Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 624989, 10 pages
doi:10.1155/2008/624989

Research Article

Gauss-Lobatto Formulae and Extremal Problems with Polynomials

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Received 11 January 2007; Accepted 5 December 2007

Recommended by Jozsef Szabados

Using quadrature formulae of the Gauss-Lobatto type, we give some new results for extremal problems with polynomials.

1. Introduction

By $\prod_n$, we denote the space of polynomials of degree not greater than n. To obtain our results, we need the following results of Duffin and Schaeffer [1] and of Gautschi and Notaris [2].

Lemma 1.1 (Duffin and Schaeffer). *If $q(x) = c\cdot\prod_{i=1}^{n}(x - x_i)$ is a polynomial of degree n with n distinct real zeroes and if $p \in \prod_n$ is such that*

$$|p'(x_i)| \le |q'(x_i)|, \quad i = \overline{1, n}, \tag{1.1}$$

then for $k = \overline{1, n-1}$,

$$|p^{(k+1)}(x)| \le |q^{(k+1)}(x)|, \tag{1.2}$$

whenever $q^{(k)}(x) = 0$.

Lemma 1.2 (Gautschi and Notaris). *A real polynomial r of exact degree 2 satisfies $r(x) > 0$ for $-1 \le x \le 1$ if and only if*

$$r(x) = b(b - 2a)x^2 + 2c(b - a)x + a^2 + c^2 \tag{1.3}$$

with $0 < a < b$, $|c| < b - a$, $b \ne 2a$.

By $P_n^{(\alpha,\beta)}(x)$, where n is a nonnegative whole number and $\alpha,\beta > -1$, we denote the nth Jacobi polynomial. It is known that Jacobi polynomials with the same parameters α and β are orthogonal on $[-1,1]$ with respect to the weight function $\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}$.

We will need the following properties of Jacobi polynomials [3]:

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \tag{1.4}$$

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n}, \tag{1.5}$$

$$\frac{d}{dx}\left\{P_n^{(\alpha,\beta)}\right\}(x) = \frac{1}{2}\cdot(n+\alpha+\beta+1)\cdot P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{1.6}$$

Let $\widetilde{P}_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n, normalized to have the leading coefficient equal to 1. Then

$$\widetilde{P}_n^{(\alpha,\beta)}(x) = 2^n n! \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\cdot P_n^{(\alpha,\beta)}(x). \tag{1.7}$$

From the relations (1.6) and (1.7), we obtain

$$\frac{d}{dx}\left\{\widetilde{P}_n^{(\alpha,\beta)}\right\}(x) = n\cdot\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{1.8}$$

The Jacobi polynomials orthogonal on $[-1,1]$ with respect to the weight function $\rho(x) = 1/\sqrt{1-x^2}$ are the so-called Chebyshev polynomials of first kind. These polynomials are given by

$$T_n(x) = \cos(n \arccos x), \quad x \in (-1,1),\ n = 0,1,2\ldots, \tag{1.9}$$

and $\widetilde{T}_n = (1/2^{n-1})T_n$ are the Chebyshev polynomials of first kind of degree n with the leading coefficient equal to 1.

The Jacobi polynomials orthogonal on $[-1,1]$ with respect to the weight function $\rho(x) = \sqrt{1-x^2}$ are the so-called Chebyshev polynomials of second kind. These polynomials are given by

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}, \quad x \in (-1,1),\ n = 0,1,2\ldots, \tag{1.10}$$

and $\widetilde{U}_n = (1/2^n)U_n$ are the Chebyshev polynomials of second kind of degree n with the leading coefficient equal to 1.

Let us denote by $x_i = \cos((2i-1)\pi/2n)$, $i = \overline{1,n}$, the zeroes of T_n, the Chebyshev polynomial of the first kind.

The following problem was raised by Turán.

Problem 1. Let $\phi(x) \geq 0$ for $-1 \leq x \leq 1$ and consider the class $P_{n,\varphi}$ of all polynomials of degree n such that $|p_n(x)| \leq \varphi(x)$ for $-1 \leq x \leq 1$. How large can $\max_{x\in[-1,1]}|p_n^{(k)}(x)|$ be if p_n is an arbitrary polynomial in $P_{n,\varphi}$?

