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Research Article

New Inequalities of Shafer-Fink Type for Arc Hyperbolic Sine

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In this paper, we extend some Shafer-Fink-type inequalities for the inverse sine to arc hyperbolic sine, and give two simple proofs of these inequalities by using the power series quotient monotone rule.

1. Introduction

Mitrinović in [1, page 247] gives us a result as follows.

Theorem 1.1. *Let* $x > 0$*. Then*

$$\arcsin x > \frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}}. \tag{1.1}$$

Fink in [2] obtains the following theorem.

Theorem 1.2. *Let* $0 \le x \le 1$*. Then*

$$\frac{3x}{2+\sqrt{1-x^2}} \le \arcsin x \le \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{1.2}$$

Furthermore, 3 *and* π *are the best constants in* (1.2).

The author of this paper improves the upper bound of inverse sine and obtains (see [3, 4]) the following theorem.

Theorem 1.3. *Let* $0 \le x \le 1$. *Then*

$$\frac{3x}{2+\sqrt{1-x^2}} \le \frac{6\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} \le \arcsin x \le \frac{\pi\left(\sqrt{2}+(1/2)\right)\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} \le \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{1.3}$$

Furthermore, 3 *and* π, 6 *and* $\pi(\sqrt{2}+(1/2))$ *are the best constants in* (1.3).

Malešević in [5, 6] obtains the following theorem using λ-method and computer, respectively.

Theorem 1.4. *For* $x \in [0,1]$, *the following inequality is true:*

$$\arcsin x \le \frac{\left(\pi(2-\sqrt{2})/(\pi-2\sqrt{2})\right)\left(\sqrt{1+x}-\sqrt{1-x}\right)}{\left(\sqrt{2}(\pi-4)/(\pi-2\sqrt{2})\right)+\sqrt{1+x}+\sqrt{1-x}}. \tag{1.4}$$

In [7], Malešević obtains inequality (1.4) by further method on computer. Zhu in [8] shows new simple proof of inequality (1.4), and obtains the following further result.

Theorem 1.5. *Let* $0 \le x \le 1$. *Then*

$$\frac{(\alpha+2)\left(\sqrt{1+x}-\sqrt{1-x}\right)}{\alpha+\sqrt{1+x}+\sqrt{1-x}} \le \arcsin x \le \frac{(\beta+2)\left(\sqrt{1+x}-\sqrt{1-x}\right)}{\beta+\sqrt{1+x}+\sqrt{1-x}} \tag{1.5}$$

holds if and only if $\alpha \ge 4$ *and* $\beta \le \sqrt{2}(4-\pi)/(\pi-2\sqrt{2})$.

Malešević in [6] gives a new upper bound for the inverse sine, and obtains the following result.

Theorem 1.6. *If* $0 \le x \le 1$, *then*

$$\arcsin x \le \frac{(\pi/(\pi-2))x}{(2/(\pi-2))+\sqrt{1-x^2}}. \tag{1.6}$$

In fact, we can easily obtain the following result by the same method in [8].

Theorem 1.7. *Let* $0 \le x \le 1$. *Then*

$$\frac{(a+1)x}{a+\sqrt{1-x^2}} \le \arcsin x \le \frac{(b+1)x}{b+\sqrt{1-x^2}} \tag{1.7}$$

holds if and only if $a \ge 2$ *and* $b \le 2/(\pi-2)$.

Combining (1.5) and (1.7) gives the following theorem.

Theorem 1.8. *If* $0 \le x \le 1$, *then*

$$\frac{3x}{2+\sqrt{1-x^2}} \le \frac{6\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} \le \arcsin x$$
$$\le \frac{\left(\pi(2-\sqrt{2})/(\pi-2\sqrt{2})\right)\left(\sqrt{1+x}-\sqrt{1-x}\right)}{\left(\sqrt{2}(\pi-4)/(\pi-2\sqrt{2})\right)+\sqrt{1+x}+\sqrt{1-x}} \le \frac{(\pi/(\pi-2))x}{(2/(\pi-2))+\sqrt{1-x^2}}. \tag{1.8}$$

Furthermore, $2, 4, \sqrt{2}(4-\pi)/(\pi-2\sqrt{2})$, *and* $2/(\pi-2)$ *are the best constants in* (1.8).

In this paper, we obtain new lower and upper bounds of arc hyperbolic sine $\sinh^{-1}x$, and we show simple proofs of the following two Shafer-Fink-type inequalities.

Theorem 1.9. *Let* $0 \le x \le r$ *and* $r > 0$. *Then*

$$\frac{(a+1)x}{a+\sqrt{1+x^2}} \le \sinh^{-1}x \le \frac{(b+1)x}{b+\sqrt{1+x^2}} \tag{1.9}$$

holds if and only if $a \le 2$ *and* $b \ge (\sqrt{1+r^2}\sinh^{-1}r - r)/(r-\sinh^{-1}r)$.

Theorem 1.10. *Let* $0 \le x \le r$ *and* $r > 0$. *Then*

$$\frac{(\alpha+2)\sqrt{2}\left(\sqrt{1+x^2}-1\right)^{1/2}}{\alpha+\sqrt{2}\left(\sqrt{1+x^2}+1\right)^{1/2}} \le \sinh^{-1}x \le \frac{(\beta+2)\sqrt{2}\left(\sqrt{1+x^2}-1\right)^{1/2}}{\beta+\sqrt{2}\left(\sqrt{1+x^2}+1\right)^{1/2}} \tag{1.10}$$

holds if and only if $\alpha \le 4$ *and* $\beta \ge ((1+\sqrt{1+r^2})^{1/2}\sinh^{-1}r - 2(\sqrt{1+r^2}-1)^{1/2})/((\sqrt{1+r^2}-1)^{1/2} - (\sinh^{-1}r/\sqrt{2}))$.

