Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 845934, 15 pages
doi:10.1155/2008/845934

*Research Article*

# Some Generalized Error Inequalities and Applications

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Received 4 February 2008; Revised 30 May 2008; Accepted 29 July 2008

Recommended by Sever Dragomir

We present a family of four-point quadrature rule, a generalization of Gauss-two point, Simpson's 3/8, and Lobatto four-point quadrature rule for twice-differentiable mapping. Moreover, it is shown that the corresponding optimal quadrature formula presents better estimate in the context of four-point quadrature formulae of closed type. A unified treatment of error inequalities for different classes of function is also given.



## 1. Introduction

We define

$$I(f) = \int_a^b f(x)dx. \tag{1.1}$$

The problem of approximating $I(f)$ is usually referred to as numerical integration or quadrature [1]. Most numerical integration formulae are based on defining the approximation by using polynomial or piecewise polynomial interpolation. Formulae using such interpolation with evenly spaced nodes are referred to as Newton-Cotes formulae. The Gaussian quadrature formulae, which are optimal and converge rapidly by selecting the node points carefully that need not be equally spaced, are investigated in [2].

In [3–5], the quadrature problem, in particular, the investigation of error bounds of Newton-Cotes formulae, namely, the mid-point, trapezoid, and Simpson's rule have been carried out by the use of Peano kernel approach in terms of variety of norms, from an inequality point of view.

The deduction of the optimal quadrature formulae in the sense of minimal error bounds has not received the right attention as long as the work by Ujević (see [6–8] and [9, pages 153–166]), who used a new approach for obtaining optimal two-point and three-point quadrature formulae of open as well as closed type, has not appeared. Further, some error inequalities have also been presented by Ujević to ensure the applications of these optimal quadrature formulae for different classes of functions.

In this paper, we present an approach similar to that of Ujević's [6] to present some improvements and generalizations in this context.

Let us first formulate the main problem.

Consider

$$K(x,y,t)=\begin{cases}\dfrac{1}{2}(t-\alpha)^2+\alpha_1, & t\in[a,x],\\[2mm] \dfrac{1}{2}(t-\beta)^2+\beta_1, & t\in(x,y),\\[2mm] \dfrac{1}{2}(t-\gamma)^2+\gamma_1, & t\in[y,b],\end{cases} \tag{1.2}$$

as defined in [6], where $x,y\in[a+h(b-a),b-h(b-a)]$, $h\in[0,1/2]$, $x<y$, and $\alpha,\alpha_1,\beta,\beta_1,\gamma,\gamma_1\in\mathbb{R}$ are parameters which are required to be determined.

We know that the exact value of the remainder term of the integral $\int_a^b K(x,y,t)f''(t)dt$ may not be found, thus, we may proceed as

$$\left|\int_a^b K(x,y,t)f''(t)dt\right|\le\max_{t\in[a,b]}|f''(t)|\int_a^b|K(x,y,t)|dt. \tag{1.3}$$

The main aim of this paper is to present a minimal estimation of the error bound (1.3) by appropriately choosing the variables and parameters involved. Moreover, it is worth mentioning that the family of quadrature formulae thus obtained hereafter is a generalization of that presented in [6].

## 2. A generalized optimal quadrature formula

Consider the above stated error inequality problem for $a=-1$, $b=1$, so that $x,y\in[-1+2h,1-2h]$. We will try to find out an optimal quadrature formula of the form

$$\int_{-1}^{1}f(t)dt-[hf(-1)+(1-h)f(x)+(1-h)f(y)+hf(1)]=\int_{-1}^{1}K(x,y,t)f''(t)dt, \tag{2.1}$$

where $K(x,y,t)$ is defined by (1.2) with $a=-1$, $b=1$, and $x,y\in[-1+2h,1-2h]$ with $x<y$, $h\in[0,1/2]$.

The parameters $\alpha,a_1,\beta,\beta_1,\gamma,\gamma_1\in\mathbb{R}$ involved in $K(x,y,t)$ are required to be determined in a way such that the representation (2.1) is obtained.

