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Research Article

On Certain Subclasses of Meromorphic Close-to-Convex Functions

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By using the operator $D_\lambda^n f(z)$, $z \in U$, Definition 2.1, we introduce a class of meromorphic functions denoted by $\Sigma(\alpha, \lambda, n)$ and we obtain certain differential subordinations.

1. Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}, \qquad \dot{U} = U - \{0\}. \tag{1.1}$$

Let $\mathscr{H}(U)$ be the space of holomorphic function in U.
Let

$$A_n = \{f \in \mathscr{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \cdots,\ z \in U\} \tag{1.2}$$

with $A_1 = A$.
For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we let

$$\mathscr{H}[a, n] = \{f \in \mathscr{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots,\ z \in U\}. \tag{1.3}$$

Let

$$K = \left\{f \in A,\ \operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\} \tag{1.4}$$

denote the class of normalized convex functions in U.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

A function f, analytic in U, is said to be convex if it is univalent and $f(U)$ is convex.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination,

(i) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$, $z \in U$,

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (i).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U.) In order to prove the original results, we use the following lemmas.

Lemma 1.1 (see [1, Theorem 3.1.6, page 71, and the references therein]). *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\operatorname{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad z \in U, \tag{1.5}$$

then

$$p(z) \prec q(z) \prec h(z), \quad z \in U, \tag{1.6}$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt, \quad z \in U. \tag{1.7}$$

The function q is convex and the best dominant.

Lemma 1.2 (see [2, Lemma 13.5.1, page 375, and the references therein]). *Let g be a convex function in U, and let*

$$h(z) = g(z) + n\alpha z g'(z), \quad z \in U, \tag{1.8}$$

where $\alpha > 0$, and n is a positive integer.

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \quad z \in U \tag{1.9}$$

is holomorphic in U, and

$$p(z) + \alpha z p'(z) \prec h(z), \quad z \in U, \tag{1.10}$$

then

$$p(z) \prec g(z), \tag{1.11}$$

and this result is sharp.

Lemma 1.3 (see [1, Corollary 2.6.g.2, page 66]). *Let $f \in A$ and*

$$F(z) = \frac{2}{z}\int_0^z f(t)dt, \quad z \in U. \tag{1.12}$$

If

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2}, \tag{1.13}$$

then

$$F \in K. \tag{1.14}$$

Lemma 1.4 (see [3, Lemma 1.5]). *Let* $\operatorname{Re} c > 0$, *and let*

$$w = \frac{k^2 + |c|^2 - |k^2 - c^2|}{4k\operatorname{Re} c}. \tag{1.15}$$

Let h be an analytic function in U with $h(0) = 1$, and suppose that

$$\operatorname{Re}\left(\frac{zh''(z)}{h'(z)} + 1\right) > -w, \quad z \in U. \tag{1.16}$$

If $p(z) = 1 + p_k z^k + \cdots$ ($k \geq 1$ integer) is analytic in U and

$$p(z) + \frac{1}{c}zp'(z) \prec h(z), \quad z \in U, \tag{1.17}$$

then

$$p(z) \prec q(z), \quad z \in U, \tag{1.18}$$

where q is the solution of the differential equation:

$$q(z) + \frac{k}{c}zq'(z) = h(z), \quad q(z) = 1, \tag{1.19}$$

given by

$$q(z) = \frac{c}{kz^{c/k}}\int_0^z t^{c/k-1}h(t)dt. \tag{1.20}$$

Moreover, q is the best dominant.

Definition 1.5 (see [4]). For $f \in A$, $n \in \mathbb{N}^* \cup \{0\}$, the operator $S^n f$ is defined by $S^n : A \to A$

$$\begin{aligned} S^0 f(z) &= f(z), \\ S^1 f(z) &= zf'(z), \\ &\cdots \\ S^{n+1} f(z &= z\left[S^n f(z)\right]', \quad z \in U. \end{aligned} \tag{1.21}$$

Remark 1.6. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.22}$$

then

$$S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, \quad z \in U. \tag{1.23}$$

Definition 1.7 (see [5]). For $f \in A$, $n \in \mathbb{N}^* \cup \{0\}$, the operator $R^n f$ is defined by $R^n : A \to A$

$$\begin{gathered} R^0 f(z) = f(z), \\ R^1 f(z) = z f'(z), \\ \cdots \\ (n+1) R^{n+1} f(z) = z\left[R^n f(z)\right]' + n R^n f(z), \quad z \in U. \end{gathered} \tag{1.24}$$

Remark 1.8. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.25}$$

then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U. \tag{1.26}$$

2. Main results

Definition 2.1. Let $n \in \mathbb{N}^* \cup \{0\}$ and $\lambda \geq 0$. Let $D_\lambda^n f$ denote the operator defined by $D_\lambda^n : A \to A$

$$D_\lambda^n f(z) = (1-\lambda) S^n f(z) + \lambda R^n f(z), \quad z \in U, \tag{2.1}$$

where the operators $S^n f$ and $R^n f$ are given by Definitions 1.5 and 1.7, respectively.