Combining (1.9) and (1.10) gives the following.

Theorem 1.11. *Let* $0 \le x \le r$ *and* $r > 0$. *Then*

$$\frac{3x}{2+\sqrt{1+x^2}} \le \frac{6\sqrt{2}\left(\sqrt{1+x^2}-1\right)^{1/2}}{4+\sqrt{2}\left(\sqrt{1+x^2}+1\right)^{1/2}} \le \sinh^{-1}x$$
$$\le \frac{(\beta+2)\sqrt{2}\left(\sqrt{1+x^2}-1\right)^{1/2}}{\beta+\sqrt{2}\left(\sqrt{1+x^2}+1\right)^{1/2}} \le \frac{(b+1)x}{b+\sqrt{1+x^2}} \tag{1.11}$$

holds, where $2, 4$, $\beta = ((1+\sqrt{1+r^2})^{1/2}\sinh^{-1}r - 2(\sqrt{1+r^2}-1)^{1/2})/((\sqrt{1+r^2}-1)^{1/2} - (\sinh^{-1}r/\sqrt{2}))$, *and* $b = (\sqrt{1+r^2}\sinh^{-1}r - r)/(r-\sinh^{-1}r)$ *are the best constants in* (1.11).

2. Two lemmas

Lemma 2.1 (see [9–11]). *Let a_n and b_n $(n = 0, 1, 2, \ldots)$ be real numbers, and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \ldots$, and if a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, \ldots$, then the function $A(t)/B(t)$ is strictly increasing (or decreasing) on $(0, R)$.*

Lemma 2.2. *The function $F(t) = (t\cosh t - \sinh t)/(\sinh t - t)$ is increasing on $(0, +\infty)$.*

Proof. Let $F(t) = (t\cosh t - \sinh)/(\sinh t - t) := A(t)/B(t)$, where $A(t) = t\cosh t - \sinh t$, $B(t) = \sinh t - t$. Since

$$A(t) = \sum_{n=1}^{\infty} a_n t^{2n+1}, \qquad B(t) = \sum_{n=1}^{\infty} b_n t^{2n+1}, \tag{2.1}$$

where $a_n = (1/(2n)!) - (1/(2n+1)!)$ and $b_n = 1/(2n+1)! > 0$. We have $a_n/b_n = 2n$ is increasing for $n = 1, 2, \ldots$, and $F(t)$ is increasing on $(0, +\infty)$ by Lemma 2.1. □

3. Simple proofs of Theorems 1.9 and 1.10

Since (1.9) and (1.10) hold for $x = 0$, the existence of Theorems 1.9 and 1.10is ensured when proving the results as follows.

Proposition 3.1. *Let $0 < x \le r$. Then*

$$\frac{(a+1)x}{a + \sqrt{1+x^2}} \le \sinh^{-1} x \le \frac{(b+1)x}{b + \sqrt{1+x^2}} \tag{3.1}$$

holds if and only if $a \le 2$ and $b \ge (\sqrt{1+r^2}\,\sinh^{-1} r - r)/(r - \sinh^{-1} r)$.

Proposition 3.2. *Let $0 < x \le r$. Then*

$$\frac{(\alpha+2)\sqrt{2}\left(\sqrt{1+x^2}-1\right)^{1/2}}{\alpha + \sqrt{2}\left(\sqrt{1+x^2}+1\right)^{1/2}} \le \sinh^{-1} x \le \frac{(\beta+2)\sqrt{2}\left(\sqrt{1+x^2}-1\right)^{1/2}}{\beta + \sqrt{2}\left(\sqrt{1+x^2}+1\right)^{1/2}} \tag{3.2}$$

holds if and only if $\alpha \le 4$ and $\beta \ge ((1+\sqrt{1+r^2})^{1/2} \sinh^{-1} r - 2(\sqrt{1+r^2}-1)^{1/2})/((\sqrt{1+r^2}-1)^{1/2} - (\sinh^{-1} r/\sqrt{2}))$.

Proof of Propositions 3.1 and 3.2. (1) By Lemma 2.2, we have that the double inequality

$$2 = F(0^+) \le F(\sinh^{-1} x) \le F(\sinh^{-1} r) = \frac{\sqrt{1+r^2}\sinh^{-1} r - r}{r - \sinh^{-1} r} \tag{3.3}$$

holds for $x \in (0, r]$. Then Proposition 3.1 is true.

(2) By the same way, we obtain that

$$\lambda = 4 = 2F(0^+) \leq 2F\left(\frac{1}{2}\sinh^{-1}x\right) \leq 2F\left(\frac{1}{2}\sinh^{-1}r\right) = \mu \tag{3.4}$$

holds for $x \in (0, r]$, where $\mu = ((1+\sqrt{1+r^2})^{1/2}\sinh^{-1}r - 2(\sqrt{1+r^2}-1)^{1/2})/((\sqrt{1+r^2}-1)^{1/2} - (\sinh^{-1}r/\sqrt{2}))$. So the proof of Proposition 3.2 is complete. □

Remark 3.3. From the left of the double inequality (3.1), one can obtain the inequality $3\sinh t/(2+\cosh t) \leq t$ for $t \geq 0$, which can be found in [12].

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