Integrating by parts right-hand side of (2.1), we have

$$\begin{aligned}\int_{-1}^{1} K(x,y,t) f''(t)\,dt = &-\left[\frac{1}{2}(1+\alpha)^2+\alpha_1\right] f'(-1)+\left[\frac{1}{2}(1-\gamma)^2+\gamma_1\right] f'(1)\\
&+\left[\frac{1}{2}\{(x-\alpha)^2-(x-\beta)^2\}+\alpha_1-\beta_1\right] f'(x)\\
&+\left[\frac{1}{2}\{(y-\beta)^2-(y-\gamma)^2\}+\beta_1-\gamma_1\right] f'(y)-(1+\alpha)f(-1)\\
&-(1-\gamma)f(1)+(\alpha-\beta)f(x)+(\beta-\gamma)f(y)\\
&+\int_{-1}^{1} f(t)\,dt.\end{aligned} \tag{2.2}$$

For the representation (2.1), we require from (2.2)

$$\begin{aligned}\frac{1}{2}(x-\alpha)^2+\alpha_1-\frac{1}{2}(x-\beta)^2-\beta_1&=0,\\
\frac{1}{2}(y-\beta)^2+\beta_1-\frac{1}{2}(y-\gamma)^2-\gamma_1&=0,\\
\frac{1}{2}(1+\alpha)^2+\alpha_1&=0,\\
\frac{1}{2}(1-\gamma)^2+\gamma_1&=0,\\
\beta-\gamma&=-(1-h),\\
\alpha-\beta&=-(1-h),\\
1+\alpha&=h,\\
1-\gamma&=h.\end{aligned} \tag{2.3}$$

This gives through simple calculations:

$$\begin{aligned}&\alpha=-(1-h),\qquad \gamma=(1-h),\qquad \beta=0,\\
&\gamma_1=-\frac{1}{2}h^2,\\
&\alpha_1=-\frac{1}{2}h^2,\\
&\beta_1=\frac{1}{2}-h+(1-h)x\\
&\quad=\frac{1}{2}-h-(1-h)y.\end{aligned} \tag{2.4}$$

Henceforth,

$$y = -x. \tag{2.5}$$

So, we have

$$K(x,t) = \begin{cases} \dfrac{1}{2}(t + (1-h))^2 - \dfrac{1}{2}h^2, & t \in [-1, x], \\ \dfrac{1}{2}t^2 + (1-h)x - h + \dfrac{1}{2}, & t \in (x, y), \\ \dfrac{1}{2}(t - (1-h))^2 - \dfrac{1}{2}h^2, & t \in [y, 1]. \end{cases} \tag{2.6}$$

We further see that

$$\left| \int_{-1}^{1} K(x,t) f''(t)\,dt \right| \leq \|f''\|_{\infty} \int_{-1}^{1} |K(x,t)|\,dt. \tag{2.7}$$

We are now required to find an $x$ that minimizes $\int_{-1}^{1} |K(x,t)|dt$.

We next define

$$\begin{aligned} G(x) &= \int_{-1}^{1} |K(x,t)|\,dt \\ &= \frac{1}{2}\int_{-1}^{x} |(t + (1-h))^2 - h^2|\,dt + \int_{x}^{y} \left| \frac{1}{2}t^2 + (1-h)x - h + \frac{1}{2} \right| dt + \frac{1}{2}\int_{y}^{1} |(t - (1-h))^2 - h^2|\,dt, \end{aligned} \tag{2.8}$$

and consider the problem

$$\text{minimize } G(x), \quad x \in [-1 + 2h, 1 - 2h], \ h \in \left[0, \frac{1}{2}\right]. \tag{2.9}$$

Hence, we would like to find a global minimizer of $G$. Recall that a global minimizer is a point $x^*$ that satisfies

$$G(x^*) \leq G(x) \quad \forall x \in [-1 + 2h, 1 - 2h], \ h \in \left[0, \frac{1}{2}\right]. \tag{2.10}$$

We now consider the following cases.

