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*Research Article*

# Asymptotic Behavior of Solutions to Some Homogeneous Second-Order Evolution Equations of Monotone Type

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We study the asymptotic behavior of solutions to the second-order evolution equation $p(t)u''(t)+r(t)u'(t)\in Au(t)$ a.e. $t\in(0,+\infty)$, $u(0)=u_0$, $\sup_{t\geq 0}|u(t)|<+\infty$, where $A$ is a maximal monotone operator in a real Hilbert space $H$ with $A^{-1}(0)$ nonempty, and $p(t)$ and $r(t)$ are real-valued functions with appropriate conditions that guarantee the existence of a solution. We prove a weak ergodic theorem when $A$ is the subdifferential of a convex, proper, and lower semicontinuous function. We also establish some weak and strong convergence theorems for solutions to the above equation, under additional assumptions on the operator $A$ or the function $r(t)$.



## 1. Introduction

Let $H$ be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. We denote weak convergence in $H$ by $\rightharpoonup$ and strong convergence by $\to$. We will refer to a nonempty subset $A$ of $H\times H$ as a (nonlinear) possibly multivalued operator in $H$. $A$ is called monotone (resp., strongly monotone) if $(y_2-y_1,x_2-x_1)\geq 0$ (resp., $(y_2-y_1,x_2-x_1)\geq\beta|x_1-x_2|^2$ for some $\beta>0$) for all $[x_i,y_i]\in A$, $i=1,2$. $A$ is called maximal monotone if $A$ is monotone and $R(I+A)=H$, where $I$ is the identity operator on $H$.

Existence, as well as asymptotic behavior of solutions to second-order evolution equations of the form

$$\begin{gathered} p(t)u''(t)+r(t)u'(t)\in Au(t)\quad \text{a.e. on } \mathbb{R}^+, \\ u(0)=u_0,\qquad \sup_{t\geq 0}|u(t)|<+\infty, \end{gathered} \tag{1.1}$$

in the special case $p(t) \equiv 1$ and $r(t) \equiv 0$, were studied by many authors, see, for example, Barbu [1], Moroşanu [2, 3], and the references therein, Mitidieri [4, 5], Poffald and Reich [6], and Véron [7].

Véron [8, 9] studied the existence and uniqueness of solutions to (1.1) with the following assumptions on $p(t)$ and $r(t)$:

$$p \in W^{2,\infty}(0,+\infty), \qquad r \in W^{1,\infty}(0,+\infty), \\ \exists \alpha > 0 \quad \text{such that } \forall t \geq 0,\ p(t) \geq \alpha, \tag{1.2}$$

$$\int_0^{+\infty} e^{-\int_0^t (r(s)/p(s))ds} dt = +\infty. \tag{1.3}$$

The following theorem is proved in [9].

Theorem 1.1. *Assume that $A$ is a maximal monotone, $0 \in A(0)$, and (1.2) and (1.3) are satisfied. Then for each $u_0 \in D(A)$, there exists a continuously differentiable function $u \in H^2((0,+\infty);H)$, satisfying*

$$p(t)u''(t) + r(t)u'(t) \in Au(t) \quad \text{a.e. on } \mathbb{R}^+, \\ u(0) = u_0, \quad u(t) \in D(A) \quad \text{a.e. on } \mathbb{R}^+. \tag{1.4}$$

*If $u$ (resp., $v$) are solutions to (1.1) with initial conditions $u_0$ (resp., $v_0$), then for each $t \geq 0$,*

$$|u(t) - v(t)| \leq |u_0 - v_0|. \tag{1.5}$$

*In addition, $|u(t)|$ is nonincreasing.*

Véron [8, 9] also proved another existence theorem by assuming $A$ to be strongly monotone, instead of (1.3).

It is easy to show that without loss of generality, the condition $0 \in A(0)$ in Theorem 1.1 can be replaced by the more general assumption $A^{-1}(0) \neq \phi$.

