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Research Article

Bounds for Certain Delay Integral Inequalities on Time Scales

Wei Nian Li[1, 2]

[1] *Department of Applied Mathematics, Shanghai Normal University, Shanghai 200234, China*
[2] *Department of Mathematics, Binzhou University, Shandong 256603, China*

Correspondence should be addressed to Wei Nian Li, wnli@263.net

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Our aim in this paper is to investigate some delay integral inequalities on time scales by using Gronwall's inequality and comparison theorem. Our results unify and extend some delay integral inequalities and their corresponding discrete analogues. The inequalities given here can be used as handy tools in the qualitative theory of certain classes of delay dynamic equations on time scales.

1. Introduction

The unification and extension of differential equations, difference equations, q-difference equations, and so on to the encompassing theory of dynamic equations on time scales was initiated by Hilger [1] in his Ph.D. thesis in 1988. During the last few years, some integral inequalities on time scales related to certain inequalities arising in the theory of dynamic equations had been established by many scholars. For example, we refer the reader to literatures [2–8] and the references therein. However, nobody studied the delay integral inequalities on time scales, as far as we know. In this paper, we investigate some delay integral inequalities on time scales, which provide explicit bounds on unknown functions. Our results extend some known results in [9].

Throughout this paper, a knowledge and understanding of time scales and time scale notation is assumed. For an excellent introduction to the calculus on time scales, we refer the reader to monographes [10, 11].

2. Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the set of nonnegative integers, $C(M, S)$ denotes the class of all continuous functions defined on set M with range in the set S, $\mathbb{T}$ is an arbitrary time scale,

C_{rd} denotes the set of rd-continuous functions, $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{ for all } t \in \mathbb{T}\}$. We use the usual conventions that empty sums and products are taken to be 0 and 1, respectively. Throughout this paper, we always assume that $t_0 \in \mathbb{T}$, $\mathbb{T}_0 = [t_0, \infty) \bigcap \mathbb{T}$.

The following lemmas are very useful in our main results.

Lemma 2.1 (see [9]). *Assume that $p \geq q \geq 0$, $p \neq 0$, and $a \in \mathbb{R}_+$. Then*

$$a^{q/p} \leq \left(\frac{q}{p} k^{(q-p)/p} a + \frac{p-q}{p} k^{q/p}\right), \quad \text{for any } k > 0. \tag{2.1}$$

Lemma 2.2 (Gronwall's inequality [10]). *Suppose $u, b \in C_{rd}$, $m \in \mathcal{R}^+$, $m \geq 0$. Then*

$$u(t) \leq b(t) + \int_{t_0}^{t} m(t)u(t)\Delta t, \quad t \in \mathbb{T}_0, \tag{2.2}$$

implies

$$u(t) \leq b(t) + \int_{t_0}^{t} e_m(t, \sigma(s)) b(s) m(s)\Delta s, \quad t \in \mathbb{T}_0. \tag{2.3}$$

Lemma 2.3 (comparison theorem [10]). *Suppose $u, b \in C_{rd}$, $a \in \mathcal{R}^+$. Then*

$$u^{\Delta}(t) \leq a(t)u(t) + b(t), \quad t \in \mathbb{T}_0, \tag{2.4}$$

implies

$$u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau)) b(\tau)\Delta \tau, \quad t \in \mathbb{T}_0. \tag{2.5}$$

Firstly, we study the delay integral inequality on time scales of the form

$$x^p(t) \leq a(t) + \int_{t_0}^{t} b(s)x^p(s)\Delta s + c(t)\int_{t_0}^{t} \left[f(s)x^q(\tau(s)) + g(s)x^r(s)\right]\Delta s, \quad t \in \mathbb{T}_0, \tag{E}$$

with the initial condition

$$\begin{gathered} x(t) = \varphi(t), \quad t \in [\alpha, t_0] \cap \mathbb{T}, \\ \varphi(\tau(t)) \leq (a(t))^{1/p} \quad \text{for } t \in \mathbb{T}_0 \text{ with } \tau(t) \leq t_0, \end{gathered} \tag{I}$$

where p, q, and r are constants, $p \neq 0$, $p \geq q \geq 0$, $p \geq r \geq 0$, $\tau : \mathbb{T}_0 \to \mathbb{T}$, $\tau(t) \leq t$, $-\infty < \alpha = \inf\{\tau(t), t \in \mathbb{T}_0\} \leq t_0$, and $\varphi(t) \in C_{rd}([\alpha, t_0] \cap \mathbb{T}, \mathbb{R}_+)$.