(i) Let $x \in [-(1-2h), (h-1/2)/(1-h)]$. Then by symmetry, we may consider

$$\begin{aligned}
G_1(x) = &-\frac{1}{2}\int_{-1}^{-1+2h}((t+(1-h))^2-h^2)dt \\
&+\frac{1}{2}\int_{-1+2h}^{x}((t+(1-h))^2-h^2)dt \\
&+\frac{1}{2}\int_{x}^{-\sqrt{2h-1-2(1-h)x}}(t^2+2(1-h)x-2h+1)dt \\
&-\frac{1}{2}\int_{-\sqrt{2h-1-2(1-h)x}}^{0}(t^2+2(1-h)x-2h+1)dt \\
= &\frac{1}{6}-\frac{1}{2}(1-h)x^2-\frac{4}{3}(1-h)\sqrt{2h-1-2(1-h)x}x \\
&+\frac{4}{3}\left(h-\frac{1}{2}\right)\sqrt{2h-1-2(1-h)x}+\frac{4}{3}h^3-\frac{h}{2}.
\end{aligned} \tag{2.11}$$

We may note that

$$G(x) = 2G_1(x). \tag{2.12}$$

Combining (2.11) and (2.12) with (2.1) and (2.7), we get

$$\begin{aligned}
&\left|\int_{-1}^{1}f(t)dt-[hf(-1)+(1-h)f(x)+(1-h)f(-x)+hf(1)]\right| \\
&\quad\le\left[\frac{1}{3}-(1-h)x^2-\frac{8}{3}(1-h)\sqrt{2h-1-2(1-h)x}x\right. \\
&\qquad\left.+\frac{8}{3}\left(h-\frac{1}{2}\right)\sqrt{2h-1-2(1-h)x}+\frac{8}{3}h^3-h\right]\|f''\|_\infty.
\end{aligned} \tag{2.13}$$

Moreover, simple calculations show that $G_1'(x)=0$ for

$$x_{1,2} = -4+4h\pm 2\sqrt{3-6h+4h^2}. \tag{2.14}$$

It is not difficult to find that

$$G_1''(x_1) > 0, \qquad G_1''(x_2) < 0. \tag{2.15}$$

Thus, $x_1$ is the local minimizer of $G(x)$ for $x \in [-(1-2h), (h-1/2)/(1-h)]$. We have

$$\begin{aligned} G_1(x_1) = {} & \frac{52}{3}h^3 - 44h^2 + \frac{83}{2}h - \frac{83}{6} + 8(1-h)^2\sqrt{4h^2-6h+3} \\ & + \frac{2}{3}(8h^2-14h+7)\sqrt{8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}} \\ & - \frac{8}{3}(1-h)\sqrt{8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}}\sqrt{4h^2-6h+3}, \end{aligned} \tag{2.16}$$

such that

$$G(x_1) = 2G_1(x_1). \tag{2.17}$$

(ii) Next, we check the point $x_3 = (h-1/2)/(1-h)$. We find that $\min_{h\in[0,1/2]} G_1(x_1) < \min_{h\in[0,1/2]} G_1(x_3)$.

Thus, from the above considerations, we find that $x^* = -4+4h+2\sqrt{3-6h+4h^2}$ is the global minima of $G$. Therefore, we get the following conclusion.

**Theorem 2.1.** *Let $I \subset \mathbb{R}$ be an open interval such that $[-1,1] \subset I$ and let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f''$ is bounded and integrable. Then,*

$$\begin{aligned} & \int_{-1}^{1} f(t)dt \\ & = [hf(-1)+(1-h)f(-4+4h+2\sqrt{3-6h+4h^2})+(1-h)f(4-4h-2\sqrt{3-6h+4h^2})+hf(1)]+R(f), \end{aligned} \tag{2.18}$$

*where*

$$|R(f)| \le 2\Delta(h)\|f''\|_\infty, \tag{2.19}$$

$h \in [0,1/2]$, *and $\Delta(h)$ is defined as*

$$\begin{aligned} \Delta(h) = {} & \frac{52}{3}h^3 - 44h^2 + \frac{83}{2}h - \frac{83}{6} + 8(1-h)^2\sqrt{4h^2-6h+3} \\ & + \frac{2}{3}(8h^2-14h+7)\sqrt{8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}} \\ & - \frac{8}{3}(1-h)\sqrt{8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}}\sqrt{4h^2-6h+3}. \end{aligned} \tag{2.20}$$