He pointed out two cases: $\varphi(x) = \sqrt{1-x^2}$ and $\varphi(x) = 1 - x^2$.

In papers [4, 5], the author considers the solution in the weighted L^2-norm for the majorant $\varphi(x) = 1/\sqrt{1-x^2}$.

Let H be the class of real polynomials $p_{n-1} \in \prod_{n-1}$, such that

$$|p_{n-1}(x_i)| \leq \frac{1}{\sqrt{1-x_i^2}}, \quad i = 1, \dots, n, \tag{1.11}$$

where the x_i are the zeroes of the Chebyshev polynomial of first kind.

Note that $U_{n-1} \in H$.

From paper [5] was obtained the following result.

Theorem 1.3 (see [5]). *If $p_{n-1} \in H$, then one has*

$$\int_{-1}^{1} (1-x^2)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \leq 2\pi \frac{(n+k+1)!}{(n-k-2)!} \frac{k^2+n^2+3k+1}{(2k+3)(2k+1)(2k+5)}, \tag{1.12}$$

$k = 0, \dots, n-2$, *with equality for $p_{n-1} = U_{n-1}$.*

We denote by $\widetilde{H}$ the class of all real polynomials $p_{n-1} \in \prod_{n-1}$, such that

$$|p_{n-1}(x_i)| \leq \frac{1}{2^{n-1}\sqrt{1-x_i^2}}, \quad i = 1, \dots, n, \tag{1.13}$$

where the x_i are the zeroes of the Chebyshev polynomial of first kind.

Note that $\widetilde{P}_{n-1}^{(1/2,1/2)} \in \widetilde{H}$.

The next theorem can be obtained in the same way of Theorem 1.3.

Theorem 1.4. *If $p_{n-1} \in \widetilde{H}$, then one has*

$$\int_{-1}^{1} (1-x^2)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \leq \frac{\pi}{2^{2n-1}} \frac{(n+k+1)!}{(n-k-2)!} \frac{k^2+n^2+3k+1}{(2k+3)(2k+1)(2k+5)}, \tag{1.14}$$

$k = 0, \dots, n-2$, *with equality for $p_{n-1} = \widetilde{P}_{n-1}^{(1/2,1/2)}$.*

Let $\widetilde{H}^{(\alpha,\beta)}$ be the class of real polynomials $p_{n-1} \in \prod_{n-1}$, such that

$$|p_{n-1}(x_i)| \leq \left|\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x_i)\right|, \quad i = \overline{1,n}, \tag{1.15}$$

where the x_i are the zeroes of $\widetilde{P}_n^{(\alpha,\beta)}$.

Remark 1.5. For $\alpha = \beta = -1/2$, the class $\widetilde{H}^{(-1/2,-1/2)}$ coincides with the class $\widetilde{H}$.

In this paper, we want to give a generalization of these results.

2. Quadrature formulae of the Gauss-Lobatto type

In this section, we recall some general concepts about quadrature formulae and we prove some lemmas which help us in proving our result.

Let

$$\int_a^b \rho(x) f(x) dx = \sum_{i=1}^{n} A_i f(a_i) + \sum_{j=1}^{p} B_j f(b_j) + R[f] \tag{2.1}$$

be a quadrature formula, where ρ is a nonnegative weight function, $b_j \notin (a,b)$, $j = \overline{1,p}$, are fixed and distinct nodes. The nodes $a_i \in (a,b)$, $i = \overline{1,n}$, will be determined from the condition that the quadrature formula (2.1) has maximal degree of exactness. These quadrature formulae are the so-called Gauss quadrature formulae with fixed nodes.

The next theorem gives the necessary and sufficient condition, such that the quadrature formula (2.1) has maximal degree of exactness.

