.3) we can obtain (3.2), hence inequality (3.2) is equivalent to (3.3).

Corollary 3.3. *If $p > 1$, $1/p + 1/q = 1$, $n \in \mathbb{Z}$, $\alpha > 0$, $f, g \geq 0$, satisfy*

$$0 < \int_{\mathbb{R}_+^n} f^p(x)dx < \infty, \qquad 0 < \int_{\mathbb{R}_+^n} g^q(y)dy < \infty. \tag{3.21}$$

Then

$$\begin{aligned}
&\iint_{\mathbb{R}_+^n} \frac{\ln\left(\|x\|_\alpha/\|y\|_\alpha\right)}{\|x\|_\alpha^n - \|y\|_\alpha^n} f(x)g(y)dx\,dy \\
&\quad < \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\frac{\pi}{n\sin(\pi/p)}\right]^2 \left[\int_{\mathbb{R}_+^n} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^n} g^q(y)dy\right]^{1/q}, \\
&\int_{\mathbb{R}_+^n}\left[\int_{\mathbb{R}_+^n} \frac{\ln\left(\|x\|_\alpha/\|y\|_\alpha\right)}{\|x\|_\alpha^n - \|y\|_\alpha^n} f(x)dx\right]^p dy < \left[\left(\frac{\pi}{n\sin(\pi/p)}\right)^2 \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\right]^p \int_{\mathbb{R}_+^n} f^p(x)dx.
\end{aligned} \tag{3.22}$$

The constant factors $(\Gamma^n(1/\alpha)/\alpha^{n-1}\Gamma(n/\alpha))[\pi/n\sin(\pi/p)]^2$, $[(\pi/n\sin(\pi/p))^2(\Gamma^n(1/\alpha)/\alpha^{n-1}\Gamma(n/\alpha))]^p$ in (3.22) are all the best possible.

Corollary 3.4. *If $p > 1$, $1/p + 1/q = 1$, $n \in \mathbb{Z}$, $f, g \geq 0$, satisfy*

$$0 < \int_{\mathbb{R}_+^n} f^p(x)dx < \infty, \qquad 0 < \int_{\mathbb{R}_+^n} g^q(y)dy < \infty. \tag{3.23}$$

Then

$$\begin{aligned}
&\iint_{\mathbb{R}_+^n} \frac{\ln\left(\sum_{i=1}^n x_i/\sum_{i=1}^n y_i\right)}{\left(\sum_{i=1}^n x_i\right)^n - \left(\sum_{i=1}^n y_i\right)^n} f(x)g(y)dx\,dy \\
&\quad < \frac{1}{(n-1)!}\left[\frac{\pi}{n\sin(\pi/p)}\right]^2 \left[\int_{\mathbb{R}_+^n} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^n} g^q(y)dy\right]^{1/q}, \\
&\int_{\mathbb{R}_+^n}\left[\int_{\mathbb{R}_+^n} \frac{\ln\left(\sum_{i=1}^n x_i/\sum_{i=1}^n y_i\right)}{\left(\sum_{i=1}^n x_i\right)^n - \left(\sum_{i=1}^n y_i\right)^n} f(x)dx\right]^p dy \\
&\quad < \left[\frac{1}{(n-1)!}\left(\frac{\pi}{n\sin(\pi/p)}\right)^2\right]^p \int_{\mathbb{R}_+^n} f^p(x)dx,
\end{aligned} \tag{3.24}$$

where the constant factors in (3.24) are all the best possible.

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