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Research Article

Regularity Criterion for Weak Solutions to the Navier-Stokes Equations in Terms of the Gradient of the Pressure

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We prove a regularity criterion $\nabla\pi \in L^{2/3}(0,T;\mathrm{BMO})$ for weak solutions to the Navier-Stokes equations in three-space dimensions. This improves the available result with $L^{2/3}(0,T;L^{\infty})$.

1. Introduction

We study the regularity condition of weak solutions to the Navier-Stokes equations

$$u_t - \Delta u + (u\cdot\nabla)u + \nabla\pi = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{1.2}$$

$$u|_{t=0} = u_0(x), \quad x\in\mathbb{R}^3. \tag{1.3}$$

Here, u is the unknown velocity vector and π is the unknown scalar pressure.

For $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$, Leray [1] constructed global weak solutions. The smoothness of Leray's weak solutions is unknown. While the existence of regular solutions is still an open problem, there are many interesting sufficient conditions which guarantee that a given weak solution is smooth. A well-known condition states that if

$$u\in L^r(0,T;L^s(\mathbb{R}^3)) \quad \text{with } \frac{2}{r}+\frac{3}{s}=1,\ \ 3\le s\le\infty, \tag{1.4}$$

then the solution u is actually regular [2–8]. A similar condition

$$\omega := \operatorname{curl} u \in L^r(0,T;L^s(\mathbb{R}^3)), \quad \text{with } \frac{2}{r}+\frac{3}{s}=2,\ \frac{3}{2}\le s\le\infty, \tag{1.5}$$

also implies the regularity as shown by Beirão da Veiga [9].

As regards (1.4) and (1.5) for $s=\infty$, Kozono et al. made an improvement to the following condition:

$$u \in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{1.6}$$

or

$$\omega \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{1.7}$$

where $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space. On the other hand, Chae and Lee [10] proposed another regularity criterion in terms of the pressure. They showed that if the pressure π satisfies

$$\pi \in L^r(0,T;L^s(\mathbb{R}^3)) \quad \text{with } \frac{2}{r}+\frac{3}{s}<2,\ \frac{3}{2}< s\le\infty, \tag{1.8}$$

then u is smooth. Berselli and Galdi [11] have extended the range of r and s to $2/r+3/s=2$ and $3/2<s\le\infty$. When $s=\infty$, Chen and Zhang [12] (also see Fan et al. [13]) refined it to the following condition:

$$\pi \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)). \tag{1.9}$$

Zhou [14] (see also Struwe [15]) proposed the following criterion in terms of the gradient of the pressure:

$$\nabla\pi \in L^r(0,T;L^s(\mathbb{R}^3)) \quad \text{with } \frac{2}{r}+\frac{3}{s}=3,\ 1<s\le\infty. \tag{1.10}$$

The aim of this paper is to refine (1.10) when $s=\infty$. We will use the following interpolation inequality:

$$\|u\|^2_{L^{2p}(\mathbb{R}^n)} \le C\|u\|_{L^p(\mathbb{R}^n)}\|u\|_{\mathrm{BMO}}, \quad 1\le p<\infty, \tag{1.11}$$

which follows from the bilinear estimates

$$\|fg\|_{L^p} \le C(\|f\|_{L^p}\|g\|_{\mathrm{BMO}}+\|g\|_{L^p}\|f\|_{\mathrm{BMO}}), \quad 1\le p<\infty, \tag{1.12}$$

due to Kozono and Taniuchi [16]. Here, BMO is the space of functions of bounded mean oscillations.

Definition 1.1. Let $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$. The function u is called a Leray weak solution of (1.1)–(1.3) in $(0,T)$ if u satisfies the following properties.

(1) $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$.

(2) Equation (1.1) and (1.2) hold in the distributional sense, and

$$u(t) \longrightarrow u_0 \quad \text{weakly in } L^2 \text{ as } t \longrightarrow 0. \tag{1.13}$$

(3) The energy inequality is

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds \le \|u_0\|_{L^2}^2, \quad \text{for any } t \in [0,T]. \tag{1.14}$$

Our main result reads as follows.

Theorem 1.2. *Let $u_0 \in (L^2 \cap L^4)(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$. Suppose that u is a Leray weak solution of (1.1)–(1.3) in $(0,T)$. If the gradient of the pressure satisfies the condition*

$$\nabla \pi \in L^{2/3}(0,T;\mathrm{BMO}), \tag{1.15}$$

then u is smooth in $(0,T]$.

Remark 1.3. If the interpolation inequality

$$\|u\|_{L^{2p}(\mathbb{R}^n)}^2 \le C\|u\|_{L^p}\|u\|_{\dot{B}^0_{\infty,\infty}} \tag{1.16}$$

is true, then as in the argument below, (1.15) may be improved to the following condition:

$$\nabla \pi \in L^{2/3}(0,T;\dot{B}^0_{\infty,\infty}). \tag{1.17}$$

Remark 1.4. Inequality (1.11) plays an important role in our proof. Chen and Zhu [17] extended (1.11) to the following inequality:

$$\|u\|_{L^q} \le C\|u\|_{L^r}^{r/q}\|u\|_{\mathrm{BMO}}^{1-r/q}, \quad 1 \le r < q < \infty, \tag{1.18}$$

and used (1.18) to obtain (1.12). Kozono and Wadade [18] give another proof of (1.18). Here, we give an elementary and short proof of (1.18) by (1.11).