Theorem 2.1 (see [6]). *The maximal degree of exactness, $r = 2n + p - 1$, of quadrature formula* (2.1) *is obtained if and only if the nodes a_i, $i = \overline{1,n}$, are the zeroes of an orthogonal polynomial of degree n with respect to the weight function $w(x) = \rho(x) \cdot \prod_{j=1}^{p} |x - b_j|$, $x \in (a,b)$.*

Let

$$\int_a^b \rho(x) f(x) dx = \sum_{i=1}^{n} \widetilde{A}_i f(a_i) + \sum_{j=1}^{p} \widetilde{B}_j f(b_j) + \sum_{j=1}^{p} \widetilde{C}_j f'(b_j) + \widetilde{R}[f] \tag{2.2}$$

be a quadrature formula.

Similarly, the next theorem gives the necessary and sufficient condition, such that the quadrature formula (2.2) has maximal degree of exactness.

Theorem 2.2 (see [6]). *The maximal degree of exactness, $r = 2n + 2p - 1$, of quadrature formula* (2.2) *is obtained if and only if the nodes a_i, $i = \overline{1,n}$, are the zeroes of an orthogonal polynomial of degree n with respect to the weight function $w(x) = \rho(x) \cdot \prod_{j=1}^{p} (x - b_j)^2$, $x \in (a,b)$.*

Remark 2.3. The coefficients A_i, $\widetilde{A}_i$, $i = \overline{1,n}$, from Gauss quadrature formulae (2.1) and (2.2) are positive.

The Gauss-Lobatto quadrature formulae are the Gauss quadrature formulae with two fixed nodes, namely, $b_1 = a$, $b_2 = b$. In this paper, we will consider the case $(a,b) = (-1,1)$ and the weight function is $\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}$. These formulae of numerical integration are called the Gauss-Jacobi-Lobatto quadrature formulae.

Lemma 2.4. *For any given n and k, $0 \le k \le n-1$, let $y_i^{(k)}$, $i = \overline{1, n-k-1}$, be the zeroes of $P_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$. Then the quadrature formulae*

$$\int_{-1}^{1} (1-x)^{k+\alpha} (1+x)^{k+\beta} f(x) dx = B_1 f(-1) + B_2 f(1) + \sum_{i=1}^{n-k-1} A_i f\left(y_i^{(k)}\right) + R[f], \tag{2.3}$$

where

$$B_1 = 2^{2k+\alpha+\beta+1} \cdot \frac{\Gamma(k+\beta+1)\Gamma(n+\alpha+1)\Gamma(n-k)\Gamma(k+\beta+2)}{\Gamma(n+\alpha+\beta+k+2)\Gamma(n+\beta+1)}, \tag{2.4}$$

$$B_2 = 2^{2k+\alpha+\beta+1} \cdot \frac{\Gamma(k+\alpha+1)\Gamma(n+\beta+1)\Gamma(n-k)\Gamma(k+\alpha+2)}{\Gamma(n+\alpha+\beta+k+2)\Gamma(n+\alpha+1)}, \tag{2.5}$$

$$A_i > 0, \tag{2.6}$$

$$\int_{-1}^{1}(1-x)^{k+\alpha}(1+x)^{k+\beta}f(x)dx = \widetilde{B}_1 f(-1) + \widetilde{B}_2 f(1) + \widetilde{C}_1 f'(-1) + \widetilde{C}_2 f'(1) + \sum_{i=1}^{n-k-2}\widetilde{A}_i f\left(y_i^{(k+1)}\right) + \widetilde{R}[f], \tag{2.7}$$

where

$$\widetilde{B}_1 = \widetilde{C}_1 \cdot \left\{1 + \frac{(n-k-2)(n+k+\alpha+\beta+3)}{2(\beta+k+3)} + \frac{(n-k-1)(n+k+\alpha+\beta+2)}{2(k+\beta+1)}\right\}, \tag{2.8}$$

$$\widetilde{B}_2 = -\widetilde{C}_2 \cdot \left\{1 + \frac{(n-k-2)(n+k+\alpha+\beta+3)}{2(\alpha+k+3)} + \frac{(n-k-1)(n+k+\alpha+\beta+2)}{2(k+\alpha+1)}\right\}, \tag{2.9}$$