Theorem 2.4. *Assume that* $x(t), a(t), b(t), c(t), f(t), g(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$*. If* $a(t)$ *and* $c(t)$ *are nondecreasing for* $t \in \mathbb{T}_0$*, then the inequality* (E) *with the initial condition* (I) *implies*

$$x(t) \leq \left\{ e_b(t,t_0)\left[a(t) + c(t)\left(F(t) + \int_{t_0}^{t} e_G(t,\sigma(s))F(s)G(s)\Delta s\right)\right]\right\}^{1/p}, \tag{2.6}$$

for any $k > 0$*,* $t \in \mathbb{T}_0$*, where*

$$F(t) = \int_{t_0}^{t}\left\{\frac{f(s)\left[e_b(s,t_0)\right]^{q/p}\left[k(p-q)+qa(s)\right]}{pk^{(p-q)/p}} + \frac{g(s)\left[e_b(s,t_0)\right]^{r/p}\left[k(p-r)+ra(s)\right]}{pk^{(p-r)/p}}\right\}\Delta s, \tag{2.7}$$

$$G(t) = c(t)\left[\frac{qf(t)\left[e_b(t,t_0)\right]^{q/p}}{pk^{(p-q)/p}} + \frac{rg(t)\left[e_b(t,t_0)\right]^{r/p}}{pk^{(p-r)/p}}\right], \quad t \in \mathbb{T}_0. \tag{2.8}$$

Proof. Define a function $z(t)$ by

$$z(t) = \left\{a(t) + \int_{t_0}^{t} b(s)x^p(s)\Delta s + c(t)\int_{t_0}^{t}\left[f(s)x^q(\tau(s)) + g(s)x^r(s)\right]\Delta s\right\}^{1/p}, \quad t \in \mathbb{T}_0. \tag{2.9}$$

It is easy to see that $z(t)$ is a nonnegative and nondecreasing function, and

$$x(t) \leq z(t), \quad t \in \mathbb{T}_0. \tag{2.10}$$

Therefore,

$$x(\tau(t)) \leq z(\tau(t)) \leq z(t), \quad \text{for } t \in \mathbb{T}_0 \text{ with } \tau(t) > t_0. \tag{2.11}$$

On the other hand, using the initial condition (I), we have

$$x(\tau(t)) = \varphi(\tau(t)) \leq (a(t))^{1/p} \leq z(t), \quad \text{for } t \in \mathbb{T}_0 \text{ with } \tau(t) \leq t_0. \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$x(\tau(t)) \leq z(t), \quad t \in \mathbb{T}_0. \tag{2.13}$$

It follows from (2.9), (2.10), and (2.13) that

$$z^p(t) \leq a(t) + \int_{t_0}^{t} b(s)z^p(s)\Delta s + c(t)\int_{t_0}^{t}\left[f(s)z^q(s) + g(s)z^r(s)\right]\Delta s, \quad t \in \mathbb{T}_0. \tag{2.14}$$

Define a function $w(t)$ by

$$w(t) = a(t) + c(t)u(t), \tag{2.15}$$

where

$$u(t) = \int_{t_0}^{t} \left[f(s)z^q(s) + g(s)z^r(s)\right]\Delta s, \quad t \in \mathbb{T}_0. \tag{2.16}$$

Then (2.14) can be restated as

$$z^p(t) \leq w(t) + \int_{t_0}^{t} b(s)z^p(s)\Delta s, \quad t \in \mathbb{T}_0. \tag{2.17}$$

Obviously, $w \in C_{\rm rd}(\mathbb{T}_0, \mathbb{R}_+)$, $b(t) \geq 0$, $b \in \mathcal{R}^+$. Using Lemma 2.2, from (2.17), we obtain

$$z^p(t) \leq w(t) + \int_{t_0}^{t} e_b(t, \sigma(s))w(s)b(s)\Delta s, \quad t \in \mathbb{T}_0. \tag{2.18}$$

Noting that $w(t)$ is nondecreasing, from (2.18), we have

$$z^p(t) \leq w(t) + w(t)\int_{t_0}^{t} e_b(t, \sigma(s))b(s)\Delta s = w(t)\left[1 + \int_{t_0}^{t} e_b(t, \sigma(s))b(s)\Delta s\right], \quad t \in \mathbb{T}_0. \tag{2.19}$$