*Proof.* From the above discussion, we find that (2.18) holds with

$$R(f) = \int_{-1}^{1} K(-4+4h+2\sqrt{3-6h+4h^2}, t) f''(t)dt, \tag{2.21}$$

and $K(x,t)$ is given by (2.6) with $y=-x$. We further have

$$\begin{aligned}|R(f)| &\le \|f''\|_\infty \int_{-1}^{1} |K(-4+4h+2\sqrt{3-6h+4h^2},t)|dt \\ &= G(-4+4h+2\sqrt{3-6h+4h^2})\|f''\|_\infty.\end{aligned} \tag{2.22}$$

Since $G(-4+4h+2\sqrt{3-6h+4h^2}) = 2G_1(-4+4h+2\sqrt{3-6h+4h^2})$, thus (2.19) holds. □

We would like now to mention here some special cases of (2.13).

*Remark 2.2.* As it has been mentioned in [6], we recapture the Gauss two-point quadrature formula for $h=0$ and $x=-\sqrt{3}/3$.

*Remark 2.3.* It may be noted that for $h = 1/6$ and $x = -\sqrt{5}/5$, we get Lobbato four-point quadrature rule as follows:

$$\int_{-1}^{1} f(t)dt = \frac{1}{6}\left[f(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f(1)\right]+R_1(f), \tag{2.23}$$

where

$$|R_1(f)| \le C_1\|f''\|_\infty, \tag{2.24}$$

and $C_1 = 1/81 + (4/27)(\sqrt{-6+3\sqrt{5}})(\sqrt{5}-2) \approx 0.0418$.

*Remark 2.4.* For $h=1/4$ and $x=-1/3$, we get 3/8 Simpson's rule as follows:

$$\int_{-1}^{1} f(t)dt = \frac{1}{4}\left[f(-1)+3f\left(-\frac{1}{3}\right)+3f\left(\frac{1}{3}\right)+f(1)\right]+R_2(f), \tag{2.25}$$

where

$$|R_2(f)| \le C_2\|f''\|_\infty, \tag{2.26}$$

and $C_2 = 1/24 \approx 0.0417$.

*Remark 2.5.* Keeping in view the above special cases, (2.13) may be considered as a generalization of Gauss two-point, Simpson's 3/8 and Lobatto four-point quadrature rule for twice differentiable mappings.

*Remark 2.6.* For $h=1/5$, $\Delta(h)$ attains its minimum value.

**Corollary 2.7.** *Let the assumptions of Theorem 2.1 hold. Then, one has the following optimal quadrature rule:*

$$\int_{-1}^{1} f(t)dt = \frac{1}{5}\left[f(-1) + 4f\left(-\frac{2}{5}\right) + 4f\left(\frac{2}{5}\right) + f(1)\right] + R_3(f), \tag{2.27}$$

$$|R_3(f)| \leq C_3\|f''\|_\infty, \tag{2.28}$$

*where* $C_3 = 14/375 \approx 0.0373$.

*Remark 2.8.* The comparison of (2.23), (2.25), and (2.27) shows that the latter presents a much better estimate in the context of four-point quadrature rules of closed type.

By considering the problem on the interval $[a,b]$, the following theorem is obvious.

**Theorem 2.9.** *Let $I \subset \mathbb{R}$ be an open interval such that $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be a twice-differentiable function such that $f''$ is bounded and integrable. Then,*

$$\int_a^b f(t)dt = \frac{1}{2}(b-a)[hf(a) + (1-h)f(x_1) + (1-h)f(x_2) + hf(b)] + R(f), \tag{2.29}$$

*where*

$$x_1 = \frac{b-a}{2}x^* + \frac{a+b}{2}, \qquad x_2 = -\frac{b-a}{2}x^* + \frac{a+b}{2}, \tag{2.30}$$

*with*

$$\begin{gathered} x^* = -4 + 4h + 2\sqrt{3 - 6h + 4h^2}, \\ |R(f)| \leq \frac{1}{4}\Delta(h)(b-a)^3\|f''\|_\infty, \end{gathered} \tag{2.31}$$