$$\widetilde{C}_1 = 2^{2k+\alpha+\beta+2} \cdot \frac{\Gamma(k+\beta+2)\Gamma(n+\alpha+1)\Gamma(n-k-1)\Gamma(\beta+k+3)}{\Gamma(n+\alpha+\beta+k+3)\Gamma(n+\beta+1)}, \tag{2.10}$$

$$\widetilde{C}_2 = -2^{2k+\alpha+\beta+2} \cdot \frac{\Gamma(k+\alpha+2)\Gamma(n+\beta+1)\Gamma(n-k-1)\Gamma(\alpha+k+3)}{\Gamma(n+\alpha+\beta+k+3)\Gamma(n+\alpha+1)}, \tag{2.11}$$

$$\widetilde{A}_i > 0, \tag{2.12}$$

have the degree of exactness equal to $2n-2k-1$.

Proof. If in the quadrature formula of the Gauss-type (2.1) we consider $a=-1$, $b=1$, $\rho(x)=(1-x)^{k+\alpha}(1+x)^{k+\beta}$, $n \to n-k-1$, $p=2$, $b_1=-1$, $b_2=1$, then by Theorem 2.1, the quadrature formula (2.3) has the maximal degree of exactness, $r=2n-2k-1$.

In order to compute the coefficients B_1 and B_2, we need the following formulae:

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\lambda}P_m^{(\alpha,\beta)}(x)dx = \frac{(-1)^m 2^{\alpha+\lambda+1}\Gamma(\lambda+1)\Gamma(m+\alpha+1)\Gamma(\beta-\lambda+m)}{\Gamma(m+1)\Gamma(\beta-\lambda)\Gamma(m+\alpha+\lambda+2)}, \quad \lambda<\beta, \tag{2.13}$$

$$\int_{-1}^{1}(1-x)^{\lambda}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)dx = \frac{2^{\beta+\lambda+1}\Gamma(\lambda+1)\Gamma(m+\beta+1)\Gamma(\alpha-\lambda+m)}{\Gamma(m+1)\Gamma(\alpha-\lambda)\Gamma(m+\beta+\lambda+2)}, \quad \lambda<\alpha. \tag{2.14}$$

If in the quadrature formula (2.3) we consider $f(x)=(1+x)P_{n-k-1}^{(\alpha+k+1,\beta+k+1)}(x)$, then by using the relation (2.14) we obtain (2.5), while by using $f(x)=(1-x)P_{n-k-1}^{(\alpha+k+1,\beta+k+1)}(x)$ and the relation (2.13) we obtain (2.4).

If in the quadrature formula of the Gauss-type (2.2) we consider $a=-1$, $b=1$, $\rho(x)=(1-x)^{k+\alpha}(1+x)^{k+\beta}$, $n \to n-k-2$, $p=2$, $b_1=-1$, $b_2=1$, then by Theorem 2.2, the quadrature formula (2.7) has maximal degree of exactness $r=2n-2k-1$.

If in the quadrature formula (2.7) we consider $f(x) = (1-x)(1+x)^2 P_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, respectively $f(x) = (1-x)^2(1+x)P_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, then by using the formulae (2.13) and (2.14) we obtain the coefficients (2.11) and (2.10).

If in the quadrature formula (2.7) we choose $f(x) = (1+x)^2 P_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, respectively $f(x) = (1-x)^2 P_{n-k-2}^{(\alpha+k+2,\beta+k+2)}(x)$, then by using the formulae (2.13) and (2.14) we obtain the coefficients (2.9) and (2.8). □