For given $1 \le r < q < \infty$, there exists a positive integer n and $\theta \in (0,1)$ such that $r < q < 2^n r$ and $1/q = \theta \cdot (1/r) + (1-\theta)\cdot(1/2^n r) = (\theta + (1-\theta)/2^n)\cdot(1/r)$. By the Hölder inequality, we have

$$\|u\|_{L^q} \le \|u\|_{L^r}^{\theta}\|u\|_{L^{2^n r}}^{1-\theta}. \tag{1.19}$$

Using (1.11) for $p = 2^{n-1}r, 2^{n-2}r, \ldots, r$, n times and plugging them into (1.6), we find that

$$\begin{aligned}\|u\|_{L^q} &\leq C\|u\|_{L^r}^{\theta+(1-\theta)/2^n}\|u\|_{\mathrm{BMO}}^{(1-\theta)(1/2+1/2^2+\cdots+1/2^n)}\\ &= C\|u\|_{L^r}^{\theta+(1-\theta)/2^n}\|u\|_{\mathrm{BMO}}^{(1-1/2^n)(1-\theta)}\\ &= C\|u\|_{L^r}^{r/q}\|u\|_{\mathrm{BMO}}^{1-r/q},\end{aligned} \tag{1.20}$$

which proves (1.18).

Remark 1.5. From Remark 1.4, we know that if (1.16) holds true, then we have

$$\|u\|_{L^q} \leq C\|u\|_{L^r}^{r/q}\|u\|_{\dot{B}^0_{\infty,\infty}}^{1-r/q}, \quad 1 \leq r < q < \infty. \tag{1.21}$$

2. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. First, we recall the following result according to Giga [5].

Proposition 2.1 (see [5]). *Suppose $u_0 \in L^s(\mathbb{R}^3)$, $s \geq 3$; then there exists T and a unique classical solution $u \in (L^\infty \cap C)([0,T); L^s)$. Moreover, let $(0,T^*)$ be the maximal interval such that u solves* (1.1)–(1.3) *in $C((0,T^*); L^s)$, $s > 3$. Then, for any $t \in (0,T^*)$,*

$$\|u(t)\|_{L^s} \geq \frac{C}{(T^*-t)^{(s-3)/2s}} \tag{2.1}$$

with the constant C independent of T^* and s.

Next, we derive a priori estimates for smooth solutions of (1.1)–(1.3). To this end, multiplying (1.1) by $|u|^2u$, integrating by parts, and using (1.2), (1.11) for $p = 2$, we see that

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \int|\nabla u|^2|u|^2dx + \frac{1}{4}\int|\nabla|u|^2|^2dx &= -\int\nabla\pi\cdot|u|^2u\,dx \leq \|\nabla\pi\|_{L^4}\|u\|_{L^4}^3\\ &\leq C\|\nabla\pi\|_{L^2}^{1/2}\|\nabla\pi\|_{\mathrm{BMO}}^{1/2}\|u\|_{L^4}^3\\ &\leq C\|u\nabla u\|_{L^2}^{1/2}\|\nabla\pi\|_{\mathrm{BMO}}^{1/2}\|u\|_{L^4}^3\\ &\leq \frac{1}{2}\int|u\cdot\nabla u|_{L^2}^2 + C\|\nabla\pi\|_{\mathrm{BMO}}^{2/3}\|u\|_{L^4}^4,\end{aligned} \tag{2.2}$$

which yields

$$\|u\|_{L^4} \leq \|u_0\|_{L^4}\exp\left(C\int_0^T\|\nabla\pi\|_{\mathrm{BMO}}^{2/3}dt\right), \tag{2.3}$$

by Gronwall's inequality. Here, we have used the estimate

$$\|\nabla\pi\|_{L^2} \leq C\|u \cdot \nabla u\|_{L^2}. \tag{2.4}$$

Now, we are in a position to complete the proof of Theorem 1.2. From Proposition 2.1, it follows that there exists $T^* > 0$ and the smooth solution v of (1.1)–(1.3) satisfies

$$v(t) \in (L^\infty \cap C)([0,T^*); L^4), \quad v(0) = u_0. \tag{2.5}$$

Since the weak solution u satisfies the energy inequality, we may apply Serrin's uniqueness criterion [19] to conclude that

$$u \equiv v \quad \text{on } [0,T^*). \tag{2.6}$$

Thus, it is sufficient to show that $T^* = T$. Suppose that $T^* < T$. Without loss of generality, we may assume that T^* is the maximal existence time for $v(t)$. By Proposition 2.1 again, we find that

$$\|u(t)\|_{L^4} \geq \frac{C}{(T^* - t)^{1/8}} \quad \text{for any } t \in (0,T^*). \tag{2.7}$$

On the other hand, from (2.3), we know that

$$\sup_{0\leq t\leq T^*} \|u(t)\|_{L^4} \leq \|u_0\|_{L^4} \exp\left(C\int_0^T \|\nabla\pi\|_{\mathrm{BMO}}^{2/3} dt\right), \tag{2.8}$$

which contradicts with (2.7). Thus, $T^* = T$.

This completes the proof.

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