By [10, Theorems 2.39 and 2.36(i)], we obtain

$$\int_{t_0}^{t} e_b(t, \sigma(s))b(s)\Delta s = e_b(t, t_0) - e_b(t, t) = e_b(t, t_0) - 1, \quad t \in \mathbb{T}_0. \tag{2.20}$$

It follows from (2.19) and (2.20) that

$$z^p(t) \leq w(t)e_b(t, t_0) = e_b(t, t_0)\left(a(t) + c(t)u(t)\right), \quad t \in \mathbb{T}_0. \tag{2.21}$$

Using Lemma 2.1, from (2.21), for any $k > 0$, we easily obtain

$$\begin{aligned} z^q(t) &\leq \left[e_b(t, t_0)\right]^{q/p}\left(a(t) + c(t)u(t)\right)^{q/p} \\ &\leq \left[e_b(t, t_0)\right]^{q/p}\left[\frac{k(p-q) + qa(t)}{pk^{(p-q)/p}} + \frac{qc(t)u(t)}{pk^{(p-q)/p}}\right], \quad t \in \mathbb{T}_0, \end{aligned} \tag{2.22}$$

$$\begin{aligned} z^r(t) &\leq \left[e_b(t, t_0)\right]^{r/p}\left(a(t) + c(t)u(t)\right)^{r/p} \\ &\leq \left[e_b(t, t_0)\right]^{r/p}\left[\frac{k(p-r) + ra(t)}{pk^{(p-r)/p}} + \frac{rc(t)u(t)}{pk^{(p-r)/p}}\right], \quad t \in \mathbb{T}_0. \end{aligned} \tag{2.23}$$

Combining (2.16), (2.22), and (2.23), we have

$$u(t) \leq \int_{t_0}^{t} \left[f(s)\left[e_b(s,t_0)\right]^{q/p}\left(\frac{k(p-q)+qa(s)}{pk^{(p-q)/p}} + \frac{qc(s)u(s)}{pk^{(p-q)/p}}\right) + g(s)\left[e_b(s,t_0)\right]^{r/p}\left(\frac{k(p-r)+ra(s)}{pk^{(p-r)/p}} + \frac{rc(s)u(s)}{pk^{(p-r)/p}}\right)\right]\Delta s$$
$$= F(t) + \int_{t_0}^{t} G(s)u(s)\Delta s, \quad t \in \mathbb{T}_0, \tag{2.24}$$

where $F(t)$ and $G(t)$ are defined by (2.7) and (2.8), respectively. Using Lemma 2.2, from (2.24), we have

$$u(t) \leq F(t) + \int_{t_0}^{t} e_G(t,\sigma(s))F(s)G(s)\Delta s, \quad t \in \mathbb{T}_0. \tag{2.25}$$

Therefore, the desired inequality (2.6) follows from (2.10), (2.22), and (2.25). This completes the proof. □

Theorem 2.5. *Suppose that all assumptions of Theorem 2.4 hold. Then the inequality* (E) *with the initial condition* (I) *implies*

$$x(t) \leq \left[e_b(t,t_0)\left(a(t) + c(t)F(t)e_G(t,t_0)\right)\right]^{1/p}, \tag{2.26}$$

for any $k > 0$, $t \in \mathbb{T}_0$, *where* $F(t)$ *and* $G(t)$ *are defined by* (2.7) *and* (2.8), *respectively.*

Proof. As in the proof of Theorem 2.4, we obtain (2.25). It is easy to see that $F(t)$ is nondecreasing for $t \in \mathbb{T}_0$. Therefore, by [10, Theorems 2.39 and 2.36(i)], we have

$$u(t) \leq F(t)e_G(t,t_0), \quad t \in \mathbb{T}_0. \tag{2.27}$$

The desired inequality (2.26) follows from (2.10), (2.22), and (2.27). The proof is complete. □

Remark 2.6. Let $\mathbb{T} = \mathbb{R}$. If $b(t) = 0$, then Theorem 2.5 reduces to [9, Theorem 2.3]. Letting $\mathbb{T} = \mathbb{Z}$, from Theorem 2.5, we easily establish the following result.