In Section 2, we present our main results on the asymptotic behavior of solutions to (1.1).

## 2. Main results

In this section, we study the asymptotic behavior of solutions to the evolution equation (1.1) under appropriate assumptions on the operator $A$ and the functions $p(t)$ and $r(t)$, similar to those assumed by Véron [8, 9], implying the existence of solutions to (1.1). Throughout the paper, we assume that (1.2) holds and $A^{-1}(0) \neq \phi$.

First we prove two lemmas.

Lemma 2.1. *Assume that $u(t)$ is a solution to (1.1). Then for each $p \in A^{-1}(0)$, $|u(t) - p|$ is either nonincreasing, or eventually increasing.*

*Proof.* Let $p \in A^{-1}(0)$. By monotonicity of $A$ and (1.1), we have

$$\big(p(t)u''(t) + r(t)u'(t), u(t) - p\big) \geq 0 \quad \text{a.e. on } (0,+\infty). \tag{2.1}$$

It follows that

$$p(t)\frac{d^2}{dt^2}|u(t)-p|^2+r(t)\frac{d}{dt}|u(t)-p|^2\geq 0. \tag{2.2}$$

Dividing both sides of the above inequality by $p(t)$ and multiplying by $e^{\int_0^t(r(s)/p(s))ds}$, we obtain

$$\frac{d}{dt}\left(e^{\int_0^t(r(s)/p(s))ds}\frac{d}{dt}|u(t)-p|^2\right)\geq 0. \tag{2.3}$$

We consider two cases.

If $(d/dt)|u(t)-p|^2\leq 0$ for each $t>0$, then $|u(t)-p|^2$ is nonincreasing. Otherwise, there exists $t_0>0$ such that $(d/dt)|u(t)-p|^2_{|t=t_0}>0$. Integrating (2.3), we get for each $t\geq t_0$ that

$$e^{\int_0^t(r(s)/p(s))ds}\frac{d}{dt}|u(t)-p|^2\geq 2e^{\int_0^{t_0}(r(s)/p(s))ds}(u'(t_0),u(t_0)-p)>0. \tag{2.4}$$

Hence, $(d/dt)|u(t)-p|^2>0$ for each $t>t_0$. This means that $|u(t)-p|$ is eventually increasing. □

Note that in the proof of Lemma 2.1, we did not use the boundedness of $u$.

LEMMA 2.2. *Suppose that $u(t)$ is a solution to (1.1). Then for each $p\in A^{-1}(0)$, $\lim_{t\to+\infty}|u(t)-p|^2$ exists and $\liminf_{t\to+\infty}(d/dt)|u(t)-p|^2\leq 0$. In addition, if either (1.3) is satisfied or $A$ is strongly monotone, then $|u(t)-p|^2$ is nonincreasing.*

*Proof.* The existence of $\lim_{t\to+\infty}|u(t)-p|^2$ follows from Lemma 2.1.

By contradiction, assume that $\liminf_{t\to+\infty}(d/dt)|u(t)-p|^2>0$. Then there exist $t_0>0$ and $\lambda>0$, such that for each $t\geq t_0$,

$$\frac{d}{dt}|u(t)-p|^2\geq\lambda. \tag{2.5}$$

Integrating from $t=t_0$ to $t=T$, we get

$$|u(T)-p|^2-|u(t_0)-p|^2\geq\lambda T-\lambda t_0. \tag{2.6}$$

Letting $T\to+\infty$, we deduce that $u$ is not bounded, a contradiction. If in addition (1.3) is satisfied, assume that $|u(t)-p|$ is eventually increasing. Then there exists $t_0>0$ such that $(u'(t_0),u(t_0)-p)>0$. Dividing both sides of (2.4) by $e^{\int_0^t(r(s)/p(s))ds}$ and integrating from $t=t_0$ to $t=T$, we get

$$|u(T)-p|^2-|u(t_0)-p|^2\geq 2e^{\int_0^{t_0}(r(s)/p(s))ds}(u'(t_0),u(t_0)-p)\int_{t_0}^T e^{-\int_0^t(r(s)/p(s))ds}dt. \tag{2.7}$$

Letting $T\to+\infty$, we obtain a contradiction to assumption (1.3). This implies that $|u(t)-p|$ is nonincreasing.