$h \in [0, 1/2]$, *and* $\Delta(h)$ *is as defined above.*

## 3. Generalized error inequalities

From the basic properties of the $L_p(a,b)$ spaces for $p = 1, 2, \infty$, we know that $L_2(a,b)$ is a Hilbert space with the inner product defined as

$$\langle f, g\rangle_2 = \int_a^b f(t)g(t)dt. \tag{3.1}$$

We now define $X = (L_2(a,b), \langle\cdot,\cdot\rangle_2)$. In the space $X$, the norm $\|\cdot\|_2$ is defined in the usual manner as

$$\|f\|_2 = \left(\int_a^b f^2(t)dt\right)^{1/2}. \tag{3.2}$$

Let us also consider $Y = (L_2(a,b), \langle\cdot,\cdot\rangle)$, where the inner product $\langle\cdot,\cdot\rangle$ is defined by

$$\langle f,g\rangle = \frac{1}{b-a}\int_a^b f(t)g(t)dt \tag{3.3}$$

with the corresponding norm $\|\cdot\|$ defined by

$$\|f\| = \sqrt{\langle f,f\rangle}. \tag{3.4}$$

We know that the Chebyshev functional is defined as

$$T(f,g) = \langle f,g\rangle - \langle f,e\rangle\langle g,e\rangle, \tag{3.5}$$

where $f,g \in L_2(a,b)$ and $e = 1$ which satisfies the pre-Grüss inequality [4, page 296] or [5, page 209]:

$$T^2(f,g) \le T(f,f)T(g,g). \tag{3.6}$$

Let us denote

$$\sigma(f) = \sigma(f;a,b) = \sqrt{(b-a)T(f,f)} \tag{3.7}$$

as defined in [6]. Moreover, the space $L_1(a,b)$ is a Banach space with the norm

$$\|f\|_1 = \int_a^b |f(t)|dt, \tag{3.8}$$

and the space $L_\infty(a,b)$ is also a Banach space with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t\in[a,b]} |f(t)|. \tag{3.9}$$

So, if $f \in L_1(a,b)$ and $g \in L_\infty(a,b)$, then we have

$$|\langle f,g\rangle_2| \le \|f\|_1\|g\|_\infty. \tag{3.10}$$

Finally, we define

$$\begin{aligned} J(f) &= J(f;a,b;h) \\ &= \int_a^b f(t)dt - \frac{1}{2}(b-a)[hf(a) + (1-h)f(x_1) + (1-h)f(x_2) + hf(b)], \end{aligned} \tag{3.11}$$

where $x_1$ and $x_2$ are given by (2.30).

We would also like to mention the following lemma [10].

**Lemma 3.1.** *Let*

$$f(t)=\begin{cases} f_1(t), & t\in[a,x_1],\\ f_2(t), & t\in[x_1,x_2],\\ f_3(t), & t\in[x_2,b],\end{cases} \tag{3.12}$$

*where* $a<x_1<x_2<b$, $f_1\in C^1(a,x_1)$, $f_2\in C^1(x_1,x_2)$, $f_3\in C^1(x_2,b)f_1(x_1)=f_2(x_1)$, *and* $f_2(x_2)=f_3(x_2)$. *If*

$$\sup_{t\in(a,x_1)}|f_1'(t)|<\infty,\qquad \sup_{t\in(x_1,x_2)}|f_2'(t)|<\infty,\qquad \sup_{t\in(x_2,b)}|f_3'(t)|<\infty, \tag{3.13}$$

*then the function* $f$ *is an absolutely continuous function.*

**Theorem 3.2.** *Let* $f:[-1,1]\to\mathbb{R}$ *be a function such that* $f'\in L_1(-1,1)$. *If there exists a real number* $\gamma_1$, *such that* $\gamma_1\le f'(t)$, $t\in[-1,1]$, *then*

$$|J(f;-1,1;h)|\le 2\Delta_0(h)(S-\gamma_1), \tag{3.14}$$

*and if there exists a real number* $\Gamma_1$, *such that* $f'(t)\le\Gamma_1$, $t\in[-1,1]$, *then*

$$|J(f;-1,1;h)|\le 2\Delta_0(h)(\Gamma_1-S), \tag{3.15}$$

*where* $J(f;-1,1;h)$ *is defined by* (3.11), $S=(f(1)-f(-1))/2$, *and* $h\in[0,1/2]$. *If there exist real numbers* $\gamma_1$, $\Gamma_1$, *such that* $\gamma_1\le f'(t)\le\Gamma_1$, $t\in[-1,1]$, *then*

$$|J(f;-1,1;h)|\le \frac{1}{2}\Delta_1(h)(\Gamma_1-\gamma_1), \tag{3.16}$$

$\Delta_0(h)$ *and* $\Delta_1(h)$ *are defined as*

$$\begin{aligned}\Delta_0(h)&=2\sqrt{4h^2-6h+3}-3(1-h),\\ \Delta_1(h)&=58h^2-98h+49-28(1-h)\sqrt{4h^2-6h+3}.\end{aligned} \tag{3.17}$$