Corollary 2.7. *Assume that* $x(n)$, $a(n)$, $b(n)$, $c(n)$, $f(n)$, $g(n)$ *are nonnegative functions defined for* $n \in \mathbb{N}_0$. *If* $a(n)$ *and* $c(n)$ *are nondecreasing in* $\mathbb{N}_0$, *and* $x(n)$ *satisfies the following delay discrete inequality:*

$$x^p(n) \leq a(n) + \sum_{s=0}^{n-1} b(s)x^p(s) + c(n)\sum_{s=0}^{n-1}\left[f(s)x^q(s-\rho) + g(s)x^r(s)\right], \quad n \in \mathbb{N}_0, \tag{E1}$$

with the initial condition

$$x(n) = \varphi(n), \quad n \in \{-\rho, \ldots, -1, 0\},$$
$$\varphi(n-\rho) \le (a(n))^{1/p} \quad \textit{for } n \in \mathbb{N}_0 \textit{ with } n-\rho \le 0, \tag{I1}$$

where p, q, r, and ρ are constants, $p \neq 0$, $p \ge q \ge 0$, $p \ge r \ge 0$, $\rho \in \mathbb{N}_0$, $\varphi(n) \in \mathbb{R}_+$, $n \in \{-\rho, \ldots, -1, 0\}$, then

$$x(n) \le \left\{ \prod_{s=0}^{n-1} (1+b(s)) \left[a(n) + c(n) H(n) \prod_{s=0}^{n-1} (1+J(s)) \right] \right\}^{1/p}, \tag{2.28}$$

for any $k > 0$, $n \in \mathbb{N}_0$,

$$H(n) = \sum_{s=0}^{n-1} \left\{ \frac{f(s) \left[\prod_{t=0}^{s-1} (1+b(t))\right]^{q/p} [k(p-q) + q a(s)]}{p k^{(p-q)/p}} + \frac{g(s) \left[\prod_{t=0}^{s-1} (1+b(t))\right]^{r/p} [k(p-r) + r a(s)]}{p k^{(p-r)/p}} \right\}, \tag{2.29}$$

$$J(n) = c(n) \left\{ \frac{q f(n) \left[\prod_{s=0}^{n-1} (1+b(s))\right]^{q/p}}{p k^{(p-q)/p}} + \frac{r g(n) \left[\prod_{s=0}^{n-1} (1+b(s))\right]^{r/p}}{p k^{(p-r)/p}} \right\}, \quad n \in \mathbb{N}_0.$$

Next, using the Chain Rule, we consider a special case of the delay integral inequality (E) of the form

$$x^p(t) \le C + \int_{t_0}^{t} b(s) x^p(s) \Delta s + \int_{t_0}^{t} f(s) x^{p-1}(\tau(s)) \Delta s, \quad t \in \mathbb{T}_0, \tag{E'}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\alpha, t_0] \cap \mathbb{T},$$
$$\varphi(\tau(t)) \le C^{1/p} \quad \text{for } t \in \mathbb{T}_0 \text{ with } \tau(t) \le t_0, \tag{I'}$$

where C and $p \ge 1$ are positive constants, $\tau(t)$, α, and $\varphi(t)$ are defined as in (I).

Theorem 2.8. *Assume that $x(t), b(t), f(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$. Then the inequality (E') with the initial condition (I') implies*

$$x(t) \le C^{1/p} e_{b/p}(t, t_0) + \frac{1}{p} \int_{t_0}^{t} e_{b/p}(t, \sigma(s)) f(s) \Delta s, \quad t \in \mathbb{T}_0. \tag{2.30}$$

Proof. Define a function $w(t)$ by

$$w^p(t) = C + \int_{t_0}^{t} b(s)x^p(s)\Delta s + \int_{t_0}^{t} f(s)x^{p-1}(\tau(s))\Delta s, \quad t \in \mathbb{T}_0. \tag{2.31}$$

Using a similar way in the proof of Theorem 2.4, we easily obtain that $w(t)$ is a positive and nondecreasing function, and

$$x(t) \le w(t), \quad t \in \mathbb{T}_0, \tag{2.32}$$

$$x(\tau(t)) \le w(t), \quad t \in \mathbb{T}_0. \tag{2.33}$$

Differentiating (2.31), we obtain

$$pw^{p-1}(\theta)w^{\Delta}(t) = b(t)x^p(t) + f(t)x^{p-1}(\tau(t)), \quad t \in \mathbb{T}_0, \tag{2.34}$$

where $\theta \in [t, \sigma(t)]$.