Remark 2.2. We observe that D_λ^n is a linear operator and for

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{2.2}$$

we have

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} \left[(1-\lambda) j^n + \lambda C_{n+j-1}^{n}\right] a_j z^j. \tag{2.3}$$

Also, it is easy to observe that if we consider $\lambda = 1$ in Definition 2.1, we obtain the Ruscheweyh differential operator, and if we consider $\lambda = 0$ in Definition 2.1, we obtain the Sălăgean differential operator.

Remark 2.3. For $n = 0$,

$$D_\lambda^0 f(z) = (1-\lambda)S^0 f(z) + \lambda R^0 f(z) = f(z) = S^0 f(z) = R^0 f(z), \tag{2.4}$$

and for $n = 1$,

$$D_\lambda^1 f(z) = (1-\lambda)S^1 f(z) + \lambda R^1 f(z) = zf'(z) = S^1 f(z) = R^1 f(z). \tag{2.5}$$

Remark 2.4. If $f \in \Sigma$,

$$f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \cdots, \tag{2.6}$$

and we let

$$g(z) = z^2 f(z) = z + a_0 z^2 + a_1 z^3 + \cdots, \quad z \in U. \tag{2.7}$$

Definition 2.5. If $0 \leq \alpha < 1$, $\lambda \geq 0$, and $n \in \mathbb{N}$, let $\Sigma(\alpha, \lambda, n+1)$ denote the class of functions $f \in \Sigma$ which satisfy the inequality,

$$\operatorname{Re}\left\{ \left[D_\lambda^{n+1} g(z)\right]' + \frac{\lambda z n \left[R^n g(z)\right]''}{n+1} \right\} > \alpha, \tag{2.8}$$

where $D_\lambda^{n+1} g$ is given by Definition 2.1, g is given by (2.7), and $R^n g$ is given by Definition 1.7.

Theorem 2.6. *If $0 \leq \alpha < 1$, $\lambda \geq 0$, and $n \in \mathbb{N}$, then*

$$\Sigma(\alpha, \lambda, n+1) \subset \Sigma(\delta, \lambda, n+1), \tag{2.9}$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1-\alpha)\ln 2. \tag{2.10}$$

Proof. Let $f \in \Sigma(\alpha, \lambda, n+1)$,

$$g(z) = z^2 f(z) = z + a_0 z^2 + a_1 z^3 + \cdots, \quad g \in A. \tag{2.11}$$

Since $f \in \Sigma(\alpha, \lambda, n+1)$ by using Definition 2.5, we deduce

$$\operatorname{Re}\left\{ \left[D_\lambda^{n+1} g(z)\right]' + \frac{\lambda n z \left[R^n g(z)\right]''}{n+1} \right\} > \alpha, \quad z \in U, \tag{2.12}$$

which is equivalent to

$$\left[D_\lambda^{n+1} g(z)\right]' + \frac{\lambda n z \left[R^n g(z)\right]''}{n+1} \prec \frac{1 + (2\alpha - 1)z}{1+z} = h(z), \quad z \in U. \tag{2.13}$$

By using the properties of the operators $D_\lambda^n g$, $S^n g$, and $R^n g$, we have

$$\begin{aligned}
&\left[(1-\lambda)S^{n+1}g(z)+\lambda R^{n+1}g(z)\right]'+\frac{\lambda nz\left[R^n g(z)\right]''}{n+1}\\
&=(1-\lambda)\left[z\left(S^n g(z)\right)'\right]'+\lambda\frac{\left[z\left(R^n g(z)\right)'+nR^n g(z)\right]'}{n+1}+\frac{\lambda nz\left(R^n g(z)\right)''}{n+1}\\
&=(1-\lambda)\left[\left(S^n g(z)\right)'+z\left(S^n g(z)\right)''\right]+\lambda\frac{\left(R^n g(z)\right)'+z\left(R^n g(z)\right)''+n\left[R^n g(z)\right]'}{n+1}+\frac{\lambda nz\left(R^n g(z)\right)''}{n+1}\\
&=(1-\lambda)\left(S^n g(z)\right)'+\lambda\left(R^n g(z)\right)'+z\left[(1-\lambda)\left(S^n g(z)\right)''+\lambda\left(R^n g(z)\right)''\right],\quad z\in U.
\end{aligned}\tag{2.14}$$