*Proof.* In order to prove (3.16), let us define

$$p_1(t)=\begin{cases} t+1-h, & t\in[-1,x],\\ t, & t\in(x,y),\\ t-(1-h), & t\in[y,1],\end{cases} \tag{3.18}$$

where $x = -4 + 4h + 2\sqrt{3 - 6h + 4h^2}$ and $y = -x$. Note that since $\langle p_1, e\rangle_2 = 0$, thus

$$\begin{aligned} \langle p_1, f'\rangle_2 &= -J(f; -1, 1; h), \\ \left\langle f' - \frac{\Gamma_1 + \gamma_1}{2}, p_1 \right\rangle_2 &= \langle f', p_1\rangle_2. \end{aligned} \tag{3.19}$$

From (3.10),

$$\begin{aligned} \left| \left\langle f' - \frac{\Gamma_1 + \gamma_1}{2}, p_1 \right\rangle_2 \right| &\le \left\| f' - \frac{\Gamma_1 + \gamma_1}{2} \right\|_\infty \|p_1\|_1 \\ &\le \frac{1}{2}\Delta_1(h)(\Gamma_1 - \gamma_1), \end{aligned} \tag{3.20}$$

as

$$\begin{aligned} \left\| f' - \frac{\Gamma_1 + \gamma_1}{2} \right\|_\infty &\le \frac{\Gamma_1 - \gamma_1}{2}, \\ \|p_1\|_1 = 58h^2 - 98h + 49 &- 28(1-h)\sqrt{4h^2 - 6h + 3}. \end{aligned} \tag{3.21}$$

From (3.19) and (3.20), it may be observed that (3.16) holds. Further, it can be seen that

$$\begin{aligned} |\langle f' - \gamma_1, p_1\rangle_2| &\le \|p_1\|_\infty \|f' - \gamma_1\|_1 \\ &= 2\Delta_0(h)(S - \gamma_1), \end{aligned} \tag{3.22}$$

since

$$\begin{aligned} \|p_1\|_\infty &= 2\sqrt{4h^2 - 6h + 3} - 3(1-h), \\ \|f' - \gamma_1\|_1 &= \int_{-1}^{1} (f'(t) - \gamma_1)\,dt \\ &= f(1) - f(-1) - 2\gamma_1 \\ &= 2(S - \gamma_1). \end{aligned} \tag{3.23}$$

Hence, (3.14) holds. In the similar manner, we can prove (3.15). □

*Remark 3.3.* It may be noted that $\Delta_0(h)$ has its minimum value 0.396 at $h = 0.259$. In a similar way, it may be observed that $1/2\Delta_1(h)$ attains its minimum value 0.1698 at $h = 0.296$.

**Theorem 3.4.** *Let $f : [a, b] \to \mathbb{R}$ be a function, such that $f' \in L_1(a, b)$. If there exists a real number $\gamma_1$, such that $\gamma_1 \le f'(t)$, $t \in [a, b]$, then*

$$|J(f; a, b; h)| \le \frac{1}{2}\Delta_0(h)(S - \gamma_1)(b - a)^2, \tag{3.24}$$

*and if there exists a real number* $\Gamma_1$*, such that* $f'(t) \le \Gamma_1$*,* $t \in [a,b]$*, then*

$$|J(f;a,b;h)| \le \frac{1}{2}\Delta_0(h)(\Gamma_1 - S)(b-a)^2, \tag{3.25}$$

*where* $J(f;a,b;h)$ *is defined by* (3.11) *and* $S = (f(a) - f(b))/(b-a)$ *and* $h \in [0,1/2]$*. If there exist real numbers* $\gamma_1$*,* $\Gamma_1$*, such that* $\gamma_1 \le f'(t) \le \Gamma_1$*,* $t \in [a,b]$*, then*

$$|J(f;a,b;h)| \le \frac{1}{8}\Delta_1(h)(\Gamma_1 - \gamma_1)(b-a)^2, \tag{3.26}$$