Finally, assume that $A$ is strongly monotone, and let $p \in A^{-1}(0)$. Then we have

$$(p(t)u''(t)+r(t)u'(t),u(t)-p) \geq \beta|u(t)-p|^2. \tag{2.8}$$

This implies that

$$p(t)\frac{d^2}{dt^2}|u(t)-p|^2+r(t)\frac{d}{dt}|u(t)-p|^2 \geq 2\beta|u(t)-p|^2. \tag{2.9}$$

Suppose to the contrary that $|u(t)-p|$ is increasing for $t \geq T_0 > 0$. Let $K$ (resp., $M$) be an upper bound for $p(t)$ (resp., $|r(t)|$). Integrating both sides of this inequality from $t = T_0$ to $t = T$, we get

$$\begin{aligned}
2\beta\int_{T_0}^{T}&|u(t)-p|^2dt\\
&\leq K\left(\frac{d}{dT}|u(T)-p|^2-2(u'(T_0),u(T_0)-p)+\int_{T_0}^{T}\frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2dt\right)\\
&\leq K\left(\frac{d}{dT}|u(T)-p|^2-2(u'(T_0),u(T_0)-p)+\frac{M}{\alpha}|u(T)-p|^2-\frac{M}{\alpha}|u(T_0)-p|^2\right).
\end{aligned} \tag{2.10}$$

Since $|u(t)-p|$ is increasing for $t \geq T_0 > 0$, we have

$$\begin{aligned}
2\beta|u(T_0)-p|^2(T-T_0)&\\
\leq K\bigg(&\frac{d}{dT}|u(T)-p|^2-2(u'(T_0),u(T_0)-p)\\
&+\frac{M}{\alpha}|u(T)-p|^2-\frac{M}{\alpha}|u(T_0)-p|^2\bigg).
\end{aligned} \tag{2.11}$$

Taking lim inf as $T \to +\infty$ of both sides in the above inequality, by the first part of this lemma we deduce that $u(t)$ is unbounded, a contradiction. □

In the following, we prove a mean ergodic theorem when $A$ is the subdifferential of a proper, convex, and lower semicontinuous function.

Theorem 2.3. *Suppose that $u(t)$ is a solution to (1.1) and $A = \partial\varphi$, where $\varphi : H \to ]-\infty,+\infty]$ is a proper, convex, and lower semicontinuous function. If (1.3) is satisfied, then $\sigma_T := (1/T)\int_0^T u(t)dt \rightharpoonup p \in A^{-1}(0)$, as $T \to +\infty$.*

*Proof.* By the subdifferential inequality and (1.1), we get for each $p \in A^{-1}(0)$ that

$$\begin{aligned}
\varphi(u(t))-\varphi(p) &\leq (p(t)u''(t)+r(t)u'(t),u(t)-p)\\
&\leq \frac{p(t)}{2}\frac{d^2}{dt^2}|u(t)-p|^2+\frac{r(t)}{2}\frac{d}{dt}|u(t)-p|^2\\
&= \frac{p(t)}{2}e^{-\int_0^t(r(s)/p(s))ds}\frac{d}{dt}\left(e^{\int_0^t(r(s)/p(s))ds}\frac{d}{dt}|u(t)-p|^2\right).
\end{aligned} \tag{2.12}$$

Let $K$ be an upper bound for $p(t)/2$. Integrating the above inequality from $t = 0$ to $t = T$, and using integration by parts, we get

$$\begin{aligned}
&\int_0^T (\varphi(u(t)) - \varphi(p))\,dt \\
&\quad \le K\left(\frac{d}{dT}\left|u(T) - p\right|^2 - 2(u'(0), u(0) - p) + \int_0^T \frac{r(t)}{p(t)}\frac{d}{dt}\left|u(t) - p\right|^2 dt\right) \\
&\quad \le K\left(-2(u'(0), u(0) - p) + \int_0^T \frac{r(t)}{p(t)}\frac{d}{dt}\left|u(t) - p\right|^2 dt\right)
\end{aligned} \tag{2.13}$$