It follows from (2.32)–(2.34) that

$$pw^{p-1}(\theta)w^{\Delta}(t) \le b(t)w^p(t) + f(t)w^{p-1}(t), \quad t \in \mathbb{T}_0. \tag{2.35}$$

Noting the fact that $0 < w(t) \le w(\theta)$ and $w^{\Delta}(t) \ge 0$, from the above inequality, we have

$$pw^{p-1}(t)w^{\Delta}(t) \le b(t)w^p(t) + f(t)w^{p-1}(t), \quad t \in \mathbb{T}_0. \tag{2.36}$$

Therefore,

$$w^{\Delta}(t) \le \frac{b(t)}{p}w(t) + \frac{f(t)}{p}, \quad t \in \mathbb{T}_0. \tag{2.37}$$

By Lemma 2.3, from (2.37), we have

$$w(t) \le C^{1/p}e_{b/p}(t, t_0) + \int_{t_0}^{t} e_{b/p}(t, \sigma(s))\frac{f(s)}{p}\Delta s, \quad t \in \mathbb{T}_0. \tag{2.38}$$

Therefore, the desired inequality (2.30) follows from (2.32) and (2.38). This completes the proof of Theorem 2.8. □

Corollary 2.9. *Assume that $x(t), b(t), f(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. If $x(t)$ satisfies the following delay integral inequality:*

$$x^p(t) \le C + \int_0^t b(s)x^p(s)ds + \int_0^t f(s)x^{p-1}(\rho(s))ds, \quad t \in \mathbb{R}_+, \tag{E2}$$

with the initial condition

$$x(t) = \phi(t), \quad t \in [\beta, 0],$$
$$\phi(\rho(t)) \le C^{1/p} \quad \textit{for } t \in \mathbb{R}_+ \textit{ with } \rho(t) \le 0, \tag{I2}$$

where C *and* $p \ge 1$ *are positive constants,* $\rho(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\rho(t) \le t$, $-\infty < \beta = \inf\{\rho(t), t \in \mathbb{R}_+\} \le 0$, *and* $\phi(t) \in C([\beta, 0], \mathbb{R}_+)$, *then*

$$x(t) \le C^{1/p} \exp\left(\int_0^t \frac{b(s)}{p} ds\right) + \frac{1}{p}\int_0^t f(s) \exp\left(\int_s^t \frac{b(\tau)}{p} d\tau\right) ds, \quad t \in \mathbb{R}_+. \tag{2.39}$$

Corollary 2.10. *Assume that* $x(n)$, $b(n)$, $f(n)$ *are nonnegative functions defined for* $n \in \mathbb{N}_0$. *If* $x(n)$ *satisfies the following delay discrete inequality:*

$$x^p(n) \le C + \sum_{s=0}^{n-1} b(s) x^p(s) + \sum_{s=0}^{n-1} f(s) x^{p-1}(s - \rho), \quad n \in \mathbb{N}_0, \tag{E3}$$

with the initial condition

$$x(n) = \varphi(n), \quad n \in \{-\rho, \dots, -1, 0\},$$
$$\varphi(n - \rho) \le C^{1/p} \quad \textit{for } n \in \mathbb{N}_0 \textit{ with } n - \rho \le 0, \tag{I3}$$

where C *and* $p \ge 1$ *are positive constants,* ρ *and* $\varphi(n)$ *are defined as in* $(I1)$, *then*

$$x(n) \le C^{1/p} \prod_{s=0}^{n-1}\left(1 + \frac{b(s)}{p}\right) + \frac{1}{p}\sum_{s=0}^{n-1} f(s) \prod_{i=s+1}^{n-1}\left(1 + \frac{b(i)}{p}\right), \quad n \in \mathbb{N}_0. \tag{2.40}$$

Finally, we study the delay integral inequality on time scales of the form

$$x^p(t) \le a(t) + c(t)\int_{t_0}^t \left[f(s) x^q(s) + L(s, x(\tau(s)))\right] \Delta s, \quad t \in \mathbb{T}_0, \tag{E''}$$

with the initial condition (I), where $p \ge 1$, $0 \le q \le p$ are constants, $\tau(t)$ is as defined in the inequality (E), and $L : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