$\Delta_0(h)$ *and* $\Delta_1(h)$ *are as defined in* (3.17)*.*

**Theorem 3.5.** *Let* $f : [-1,1] \to \mathbb{R}$ *be an absolutely continuous function, such that* $f' \in L_2(-1,1)$*. Then,*

$$|J(f;-1,1;h)| \le \sqrt{\Delta_2(h)}\sigma(f';-1,1), \tag{3.27}$$

*where* $\sigma(f';-1,1)$ *is defined by* (3.7) *and*

$$\Delta_2(h) = -56h^3 + 154h^2 - 146h + \frac{146}{3} - 28(1-h)^2\sqrt{4h^2 - 6h + 3} \tag{3.28}$$

*for* $h \in [0,1/2]$*.*

*Proof.* Let $p_1$ be the same as defined above. We have

$$\langle p_1, f'\rangle_2 = -J(f;-1,1;h), \tag{3.29}$$

since $\langle p_1, e\rangle_2 = 0$, if $[a,b] = [-1,1]$. Moreover, $\langle f,g\rangle = (1/2)\langle f,g\rangle_2$ and

$$\langle p_1, f'\rangle = T(f', p_1). \tag{3.30}$$

From (3.6), it follows that

$$\begin{aligned} T(f',p_1) &\le \sqrt{T(p_1,p_1)}\sqrt{T(f',f')} \\ &= \frac{1}{2}\|p_1\|_2\sigma(f';-1,1) \\ &= \frac{1}{2}\sqrt{\Delta_2(h)}\sigma(f';-1,1), \end{aligned} \tag{3.31}$$

as

$$\|p_1\|_2^2 = -56h^3 + 154h^2 - 146h + \frac{146}{3} - 28(1-h)^2\sqrt{4h^2 - 6h + 3}. \tag{3.32}$$

Using (3.29), (3.30), (3.31), and (3.32), inequality (3.27) is proved. □

*Remark 3.6.* $\sqrt{\Delta_2(h)}$ attains its minimum value 0.2799 at $h = 0.2957$.

**Theorem 3.7.** *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function, such that $f' \in L_2(a,b)$. Then,*

$$|J(f;a,b;h)| \le \frac{1}{2\sqrt{2}}\sqrt{\Delta_2(h)}\sigma(f';a,b)(b-a)^{3/2}, \tag{3.33}$$

*where $\sigma(f';a,b)$ is defined by* (3.7) *and $\Delta_2(h)$ is as defined above.*

## 4. Applications in numerical integration

Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a subdivision of the interval $[a,b]$, such that $h_i = x_{i+1} - x_i = h = (b-a)/n$. From (3.11), we have

$$\begin{aligned} J(f) &= J(f;x_i,x_{i+1};\delta) \\ &= \int_{x_i}^{x_{i+1}} f(t)dt - \frac{h}{2}[\delta f(x_i) + (1-\delta)f(x_{1i}) + (1-\delta)f(x_{2i}) + \delta f(x_{i+1})], \end{aligned} \tag{4.1}$$

where

$$\begin{gathered} x_{1i} = \frac{h}{2}x^* + \frac{x_i + x_{i+1}}{2}, \qquad x_2 = -\frac{h}{2}x^* + \frac{x_i + x_{i+1}}{2}, \\ x^* = -4 + 4\delta + 2\sqrt{3 - 6\delta + 4\delta^2}, \quad \delta \in \left[0, \frac{1}{2}\right]. \end{gathered} \tag{4.2}$$

Summing-up the above relation from 1 to $n-1$, we get

$$\sum_{i=0}^{n-1} J(f;x_i,x_{i+1};\delta) = \int_a^b f(t)dt - \frac{h}{2}\sum_{i=0}^{n-1}[\delta f(x_i) + (1-\delta)f(x_{1i}) + (1-\delta)f(x_{2i}) + \delta f(x_{i+1})]. \tag{4.3}$$