Using (2.14) in (2.13), we obtain

$$(1-\lambda)\left(S^n g(z)\right)'+\lambda\left(R^n g(z)\right)'+z\left[(1-\lambda)\left(S^n g(z)\right)''+\lambda\left(R^n g(z)\right)''\right]\prec\frac{1+(2\alpha-1)z}{1+z},\quad z\in U.\tag{2.15}$$

Let

$$\begin{aligned}
p(z)&=\left[D_\lambda^n g(z)\right]'\\
&=(1-\lambda)\left(S^n g(z)\right)'+\lambda\left(R^n g(z)\right)'\\
&=(1-\lambda)\left(z+\sum_{j=2}^{\infty}j^n a_j z^j\right)'+\lambda\left(z+\sum_{j=2}^{\infty}C_{n+j-1}^n a_j z^j\right)'\\
&=(1-\lambda)\left(1+\sum_{j=2}^{\infty}j^{n+1}a_j z^{j-1}\right)+\lambda\left(1+\sum_{j=2}^{\infty}jC_{n+j-1}^n a_j z^{j-1}\right)\\
&=1+\sum_{j=2}^{\infty}\left[(1-\lambda)j^{n+1}+\lambda jC_{n+j-1}^n\right]a_j z^{j-1}\\
&=1+b_1z+b_2z^2+\cdots,\quad z\in U.
\end{aligned}\tag{2.16}$$

We have that $p\in\mathcal{H}[1,1]$. From (2.16), we have

$$p'(z)=(1-\lambda)\left(S^n g(z)\right)''+\lambda\left(R^n g(z)\right)''.\tag{2.17}$$

Using (2.16) and (2.17) in (2.15), we obtain

$$p(z)+zp'(z)\prec\frac{1+(2\alpha-1)z}{1+z}=h(z),\quad z\in U.\tag{2.18}$$

By using Lemma 1.1, we have

$$p(z)\prec q(z)\prec h(z),\quad z\in U,\tag{2.19}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = 2\alpha - 1 + 2(1-\alpha)\frac{\ln(1+z)}{z}, \quad z \in U. \tag{2.20}$$

The function q is convex and best dominant.

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\operatorname{Re} p(z) > \operatorname{Re} q(1) = \delta = \delta(\alpha) = 2\alpha - 1 + 2(1-\alpha)\ln 2, \tag{2.21}$$

from which we deduce that $\Sigma(\alpha, \lambda, n+1) \subset \Sigma(\delta, \lambda, n+1)$. □

Example 2.7. If $n = 0$, $\alpha = 1/2$, $\lambda \geq 0$, then $\delta(1/2) = \ln 2$, and we deduce for $f \in \Sigma$ that

$$\operatorname{Re}\left[4zf(z) + 5z^2 f'(z) + z^3 f''(z)\right] > \frac{1}{2}, \quad z \in U \tag{2.22}$$

implies

$$\operatorname{Re}\left[2zf(z) + z^2 f'(z)\right] > \ln 2, \quad z \in U. \tag{2.23}$$

Theorem 2.8. *Let r be a convex function, $r(0) = 1$, and let h be a function such that*

$$h(z) = r(z) + zr'(z), \quad z \in U. \tag{2.24}$$

If $f \in \Sigma$, g is given by (2.7)*, and the following differential subordination holds*

$$\left[D_\lambda^{n+1} g(z)\right]' + \frac{\lambda n z\left[R^n g(z)\right]''}{n+1} \prec h(z) = r(z) + zr'(z), \quad z \in U, \tag{2.25}$$

then

$$\left[D_\lambda^n g(z)\right]' \prec r(z), \quad z \in U, \tag{2.26}$$

and this result is sharp.