Theorem 2.11. *Assume that* $x(t), a(t), c(t), f(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$. *If* $a(t)$ *and* $c(t)$ *are nondecreasing for* $t \in \mathbb{T}_0$, *and*

$$0 \le L(t, x) - L(t, y) \le K(t, y)(x - y), \tag{2.41}$$

for $x \geq y \geq 0$, where $K : \mathbb{T}_0 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function, then the inequality (E'') with the initial condition (I) implies

$$x(t) \leq \left[a(t) + c(t)\left(H(t) + \int_{t_0}^{t} e_J(t, \sigma(s))H(s)J(s)\Delta s\right)\right]^{1/p}, \tag{2.42}$$

for any $k > 0$, $t \in \mathbb{T}_0$, where

$$H(t) = \int_{t_0}^{t}\left[\frac{f(s)\left[k(p-q) + qa(s)\right]}{pk^{(p-q)/p}} + L\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right)\right]\Delta s, \tag{2.43}$$

$$J(t) = \frac{qc(t)f(t)}{pk^{(p-q)/p}} + K\left(t, \frac{p-1}{p} + \frac{a(t)}{p}\right)\frac{c(t)}{p}. \tag{2.44}$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t}\left[f(s)x^q(s) + L(s, x(\tau(s)))\right]\Delta s, \quad t \in \mathbb{T}_0. \tag{2.45}$$

We easily observe that $z(t)$ is a nonnegative and nondecreasing function, and (E'') can be restated as

$$x(t) \leq \left(a(t) + c(t)z(t)\right)^{1/p}, \quad t \in \mathbb{T}_0. \tag{2.46}$$

Using Lemma 2.1, from (2.46), we have

$$x(t) \leq \left(a(t) + c(t)z(t)\right)^{1/p} \leq \frac{p-1}{p} + \frac{a(t)}{p} + \frac{c(t)z(t)}{p}, \quad t \in \mathbb{T}_0. \tag{2.47}$$

Therefore, for $t \in \mathbb{T}_0$ with $\tau(t) \geq t_0$, we obtain

$$x(\tau(t)) \leq \frac{p-1}{p} + \frac{a(\tau(t))}{p} + \frac{c(\tau(t))z(\tau(t))}{p} \leq \frac{p-1}{p} + \frac{a(t)}{p} + \frac{c(t)z(t)}{p}, \tag{2.48}$$

and for $t \in \mathbb{T}_0$ with $\tau(t) \leq t_0$, using the initial condition (I) and (2.47), we get

$$x(\tau(t)) = \varphi(\tau(t)) \leq \left(a(t)\right)^{1/p} \leq \frac{p-1}{p} + \frac{a(t)}{p} + \frac{c(t)z(t)}{p}. \tag{2.49}$$

It follows from (2.48) and (2.49) that

$$x(\tau(t)) \le \frac{p-1}{p} + \frac{a(t)}{p} + \frac{c(t)z(t)}{p}, \quad t \in \mathbb{T}_0. \tag{2.50}$$

Combining (2.45), (2.46), and (2.50), by Lemma 2.1, for any $k > 0$, we obtain

$$\begin{aligned}
z(t) &\le \int_{t_0}^{t} f(s)\left[a(s)+c(s)z(s)\right]^{q/p}\Delta s + \int_{t_0}^{t} L\left(s, \frac{p-1}{p} + \frac{a(s)}{p} + \frac{c(s)z(s)}{p}\right)\Delta s \\
&\le \int_{t_0}^{t} f(s)\left[\frac{k(p-q)+qa(s)}{pk^{(p-q)/p}} + \frac{qc(s)z(s)}{pk^{(p-q)/p}}\right]\Delta s \\
&\quad + \int_{t_0}^{t}\left\{L\left(s, \frac{p-1}{p} + \frac{a(s)}{p} + \frac{c(s)z(s)}{p}\right) - L\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right) + L\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right)\right\}\Delta s \\
&\le \int_{t_0}^{t}\left[\frac{f(s)\left[k(p-q)+qa(s)\right]}{pk^{(p-q)/p}} + L\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right)\right]\Delta s \\
&\quad + \int_{t_0}^{t}\left[\frac{qc(s)f(s)}{pk^{(p-q)/p}} + K\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right)\frac{c(s)}{p}\right]z(s)\Delta s \\
&= H(t) + \int_{t_0}^{t} J(s)z(s)\Delta s, \quad t \in \mathbb{T}_0,
\end{aligned} \tag{2.51}$$

where $H(t)$ and $J(t)$ are defined by (2.43) and (2.44), respectively.