Let us denote

$$S(f;a,b;\delta) = \sum_{i=0}^{n-1} J(f;x_i,x_{i+1};\delta). \tag{4.4}$$

**Theorem 4.1.** *Let the assumptions of Theorem 2.9 hold, then*

$$|S(f;a,b;\delta)| \le \frac{1}{4n^2}\Delta(\delta)\|f''\|_\infty(b-a)^3, \tag{4.5}$$

*where $S(f;a,b;\delta)$ is defined by* (4.4)*, $\delta \in [0,1/2]$, and $\Delta(\delta)$ is defined by* (2.20)*. However, $\pi$ is the uniform subdivision of $[a,b]$.*

**Theorem 4.2.** *Let the assumptions of Theorem 3.4 hold, then it follows that*

$$|S(f;a,b;\delta)| \le \frac{1}{8}\Delta_1(\delta)\frac{\Gamma_1-\gamma_1}{n}(b-a)^2,$$
$$|S(f;a,b;\delta)| \le \frac{1}{2n}\Delta_0(\delta)(S-\gamma_1)(b-a)^2, \tag{4.6}$$

*and if there exists a real number* $\Gamma_1$, *such that* $f'(t) \le \Gamma_1,\ t \in [a,b]$, *then*

$$|S(f;a,b;\delta)| \le \frac{1}{2n}\Delta_0(\delta)(\Gamma_1-S)(b-a)^2, \tag{4.7}$$

*where* $S(f;a,b;\delta)$ *is defined by* (4.4), $\Delta_0(\delta)$, $\Delta_1(\delta)$ *are defined by* (3.17) *and* $S = (f(a)-f(b))/(b-a)$. *However,* $\pi$ *is the uniform subdivision of* $[a,b]$.

**Theorem 4.3.** *Let the assumptions of Theorem 3.7 hold, then it follows that*

$$|S(f;a,b;\delta)| \le \frac{(b-a)^{3/2}}{2\sqrt{2}n}\sqrt{\Delta_2(\delta)}\sigma(f'), \tag{4.8}$$

*where* $S(f;a,b;\delta)$ *is defined by* (4.4), $\sigma(f')$ *is defined by* (3.7), *and* $\Delta_2(\delta)$ *is as defined by* (3.28). *However,* $\pi$ *is the uniform subdivision of* $[a,b]$.

*Proof.* Applying Theorem 3.7 on the interval $[x_i, x_{i+1}]$,

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(t)dt - \frac{h}{2}[\delta f(x_i) + (1-\delta)f(x_{1i}) + (1-\delta)f(x_{2i}) + \delta f(x_{i+1})]\right| \\ &\quad\le \frac{1}{2\sqrt{2}}\sqrt{\Delta_2(\delta)}h^{3/2}\left[\int_{x_i}^{x_{i+1}}(f'(t))^2dt - \frac{1}{h}(f(x_{i+1})-f(x_i))^2\right]^{1/2}.\end{aligned} \tag{4.9}$$

Summing over $i$ from 0 to $n-1$,

$$|S(f;a,b;\delta)| \le \frac{1}{2\sqrt{2}}\sqrt{\Delta_2(\delta)}h^{3/2}\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}(f'(t))^2dt - \frac{1}{h}(f(x_{i+1})-f(x_i))^2\right]^{1/2}. \tag{4.10}$$

Using Cauchy-Schwartz inequality and the relation $h = (b-a)/n$, we obtain the required inequality:

$$\begin{aligned}|S(f;a,b;\delta)| &\le \frac{1}{2\sqrt{2}}\sqrt{\Delta_2(\delta)}\frac{(b-a)^{3/2}}{n^{3/2}}n^{1/2}\left[\|f'\|_2^2 - \frac{n}{b-a}\sum_{i=0}^{n-1}(f(x_{i+1})-f(x_i))^2\right]^{1/2} \\ &\le \frac{1}{2\sqrt{2}}\sqrt{\Delta_2(\delta)}\frac{(b-a)^{3/2}}{n}\left[\|f'\|_2^2 - \frac{(f(b)-f(a))^2}{b-a}\right]^{1/2}.\end{aligned} \tag{4.11}$$

□

**Acknowledgment**

The authors are thankful to the referees for giving valuable comments and suggestions for the preparation of final version of this paper.

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