Proof. By using the properties of the operator $D_\lambda^n g$, we have

$$D_\lambda^{n+1} g(z) = (1-\lambda)S^{n+1} g(z) + \lambda R^{n+1} g(z). \tag{2.27}$$

By using the properties of operators $S^n g(z)$, $R^n g(z)$, and by differentiating (2.27), we obtain

$$\begin{aligned}\left[D_\lambda^{n+1} g(z)\right]' &= \left[(1-\lambda)S^{n+1} g(z) + \lambda R^{n+1} g(z)\right]' \\ &= (1-\lambda)\left[\left(S^n g(z)\right)' + z\left(S^n g(z)\right)''\right] + \lambda \frac{(n+1)\left(R^n g(z)\right)' + z\left(R^n g(z)\right)''}{n+1}.\end{aligned} \tag{2.28}$$

Using (2.28) in (2.25) and relations (2.16) and (2.17), after a simple calculation, Subordination (2.25) becomes

$$p(z) + zp'(z) \prec r(z) + zr'(z), \quad z \in U. \tag{2.29}$$

By using Lemma 1.2, we have

$$p(z) \prec r(z), \quad z \in U, \tag{2.30}$$

that is,

$$\left[D_\lambda^n g(z)\right]' \prec r(z), \quad z \in U. \tag{2.31}$$

□

Example 2.9. If $n = 0$, $\lambda \geq 0$, $r(z) = (1+z)/(1-z)$, from Theorem 2.8, we deduce that if $f \in \Sigma$ and

$$4zf(z) + 5z^2 f'(z) + z^3 f''(z) \prec \frac{1+2z-z^2}{(1-z)^2}, \quad z \in U, \tag{2.32}$$

then

$$2zf(z) + z^2 f'(z) \prec \frac{1+z}{1-z}, \quad z \in U. \tag{2.33}$$

Theorem 2.10. *Let r be a convex function, $r(0) = 1$, and*

$$h(z) = r(z) + zr'(z), \quad z \in U. \tag{2.34}$$

If $f \in \Sigma$, g is given by (2.7), *and the following differential subordination holds*

$$\left[D_\lambda^n g(z)\right]' \prec h(z) = r(z) + zr'(z), \quad z \in U, \tag{2.35}$$

then

$$\frac{D_\lambda^n g(z)}{z} \prec r(z), \quad z \in U, \tag{2.36}$$

and this result is sharp.

Proof. We let

$$p(z) = \frac{D_\lambda^n g(z)}{z}, \quad z \in U. \tag{2.37}$$

By differentiating (2.37), we obtain

$$\left[D_\lambda^n g(z)\right]' = p(z) + zp'(z), \quad z \in U. \tag{2.38}$$

Using (2.38), Subordination (2.35) becomes

$$p(z) + zp'(z) \prec r(z) + zr'(z) = h(z), \quad z \in U. \tag{2.39}$$

By using Lemma 1.2, we have

$$p(z) \prec r(z), \tag{2.40}$$

that is,

$$\frac{D_\lambda^n g(z)}{z} \prec r(z), \quad z \in U, \tag{2.41}$$

and this result is sharp. □

Example 2.11. If we let $r(z) = 1/(1-z)$, $n = 1$, $\lambda \geq 0$, then

$$h(z) = \frac{1}{(1-z)^2}, \tag{2.42}$$

and from Theorem 2.10, we deduce that if $f \in \Sigma$, and

$$4zf(z) + 5z^2 f'(z) + z^3 f''(z) \prec \frac{1}{(1-z)^2}, \quad z \in U, \tag{2.43}$$

then

$$2f(z) + zf'(z) \prec \frac{1}{1-z}, \quad z \in U, \tag{2.44}$$

and this result is sharp.

Theorem 2.12. *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$ which verifies the inequality:*

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad z \in U. \tag{2.45}$$

If $f \in \Sigma$, g is given by (2.7) *and the following differential subordination holds*

$$\left[D_\lambda^n g(z)\right]' \prec h(z), \quad z \in U, \tag{2.46}$$

then

$$\frac{D_\lambda^n g(z)}{z} \prec q(z), \quad z \in U, \tag{2.47}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)\,dt, \quad z \in U. \tag{2.48}$$

Function q is convex and the best dominant.