By Lemma 2.2, from (2.51), we have

$$z(t) \le H(t) + \int_{t_0}^{t} e_J(t, \sigma(s))H(s)J(s)\Delta s, \quad t \in \mathbb{T}_0. \tag{2.52}$$

Therefore, the desired inequality (2.42) follows from (2.46) and (2.52). The proof of Theorem 2.11 is complete. □

Noting $H(t)$, defined by (2.43), is nondecreasing for $t \in \mathbb{T}_0$, we easily obtain the following result.

Theorem 2.12. *Suppose that all assumptions of Theorem 2.11 hold. Then the inequality* (E'') *with the initial condition* (I) *implies*

$$x(t) \le \left(a(t) + c(t)H(t)e_J(t, t_0)\right)^{1/p}, \tag{2.53}$$

for any $k > 0$, $t \in \mathbb{T}_0$, *where* $H(t)$ *and* $J(t)$ *are defined by* (2.46) *and* (2.47), *respectively.*

Remark 2.13. If $\mathbb{T} = \mathbb{R}$, then Theorem 2.12 reduces to [9, Theorem 2.8]. Letting $\mathbb{T} = \mathbb{Z}$, from Theorem 2.12, we can obtain the following corollary.

Corollary 2.14. *Assume that $x(n)$, $a(n)$, $c(n)$, $f(n)$ are nonnegative functions defined for $n \in \mathbb{N}_0$. If $a(n)$ and $c(n)$ are nondecreasing in $\mathbb{N}_0$, and $x(n)$ satisfies the following delay discrete inequality:*

$$x^p(n) \le a(n) + c(n)\sum_{s=0}^{n-1}\left[f(s)x^q(s) + L\left(s, x(s-\rho)\right)\right], \quad n \in \mathbb{N}_0, \tag{E4}$$

where p, q, and ρ are constants, $p \ge 1$, $p \ge q \ge 0$, $\rho \in \mathbb{N}_0$, and $L, K : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$0 \le L(n,x) - L(n,y) \le K(n,y)(x-y), \tag{2.54}$$

for $x \ge y \ge 0$, then the inequality $(E4)$ with the initial condition $(I1)$ implies

$$x(n) \le \left(a(n) + c(n)\widetilde{H}(n)\prod_{s=0}^{n-1}\left[1 + \widetilde{J}(s)\right]\right)^{1/p}, \tag{2.55}$$

for any $k > 0$, $n \in \mathbb{N}_0$, where

$$\begin{gathered}\widetilde{H}(n) = \sum_{s=0}^{n-1}\left[\frac{f(s)\left[k(p-q) + qa(s)\right]}{pk^{(p-q)/p}} + L\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right)\right], \\ \widetilde{J}(n) = \frac{qc(n)f(n)}{pk^{(p-q)/p}} + K\left(n, \frac{p-1}{p} + \frac{a(n)}{p}\right)\frac{c(n)}{p}.\end{gathered} \tag{2.56}$$

3. Some applications

In this section, we present some applications of our results.

Example 3.1. Consider the delay dynamic equation on time scales:

$$\left(x^p(t)\right)^{\Delta} = M(t, x(t), x(\tau(t))), \quad t \in \mathbb{T}_0, \tag{3.1}$$

with the initial condition

$$\begin{gathered}x(t) = \varphi(t), \quad t \in \left[\alpha, t_0\right] \cap \mathbb{T}, \\ \varphi\left(\tau(t)\right) = C^{1/p} \quad \text{for } t \in \mathbb{T}_0 \text{ with } \tau(t) \le t_0,\end{gathered} \tag{I''}$$

where $M : \mathbb{T}_0 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $C = x^p(t_0)$ and $p > 0$ are constants, α and $\tau(t)$ are as defined in the initial condition (I), and $\varphi(t) \in C_{\mathrm{rd}}([\alpha, t_0] \cap \mathbb{T}, \mathbb{R})$.