Lemma 2.5. *Let $r(x) = b(b-2a)x^2 + 2c(b-a)x + a^2 + c^2$ be a real polynomial. For any given n and k, $0 \le k \le n-1$, let $y_i^{(k)}$, $i = \overline{1, n-k-1}$, be the zeroes of $P_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$. Then the quadrature formulae*

$$\int_{-1}^{1} r(x)(1-x)^{k+\alpha}(1+x)^{k+\beta} f(x)dx = D_1 f(-1) + D_2 f(1) + \sum_{i=1}^{n-k-1} A_i r\left(y_i^{(k)}\right) f\left(y_i^{(k)}\right) + R[f], \tag{2.15}$$

where

$$D_1 = 2^{2k+\alpha+\beta+1} \cdot \frac{\Gamma(k+\beta+1)\Gamma(n+\alpha+1)\Gamma(n-k)\Gamma(k+\beta+2)}{\Gamma(n+\alpha+\beta+k+2)\Gamma(n+\beta+1)} \cdot (a-b+c)^2,$$
$$D_2 = 2^{2k+\alpha+\beta+1} \cdot \frac{\Gamma(k+\alpha+1)\Gamma(n+\beta+1)\Gamma(n-k)\Gamma(k+\alpha+2)}{\Gamma(n+\alpha+\beta+k+2)\Gamma(n+\alpha+1)} \cdot (a-b-c)^2, \tag{2.16}$$

$$A_i > 0,$$

$$\int_{-1}^{1} r(x)(1-x)^{k+\alpha}(1+x)^{k+\beta} f(x)dx = \widetilde{D}_1 f(-1) + \widetilde{D}_2 f(1) + \widetilde{G}_1 f'(-1) + \widetilde{G}_2 f'(1) + \sum_{i=1}^{n-k-2} \widetilde{A}_i r\left(y_i^{(k+1)}\right) f\left(y_i^{(k+1)}\right) + \widetilde{R}[f], \tag{2.17}$$

where

$$\widetilde{D}_1 = \widetilde{C}_1 \cdot \left\{ 2(-b^2+2ab+bc-ac) + \left[1 + \frac{(n-k-2)(n+k+\alpha+\beta+3)}{2(\beta+k+3)} + \frac{(n-k-1)(n+k+\alpha+\beta+2)}{2(k+\beta+1)} \right] \cdot (a-b+c)^2 \right\},$$

$$\widetilde{D}_2 = \widetilde{C}_2 \cdot \left\{ 2(b^2-2ab+bc-ac) - \left[1 + \frac{(n-k-2)(n+k+\alpha+\beta+3)}{2(\alpha+k+3)} + \frac{(n-k-1)(n+k+\alpha+\beta+2)}{2(k+\alpha+1)} \right] \cdot (a-b-c)^2 \right\},$$

$$\widetilde{G}_1 = \widetilde{C}_1 \cdot (a-b+c)^2, \qquad \widetilde{G}_2 = \widetilde{C}_2 \cdot (a-b-c)^2, \qquad \widetilde{A}_i > 0, \tag{2.18}$$

with $\widetilde{C}_1$, $\widetilde{C}_2$ defined in (2.10) *and* (2.11), *have degree of exactness $2n-2k-1$.*

Proof. The proof follows directly by replacing f with rf in Lemma 2.4. □

3. Extremal problems with polynomials

In this section, we want to give exact estimations of certain weighted L^2-norms of the kth derivative of polynomials which are in the class $\widetilde{H}^{(\alpha,\beta)}$.

Remark 3.1. Since $P_{n-1}^{(\alpha+1,\beta+1)} = c\cdot\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}$, $c\in\mathbb{R}$, and $(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)})^{(k)} = \widetilde{P}_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$, it follows that for $k = \overline{0, n-1}$, the polynomials $P_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$, $\widetilde{P}_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$, and $(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)})^{(k)}$ have the same zeroes $y_j^{(k)}$, $j = \overline{1, n-k-1}$.