Proof. In order to prove Theorem 2.12, we will use Lemmas 1.3 and 1.4. We deduce the value of w from Lemma 1.4 by using the conditions of Theorem 2.12. From (2.37), Definition 2.1 and Remark 2.2, we have

$$\begin{aligned} p(z) &= \frac{D_\lambda^n g(z)}{z} \\ &= \frac{z + \sum_{j=2}^{\infty}\left[(1-\lambda)j^n + \lambda C_{n+j-1}^n\right] a_j z^j}{z} \\ &= 1 + \sum_{j=2}^{\infty}\left[(1-\lambda)j^n + \lambda C_{n+j-1}^n\right] a_j z^{j-1} \\ &= 1 + b_1 z + b_2 z + \cdots, \quad z \in U. \end{aligned} \tag{2.49}$$

Using Lemma 1.4, we deduce from (2.49) that $k = 1$. Using (2.38) in Subordination (2.46), we have

$$p(z) + zp'(z) \prec h(z), \quad z \in U. \tag{2.50}$$

From Subordination (2.50), by using Lemma 1.4, we deduce that $c = 1$. Then,

$$w = \frac{k^2 + c^2 - |k^2 - c^2|}{4k\operatorname{Re} c} = \frac{1 + 1 - |1 - 1|}{4} = \frac{1}{2}. \tag{2.51}$$

Applying Lemma 1.4, from Subordination (2.50), we obtain

$$p(z) \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U, \tag{2.52}$$

that is,

$$\frac{D_\lambda^n g(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U, \tag{2.53}$$

where q is the best dominant.

Since the function h verifies the relation (2.45), from Lemma 1.3, we deduce that q is a convex function. □

Example 2.13. If $n = 0$, $\lambda \geq 0$, $h(z) = e^{(3/2)z} - 1$, from Theorem 2.12, we deduce for $f \in \Sigma$ that if

$$4zf(z) + 5z^2 f'(z) + z^3 f''(z) \prec e^{(3/2)z} - 1, \quad z \in U, \tag{2.54}$$

then

$$2f(z) + zf'(z) \prec \frac{2}{3z}e^{(3/2)z} - \frac{2}{3z} - 1, \quad z \in U. \tag{2.55}$$

Theorem 2.14. *Let $h \in \mathscr{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$ which verifies the inequality:*

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad z \in U. \tag{2.56}$$

If $f \in \Sigma$, g is given by (2.7)*, and the following differential subordination holds*

$$\left[D_\lambda^{n+1} g(z)\right]' + \frac{\lambda n z\left[R^n g(z)\right]''}{n+1} \prec h(z), \quad z \in U, \tag{2.57}$$

then

$$\left[D_\lambda^n g(z)\right]' \prec q(z), \quad z \in U, \tag{2.58}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt \tag{2.59}$$

is convex and the best dominant.

Proof. In order to prove Theorem 2.14, we will use Lemmas 1.3 and 1.4. The value of w is obtained using the conditions of Theorem 2.14.

Using (2.16) and (2.17), Subordination (2.49) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U. \tag{2.60}$$

From Subordination (2.60), by using Lemma 1.4, we deduce that $c = 1$; and from the relation (2.16), Definition 2.1, and Remark 2.2, we obtain

$$\begin{aligned} p(z) &= \left[D_\lambda^n g(z)\right]' \\ &= \left[z + \sum_{j=2}^{\infty}\left[(1-\lambda)j^n + \lambda C_{n+j-1}^n\right]a_j z^j\right]' \\ &= 1 + \sum_{j=2}^{\infty}\left[(1-\lambda)j^n + \lambda C_{n+j-1}^n\right]\cdot j\cdot a_j\cdot z^{j-1} \\ &= 1 + c_1 z + c_2 z^2 + \cdots, \quad z \in U. \end{aligned} \tag{2.61}$$

From (2.61), by using Lemma 1.4, we deduce that $k = 1$, then

$$w = \frac{k^2 + |c|^2 - |k-c|^2}{4k\operatorname{Re} c} = \frac{1 + 1 - |1-1|^2}{4} = \frac{1}{2}. \tag{2.62}$$

Applying Lemma 1.4, from Subordination (2.60), we obtain

$$p(z) \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U, \tag{2.63}$$

that is,

$$\left[D_\lambda^n g(z)\right]' \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U, \tag{2.64}$$

where q is the best dominant.

Since the function h verifies the inequality (2.45), from Lemma 1.3, we deduce that q is a convex function. □

Example 2.15. If $n = 0$, $\lambda \geq 0$, $f \in \Sigma$, $h(z) = (2z + z^2)/(1+z)^2$, from Theorem 2.14, we deduce that if

$$4zf(z) + 5z^2 f'(z) + z^3 f''(z) \prec \frac{2z + z^2}{2(1+z)^2}, \quad z \in U, \tag{2.65}$$

then

$$2f(z) + zf'(z) \prec \frac{1}{2}z + \frac{1}{2}\frac{1}{1+z} + 1, \quad z \in U. \tag{2.66}$$

References

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