(the second inequality holds by Lemma 2.2). Let $R$ be an upper bound for $|r(t)|$, which exists by assumption (1.2). Since $|u(t) - p|$ is nonincreasing (by Lemma 2.2), we get from (2.13) that

$$\begin{aligned}
&\limsup_{T\to+\infty} \frac{1}{T}\int_0^T (\varphi(u(t)) - \varphi(p))\,dt \\
&\quad \le \limsup_{T\to+\infty} \frac{K}{T}\int_0^T \frac{r(t)}{p(t)}\frac{d}{dt}\left|u(t) - p\right|^2 dt \\
&\quad \le \frac{-KR}{\alpha}\limsup_{T\to+\infty} \frac{1}{T}\left[\left|u(T) - p\right|^2 - \left|u(0) - p\right|^2\right] = 0.
\end{aligned} \tag{2.14}$$

Since $p \in A^{-1}(0)$ and $A = \partial\varphi$, $p$ is a minimum point of $\varphi$. Convexity of $\varphi$ implies that

$$0 \le \varphi(\sigma_T) - \varphi(p) \le \frac{1}{T}\int_0^T \varphi(u(t))\,dt - \varphi(p). \tag{2.15}$$

Taking the lim sup as $T \to +\infty$ in the above inequality, we get by (2.14)

$$\limsup_{T\to+\infty} \varphi(\sigma_T) \le \varphi(p). \tag{2.16}$$

Assume that $\sigma_{T_n} \rightharpoonup q$ for some sequence $\{T_n\}$ converging to $+\infty$ as $n \to +\infty$. Since $\varphi$ is lower semicontinuous, we have

$$\liminf_{n\to+\infty} \varphi(\sigma_{T_n}) \ge \varphi(q). \tag{2.17}$$

Therefore,

$$\varphi(p) \ge \limsup_{T\to+\infty} \varphi(\sigma_T) \ge \liminf_{n\to+\infty} \varphi(\sigma_{T_n}) \ge \varphi(q). \tag{2.18}$$

Hence, $q \in A^{-1}(0)$ and by Lemma 2.2 $\lim_{t\to+\infty} |u(t) - q|^2$ exists. Now if $p$ is another weak cluster point of $\sigma_T$, then $\lim_{t\to+\infty}(|u(t) - p|^2 - |u(t) - q|^2)$ exists. It follows that $\lim_{t\to+\infty}(u(t), p - q)$ exists, hence $\lim_{T\to+\infty}(\sigma_T, p - q)$ exists. This implies that $p = q$, and therefore $\sigma_T \rightharpoonup p \in A^{-1}(0)$, as $T \to +\infty$. □

Theorem 2.4. *Let $u$ be a solution to (1.1). If (1.3) is satisfied and there exist $t_0 > 0$ and a positive constant $M$, such that $r(t) \geq -Mt^{-2}$ for $t \geq t_0$, then*

$$\lim_{T\to+\infty} \left| u(T) - \frac{1}{T}\int_0^T u(t)dt \right| = 0. \tag{2.19}$$

*Proof.* From (2.1), we have

$$|u'(t)|^2 \leq \frac{1}{2}\frac{d^2}{dt^2}|u(t)-p|^2 + \frac{1}{2}\frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2. \tag{2.20}$$

Multiplying both sides of the above inequality by $t^2$, integrating from $t = 0$ to $t = T$, and dividing by $T$, since $|u(t) - p|^2$ is nonincreasing, we get after integration by parts that

$$\frac{1}{T}\int_0^T t^2|u'(t)|^2 dt \leq -|u(T)-p|^2 + \frac{1}{T}\int_0^T |u(t)-p|^2 dt + \frac{1}{2T}\int_0^T \frac{t^2 r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2 dt. \tag{2.21}$$