Theorem 3.2. *Assume that*

$$\left|M\left(t, x(t), x\left(\tau(t)\right)\right)\right| \le f(t)\left|x^q\left(\tau(t)\right)\right| + g(t)\left|x^r(t)\right|, \tag{3.2}$$

where $f(t), g(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, *$q$ and r are constants,* $p \geq q \geq 0$, $p \geq r \geq 0$. *If $x(t)$ is a solution of* (3.1) *satisfying the initial condition* (I''), *then*

$$|x(t)| \leq \left[|C| + \overline{F}(t) + \int_{t_0}^{t} e_{\overline{G}}(t, \sigma(s))\overline{F}(s)\overline{G}(s)\Delta s\right]^{1/p}, \tag{3.3}$$

for any $k > 0$, $t \in \mathbb{T}_0$, *where*

$$\overline{F}(t) = \int_{t_0}^{t} \left\{ \frac{f(s)\left[k(p-q) + q|C|\right]}{pk^{(p-q)/p}} + \frac{g(s)\left[k(p-r) + r|C|\right]}{pk^{(p-r)/p}} \right\}\Delta s, \tag{3.4}$$

$$\overline{G}(t) = \frac{qf(t)}{pk^{(p-q)/p}} + \frac{rg(t)}{pk^{(p-r)/p}}. \tag{3.5}$$

Proof. Obviously, the solution $x(t)$ of (3.1) with the initial condition (I'') satisfies the equivalent delay integral equation on time scales

$$x^p(t) = C + \int_{t_0}^{t} M(s, x(s), x(\tau(s)))\Delta s, \quad t \in \mathbb{T}_0, \tag{3.6}$$

with the initial condition (I''). Noting the assumption (3.2), we have

$$|x^p(t)| \leq |C| + \int_{t_0}^{t} \left[f(s)|x^q(\tau(s))| + g(s)|x^r(s)|\right]\Delta s, \quad t \in \mathbb{T}_0, \tag{3.7}$$

with the initial condition (I''). Therefore, by Theorem 2.4, from (3.7), we easily obtain the estimate (3.3) of solutions of (3.1). The proof of Theorem 3.2 is complete. □

Using Theorem 2.5, we easily obtain the following result.

Theorem 3.3. *Suppose that all assumptions of Theorem 3.2 hold. If $x(t)$ is a solution of* (3.1) *satisfying the initial condition* (I''), *then*

$$|x(t)| \leq \left[|C| + \overline{F}(t)e_{\overline{G}}(t, t_0)\right]^{1/p}, \tag{3.8}$$

for any $k > 0$, $t \in \mathbb{T}_0$, *where $\overline{F}(t)$ and $\overline{G}(t)$ are defined by* (3.4) *and* (3.5), *respectively.*

Remark 3.4. The right-hand sides of (3.3) and (3.8) give us the bounds on the solution $x(t)$ of (3.1) satisfying the initial condition (I'') in terms of the known functions for any $k > 0$, $t \in \mathbb{T}_0$, respectively.

Example 3.5. Consider the delay discrete inequality as in $(E3)$ satisfying the initial condition $(I3)$ with $p = 2$, $C = 1/4$, $\rho = 2$, $\varphi(n) = 1/2$, $n \in \{-2, -1, 0\}$, $b(n) = 10^{-3}n^2$, $f(n) = 10^{-4}n$, $n \in \mathbb{N}_0$, and we compute the values of $x(n)$ from $(E3)$ and also we find the values of $x(n)$ by using the result (2.40). In our computations, we use $(E3)$ and (2.40) as equations and as we see in Table 1 the computation values as in $(E3)$ are less than the values of the result (2.40).

Table 1

n	$(E3)$	(2.43)
1	5.000000000000000e–001	5.000000000000000e–001
3	5.013992421214853e–001	5.014006000000000e–001
7	5.240341550720497e–001	5.242013057437409e–001
11	6.053588145272404e–001	6.073138820474305e–001
14	7.428258989476674e–001	7.507097542821271e–001
17	1.009578705314619e+000	1.036912536372208e+000
22	2.189862704124656e+000	2.391160696569409e+000
25	4.143517993238956e+000	4.841349883598182e+000
27	6.839504919933415e+000	8.504988064333858e+000
30	1.630102753510524e+001	2.295320353713791e+001
35	9.500824036460114e+001	1.816014350966817e+002
40	8.204195033362939e+002	2.464464322608679e+003

From Table 1, we easily find that the numerical solution agrees with the analytical solution for some discrete inequalities. The program is written in the programming Matlab 7.0.

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