Lemma 3.2. *If $p_{n-1} \in \widetilde{H}^{(\alpha,\beta)}$, then for $k = \overline{0, n-1}$, one has*

$$\left|p_{n-1}^{(k+1)}\left(y_j^{(k)}\right)\right| \leq \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_j^{(k)}\right)\right|, \tag{3.1}$$

whenever

$$\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k)}\left(y_j^{(k)}\right) = 0 \quad \textit{for } j = \overline{1, n-k-1}, \tag{3.2}$$

$$\left|p_{n-1}^{(k+1)}(1)\right| \leq \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\right|, \tag{3.3}$$

$$\left|p_{n-1}^{(k+1)}(-1)\right| \leq \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(-1)\right|. \tag{3.4}$$

Proof. By the Lagrange interpolation formula based on the zeroes of $\widetilde{P}_n^{(\alpha,\beta)}$, we can represent any algebraic polynomial p_{n-1} by

$$p_{n-1}(x) = \sum_{i=1}^{n} \frac{\widetilde{P}_n^{(\alpha,\beta)}(x)}{(x-x_i)\left(\widetilde{P}_n^{(\alpha,\beta)}\right)'(x_i)} p_{n-1}(x_i) = \frac{1}{n}\sum_{i=1}^{n} \frac{\widetilde{P}_n^{(\alpha,\beta)}(x)}{(x-x_i)} \cdot \frac{p_{n-1}(x_i)}{\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x_i)}. \tag{3.5}$$

We also have

$$\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x) = \sum_{i=1}^{n} \frac{\widetilde{P}_n^{(\alpha,\beta)}(x)}{(x-x_i)\left(\widetilde{P}_n^{(\alpha,\beta)}\right)'(x_i)} \widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x_i) = \frac{1}{n}\sum_{i=1}^{n} \frac{\widetilde{P}_n^{(\alpha,\beta)}(x)}{x-x_i}. \tag{3.6}$$

Differentiating with respect to x, we obtain

$$\begin{aligned} p_{n-1}'(x) &= \frac{1}{n}\sum_{i=1}^{n} \frac{(x-x_i)\left(\widetilde{P}_n^{(\alpha,\beta)}\right)'(x) - \widetilde{P}_n^{(\alpha,\beta)}(x)}{(x-x_i)^2} \cdot \frac{p_{n-1}(x_i)}{\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x_i)}, \\ \left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)'(x) &= \frac{1}{n}\sum_{i=1}^{n} \frac{(x-x_i)\left(\widetilde{P}_n^{(\alpha,\beta)}\right)'(x) - \widetilde{P}_n^{(\alpha,\beta)}(x)}{(x-x_i)^2}. \end{aligned} \tag{3.7}$$

Since $y_j^{(0)}$, $j = \overline{1, n-1}$, are the zeroes of $\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x)$, we have $(\widetilde{P}_n^{(\alpha,\beta)})'(y_j^{(0)}) = 0$ and

$$\begin{aligned} p_{n-1}'\left(y_j^{(0)}\right) &= \frac{1}{n}\sum_{i=1}^{n} \frac{-\widetilde{P}_n^{(\alpha,\beta)}\left(y_j^{(0)}\right)}{\left(y_j^{(0)} - x_i\right)^2} \cdot \frac{p_{n-1}(x_i)}{\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x_i)}, \\ \left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)'\left(y_j^{(0)}\right) &= \frac{1}{n}\sum_{i=1}^{n} \frac{-\widetilde{P}_n^{(\alpha,\beta)}\left(y_j^{(0)}\right)}{\left(y_j^{(0)} - x_i\right)^2}. \end{aligned} \tag{3.8}$$

We find

$$\left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)'\left(y_j^{(0)}\right)\right| = \frac{1}{n}\left|\widetilde{P}_n^{(\alpha,\beta)}\left(y_j^{(0)}\right)\right|\cdot\sum_{i=1}^{n}\frac{1}{\left(y_j^{(0)}-x_i\right)^2},$$

$$\left|p'_{n-1}\left(y_j^{(0)}\right)\right| \le \frac{1}{n}\left|\widetilde{P}_n^{(\alpha,\beta)}\left(y_j^{(0)}\right)\right|\cdot\sum_{i=1}^{n}\frac{1}{\left(y_j^{(0)}-x_i\right)^2}\cdot\frac{|p_{n-1}(x_i)|}{\left|\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x_i)\right|} \tag{3.9}$$

$$\le \frac{1}{n}\left|\widetilde{P}_n^{(\alpha,\beta)}\left(y_j^{(0)}\right)\right|\cdot\sum_{i=1}^{n}\frac{1}{\left(y_j^{(0)}-x_i\right)^2} = \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)'\left(y_j^{(0)}\right)\right|.$$