Since $|u(t) - p|^2$ is nonincreasing (by Lemma 2.2), $r(t) \geq -Mt^{-2}$ for $t \geq t_0$, and $p(t)$ is bounded from below and by $\alpha$, we get

$$\begin{aligned}\limsup_{T\to+\infty}\frac{1}{T}\int_0^T t^2|u'(t)|^2 dt &\leq \limsup_{T\to+\infty}\frac{1}{2T}\int_0^T \frac{t^2 r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2 dt \\ &\leq \frac{-M}{2\alpha}\limsup_{T\to+\infty}\frac{1}{T}\Big[|u(T)-p|^2 - |u(t_0)-p|^2\Big] = 0.\end{aligned} \tag{2.22}$$

Integrating by parts and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\left|u(t) - \frac{1}{t}\int_0^t u(s)ds\right|^2 &= \left|\frac{1}{t}\int_0^t su'(s)ds\right|^2 \leq \left(\frac{1}{t}\int_0^t s|u'(s)|ds\right)^2 \\ &\leq \frac{1}{t^2}\left(\int_0^t ds\right)\left(\int_0^t s^2|u'(s)|^2 ds\right) = \frac{1}{t}\int_0^t s^2|u'(s)|^2 ds.\end{aligned} \tag{2.23}$$

Thus by (2.22),

$$\limsup_{t\to+\infty}\left|u(t) - \frac{1}{t}\int_0^t u(s)ds\right|^2 \leq \limsup_{t\to+\infty}\frac{1}{t}\int_0^t s^2|u'(s)|^2 ds = 0. \tag{2.24}$$

□

As a corollary to Theorem 2.4, we have the following weak convergence theorem.

Theorem 2.5. *Suppose that the assumptions in Theorems 2.3 and 2.4 are satisfied. Then $u(t) \rightharpoonup p \in A^{-1}(0)$ as $t \to +\infty$.*

In our next theorem, we prove the strong convergence of $u$ by assuming $A$ to be strongly monotone.

Theorem 2.6. *Assume that the operator $A$ is strongly monotone, and let $u$ be a solution to (1.1). Then $u(t)$ converges strongly to $p \in A^{-1}(0)$ as $t \to +\infty$.*

*Proof.* By the strong monotonicity of $A$, and for $p \in A^{-1}(0)$ (in this case $A^{-1}(0)$ is a singleton), we have

$$(p(t)u''(t)+r(t)u'(t),u(t)-p) \geq \beta |u(t)-p|^2. \tag{2.25}$$

Let $K$ be an upper bound for $p(t)$. Integrating this inequality from $t=0$ to $t=T$ and using Lemma 2.2, we obtain

$$2\beta\int_0^T |u(t)-p|^2 dt \leq K\left(\frac{d}{dT}|u(T)-p|^2 - 2(u'(0),u(0)-p) + \int_0^T \frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2 dt\right). \tag{2.26}$$

Let $R$ be an upper bound for $|r(t)|$, which exists by assumption (1.2). Dividing both sides of this inequality by $T$ and using Lemma 2.2, we get

$$\begin{aligned} 2\beta \lim_{T\to+\infty} |u(T)-p|^2 &= \limsup_{T\to+\infty} \frac{\beta}{T}\int_0^T |u(t)-p|^2 dt \\ &\leq \limsup_{T\to+\infty} \frac{K}{T}\int_0^T \frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2 dt \\ &\leq \frac{-KR}{\alpha}\limsup_{T\to+\infty}\frac{1}{T}\Big[|u(T)-p|^2 - |u(0)-p|^2\Big] = 0. \end{aligned} \tag{2.27}$$

This completes the proof of the theorem. □

Now, we apply our results to an example presented by Véron [8] and Apreutesei [10].

*Example 2.7.* Let $H = L^2(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma$. Let $j : \mathbb{R} \to (-\infty,+\infty]$ be proper, convex, and lower semicontinuous and $\beta = \partial j$. We assume for simplicity that $0 \in \beta(0)$. Define

$$Au = -\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \tag{2.28}$$

with

$$D(A) = \left\{u \in H^2(\Omega),\ \frac{-\partial u}{\partial \eta}(x) \in \beta(u(x)) \text{ a.e. on } \Gamma\right\}, \tag{2.29}$$

where $((\partial u/\partial \eta)(x))$ is the outward normal derivative to $\Gamma$ at $x \in \Gamma$. We know that $A = \partial\phi$, where $\phi : L^2(\Omega) \to (-\infty,+\infty]$ is the Brézis functional:

$$\phi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Gamma \beta(u(x))d\sigma & \text{if } u \in H^1(\Omega),\ \beta(u) \in L^1(\Gamma), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.30}$$

Consider the following equation:

$$p(t)\frac{\partial^2 u}{\partial t^2}(t,x)+r(t)\frac{\partial u}{\partial t}(t,x)+\sum_i \frac{\partial^2 u}{\partial x_i^2}(t,x)=0 \quad \text{a.e. on } \mathbb{R}^+\times\Omega,$$
$$-\frac{\partial u}{\partial \eta}(t,x)\in\beta u(t,x) \quad \text{a.e. on } \mathbb{R}^+\times\Gamma, \tag{2.31}$$
$$u(0,x)=u_0(x) \quad \text{a.e. on } \Omega.$$

Assume that $p(t)$ and $r(t)$ are real functions satisfying (1.2) and (1.3). Then Theorem 2.3 implies the weak mean ergodic convergence of $u(t,\cdot)$. In addition, if $r(t)\geq -Mt^{-2}$ eventually, Corollary 2.5 implies the weak convergence of the solution to the above equation.

## References


[1] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei Republicii Socialiste România, Bucharest, Romania; Noordhoff International, Leiden, The Netherlands, 1976.

[2] G. Moroşanu, *Nonlinear Evolution Equations and Applications*, vol. 26 of *Mathematics and Its Applications (East European Series)*, D. Reidel, Dordrecht, The Netherlands; Editura Academiei, Bucharest, Romania, 1988.

[3] G. Moroşanu, "Asymptotic behaviour of solutions of differential equations associated to monotone operators," *Nonlinear Analysis*, vol. 3, no. 6, pp. 873–883, 1979.

[4] E. Mitidieri, "Asymptotic behaviour of some second order evolution equations," *Nonlinear Analysis*, vol. 6, no. 11, pp. 1245–1252, 1982.

[5] E. Mitidieri, "Some remarks on the asymptotic behaviour of the solutions of second order evolution equations," *Journal of Mathematical Analysis and Applications*, vol. 107, no. 1, pp. 211–221, 1985.

[6] E. I. Poffald and S. Reich, "An incomplete Cauchy problem," *Journal of Mathematical Analysis and Applications*, vol. 113, no. 2, pp. 514–543, 1986.

[7] L. Véron, "Un exemple concernant le comportement asymptotique de la solution bornée de l'équation $d^2u/dt^2\in\partial\varphi(u)$," *Monatshefte für Mathematik*, vol. 89, no. 1, pp. 57–67, 1980.

[8] L. Véron, "Problèmes d'évolution du second ordre associés à des opérateurs monotones," *Comptes Rendus de l'Académie des Sciences de Paris. Série A*, vol. 278, pp. 1099–1101, 1974.

[9] L. Véron, "Équations d'évolution du second ordre associées à des opérateurs maximaux monotones," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 75, no. 2, pp. 131–147, 1975/1976.

[10] N. C. Apreutesei, "Second-order differential equations on half-line associated with monotone operators," *Journal of Mathematical Analysis and Applications*, vol. 223, no. 2, pp. 472–493, 1998.



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