Now, applying the Duffin-Schaeffer lemma, we have

$$\left|p_{n-1}^{(k+1)}\left(y_j^{(k)}\right)\right| \le \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_j^{(k)}\right)\right|, \quad j=\overline{1,n-k-1}. \tag{3.10}$$

By the Lagrange interpolation formula based on the zeroes $y_j^{(k)}$, $j=\overline{1,n-k-1}$, of $(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)})^{(k)}$, we can represent the polynomials $p_{n-1}^{(k+1)}$ and $(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)})^{(k+1)}$ by

$$p_{n-1}^{(k+1)}(x) = \sum_{j=1}^{n-k-1}\frac{\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k)}(x)}{\left(x-y_j^{(k)}\right)\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_j^{(k)}\right)}\cdot p_{n-1}^{(k+1)}\left(y_j^{(k)}\right),$$

$$\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(x) = \sum_{j=1}^{n-k-1}\frac{\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k)}(x)}{x-y_j^{(k)}}. \tag{3.11}$$

Since

$$\left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\right| = \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k)}(1)\right|\cdot\sum_{j=1}^{n-k-1}\frac{1}{1-y_j^{(k)}}, \tag{3.12}$$

using relation (3.1), we have

$$\left|p_{n-1}^{(k+1)}(1)\right| \le \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k)}(1)\right|\cdot\sum_{j=1}^{n-k-1}\frac{1}{1-y_j^{(k)}}\cdot\frac{\left|p_{n-1}^{(k+1)}\left(y_j^{(k)}\right)\right|}{\left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_j^{(k)}\right)\right|} \tag{3.13}$$

$$\le \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k)}(1)\right|\cdot\sum_{j=1}^{n-k-1}\frac{1}{1-y_j^{(k)}} = \left|\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\right|.$$

We recall that the zeroes of the orthogonal polynomial on an interval $[a,b]$ are real, distinct, and are located in the interval (a,b). In our case, we have $y_j^{(k)} \in (-1,1)$.

The relation (3.4) can be obtained in a similar way, so the proof is completed. □

Lemma 3.3. *The following formulae hold:*

$$\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)=\frac{2^{n-k-2}(n-1)!\Gamma(n+\alpha+\beta+k+3)\Gamma(n+\alpha+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(n-k-1)\Gamma(k+\alpha+3)},$$
$$\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(-1)=(-1)^{n-k-2}\frac{2^{n-k-2}(n-1)!\Gamma(n+\alpha+\beta+k+3)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(n-k-1)\Gamma(k+\beta+3)}. \tag{3.14}$$

Proof. Relation (1.8) yields

$$\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k)}(x)=\frac{(n-1)!}{(n-k-1)!}\cdot\widetilde{P}_{n-k-1}^{(\alpha+k+1,\beta+k+1)}(x). \tag{3.15}$$

The proof is completed by using relations (1.4), (1.5), and (1.7). □

Theorem 3.4. *If* $p_{n-1}\in\widetilde{H}^{(\alpha,\beta)}$, *then*

$$\begin{aligned}\int_{-1}^{1}&(1-x)^{k+\alpha}(1+x)^{k+\beta}\left[p_{n-1}^{(k+1)}(x)\right]^2dx\\&\leq 2^{2n+\alpha+\beta-2}\left[\frac{(n-1)!}{\Gamma(2n+\alpha+\beta+1)}\right]^2\cdot\frac{\Gamma(n+\alpha+\beta+k+3)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n-k-1)}\\&\cdot\left[\frac{1}{k+\beta+2}-\frac{(n+\alpha+\beta+k+3)(n-k-2)}{2(k+\beta+2)(k+\beta+3)}+\frac{(n-k-1)(n+k+\alpha+\beta+2)}{2(k+\beta+1)(k+\beta+2)}\right.\\&\left.+\frac{1}{k+\alpha+2}-\frac{(n+\alpha+\beta+k+3)(n-k-2)}{2(k+\alpha+2)(k+\alpha+3)}+\frac{(n-k-1)(n+k+\alpha+\beta+2)}{2(k+\alpha+1)(k+\alpha+2)}\right]\end{aligned} \tag{3.16}$$

holds for all $k=\overline{0,n-2}$, *with equality for* $p_{n-1}=\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}$.

Proof. According to Lemma 2.4 and positivity of the coefficients in the quadrature formulae, we have

$$\begin{aligned}\int_{-1}^{1}&(1-x)^{k+\alpha}(1+x)^{k+\beta}\left[p_{n-1}^{(k+1)}(x)\right]^2dx\\&=B_1\left(p_{n-1}^{(k+1)}(-1)\right)^2+B_2\left(p_{n-1}^{(k+1)}(1)\right)^2+\sum_{i=1}^{n-k-1}A_i\left(p_{n-1}^{(k+1)}\left(y_i^{(k)}\right)\right)^2\\&\leq B_1\left[\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(-1)\right]^2+B_2\left[\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\right]^2+\sum_{i=1}^{n-k-1}A_i\left[\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_i^{(k)}\right)\right]^2\\&=\int_{-1}^{1}(1-x)^{k+\alpha}(1+x)^{k+\beta}\left[\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x)\right)^{(k+1)}\right]^2dx\\&=\widetilde{B}_1\left[\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(-1)\right]^2+\widetilde{B}_2\left[\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\right]^2\\&\quad+2\widetilde{C}_1\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(-1)\cdot\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+2)}(-1)+2\widetilde{C}_2\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}(1)\cdot\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+2)}(1)\\&\quad+\sum_{i=1}^{n-k-2}\widetilde{A}_i\left[\left(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}\right)^{(k+1)}\left(y_i^{(k+1)}\right)\right]^2.\end{aligned} \tag{3.17}$$

Since $(\widetilde{P}_{n-1}^{(\alpha+1,\beta+1)})^{(k+1)}(y_i^{(k+1)}) = 0$, $i = \overline{1, n-k-2}$, and by using Lemma 3.3 we obtain the inequality (3.16). □

Remark 3.5. If we choose $\alpha = \beta = -1/2$ in the above theorem, we obtain Theorem 1.4.

Theorem 3.6. *If $p_{n-1} \in \widetilde{H}^{(\alpha,\beta)}$, and if $r(x) = b(b-2a)x^2 + 2c(b-a)x + a^2 + c^2$ is a real polynomial with $0 < a < b$, $|c| < b - a$, $b \neq 2a$, then*

$$\begin{aligned}
&\int_{-1}^{1} r(x)(1-x)^{k+\alpha}(1+x)^{k+\beta}\left[p_{n-1}^{(k+1)}(x)\right]^2 dx \\
&\leq 2^{2n+\alpha+\beta-2}\left[\frac{(n-1)!}{\Gamma(2n+\alpha+\beta+1)}\right]^2 \cdot \frac{\Gamma(n+\alpha+\beta+k+3)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n-k-1)} \\
&\quad \cdot \left\{ \frac{2(-b^2+2ab+bc-ac)}{k+\beta+2} + \left[1 - \frac{(n+\alpha+\beta+k+3)(n-k-!2)}{2(k+\beta+3)} + \frac{(n-k-1)(n+k+\alpha+\beta+2)}{2(k+\beta+1)}\right] \right. \\
&\quad \cdot \frac{(a-b+c)^2}{k+\beta+2} - \frac{2(b^2-2ab+bc-ac)}{k+\alpha+2} \\
&\quad \left. + \left[1 - \frac{(n+\alpha+\beta+k+3)(n-k-2)}{2(k+\alpha+3)} + \frac{(n-k-1)(n+k+\alpha+\beta+2)}{2(k+\alpha+1)}\right] \cdot \frac{(a-b-c)^2}{k+\alpha+2} \right\}
\end{aligned} \tag{3.18}$$

holds for all $k = \overline{0, n-2}$, with equality for $p_{n-1} = \widetilde{P}_{n-1}^{(\alpha+1,\beta+1)}$.

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