

Research Article

On Pečarić-Rajić-Dragomir-Type Inequalities in Normed Linear Spaces

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We establish some generalizations of the recent Pečarić-Rajić-Dragomir-type inequalities by providing upper and lower bounds for the norm of a linear combination of elements in a normed linear space. Our results provide new estimates on inequalities of this type.

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1. Introduction

In the recent paper [1], Pečarić and Rajić proved the following inequality for n nonzero vectors x_k , $k \in \{1, \dots, n\}$ in the real or complex normed linear space $(X, \|\cdot\|)$:

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \|x_j\| - \|x_k\| \right] \right\} \\ \leq \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \|x_j\| - \|x_k\| \right] \right\} \end{aligned} \quad (1.1)$$

and showed that this inequality implies the following refinement of the generalised triangle

inequality obtained by Kato et al. in [2]:

$$\min_{k \in \{1, \dots, n\}} \{\|x_k\|\} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right] \leq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \leq \max_{k \in \{1, \dots, n\}} \{\|x_k\|\} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right]. \quad (1.2)$$

The inequality (1.2) can also be obtained as a particular case of Dragomir's result established in [3]:

$$\begin{aligned} \max_{1 \leq j \leq n} \{\|x_j\|\} \left[\sum_{j=1}^n \|x_j\|^{p-1} - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|^p \right] &\geq \sum_{j=1}^n \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^n x_j \right\|^p \\ &\geq \min_{1 \leq j \leq n} \{\|x_j\|\} \left[\sum_{j=1}^n \|x_j\|^{p-1} - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|^p \right], \end{aligned} \quad (1.3)$$

where $p \geq 1$ and $n \geq 2$.

Notice that, in [3], a more general inequality for convex functions has been obtained as well.

Recently, the following inequality which is more general than (1.1) was given by Dragomir [4]:

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\| \right\} \\ \leq \left\| \sum_{j=1}^n \alpha_j x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\| \right\}. \end{aligned} \quad (1.4)$$

The main aim of this paper is to establish further generalizations of these Pečarić-Rajić- Dragomir-type inequalities (1.1), (1.2), (1.3), and (1.4) by providing upper and lower bounds for the norm of a linear combination of elements in the normed linear space. Our results provide new estimates on such type of inequalities.

2. Main Results

Theorem 2.1. Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . If $\alpha_{i_1, \dots, i_n} \in \mathbb{K}$ and $x_{i_1, \dots, i_n} \in X$ for $i_1, \dots, i_n \in \{1, \dots, n\}$ with $n \geq 2$, then

$$\begin{aligned} & \max_{\substack{k_j \in \{1, \dots, n\} \\ j=1, \dots, n}} \left\{ \left| \alpha_{k_1, \dots, k_n} \right| \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} \right\| - \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \left| \alpha_{i_1, \dots, i_n} - \alpha_{k_1, \dots, k_n} \right| \|x_{i_1, \dots, i_n}\| \right\} \\ & \leq \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \alpha_{i_1, \dots, i_n} x_{i_1, \dots, i_n} \right\| \\ & \leq \min_{\substack{k_j \in \{1, \dots, n\} \\ j=1, \dots, n}} \left\{ \left| \alpha_{k_1, \dots, k_n} \right| \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} \right\| + \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \left| \alpha_{i_1, \dots, i_n} - \alpha_{k_1, \dots, k_n} \right| \|x_{i_1, \dots, i_n}\| \right\}. \end{aligned} \quad (2.1)$$

Proof. Observe that, for any fixed $k_j \in \{1, \dots, n\}$, $j = 1, \dots, n$, we have

$$\sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \alpha_{i_1, \dots, i_n} x_{i_1, \dots, i_n} = \alpha_{k_1, \dots, k_n} \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} + \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n (\alpha_{i_1, \dots, i_n} - \alpha_{k_1, \dots, k_n}) x_{i_1, \dots, i_n}. \quad (2.2)$$

Taking the norm in (2.2) and utilizing the triangle inequality, we have

$$\begin{aligned} \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \alpha_{i_1, \dots, i_n} x_{i_1, \dots, i_n} \right\| & \leq \left\| \alpha_{k_1, \dots, k_n} \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} \right\| + \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n (\alpha_{i_1, \dots, i_n} - \alpha_{k_1, \dots, k_n}) x_{i_1, \dots, i_n} \right\| \\ & \leq \left| \alpha_{k_1, \dots, k_n} \right| \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} \right\| + \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \left| \alpha_{i_1, \dots, i_n} - \alpha_{k_1, \dots, k_n} \right| \|x_{i_1, \dots, i_n}\|, \end{aligned} \quad (2.3)$$

which, on taking the minimum over $k_j \in \{1, \dots, n\}$, $j = 1, \dots, n$, produces the second inequality in (2.1).

Next, by (2.2) we have obviously

$$\sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \alpha_{i_1, \dots, i_n} x_{i_1, \dots, i_n} = \alpha_{k_1, \dots, k_n} \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} - \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n (\alpha_{k_1, \dots, k_n} - \alpha_{i_1, \dots, i_n}) x_{i_1, \dots, i_n}. \quad (2.4)$$

On utilizing the continuity property of the norm we also have

$$\begin{aligned}
& \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \alpha_{i_1, \dots, i_n} x_{i_1, \dots, i_n} \right\| \\
& \geq \left\| \alpha_{k_1, \dots, k_n} \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} \right\| - \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n (\alpha_{i_1, \dots, i_n} - \alpha_{k_1, \dots, k_n}) x_{i_1, \dots, i_n} \right\| \\
& \geq \left\| \alpha_{k_1, \dots, k_n} \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} \right\| - \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n (\alpha_{i_1, \dots, i_n} - \alpha_{k_1, \dots, k_n}) x_{i_1, \dots, i_n} \right\| \\
& \geq |\alpha_{k_1, \dots, k_n}| \left\| \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n x_{i_1, \dots, i_n} \right\| - \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n |\alpha_{i_1, \dots, i_n} - \alpha_{k_1, \dots, k_n}| \|x_{i_1, \dots, i_n}\|,
\end{aligned} \tag{2.5}$$

which, on taking the maximum over $k_j \in \{1, \dots, n\}$, $j = 1, \dots, n$, produces the first part of (2.1) and the theorem is completely proved. \square

Remark 2.2. (i) In case the multi-indices i_1, \dots, i_n and k_1, \dots, k_n reduce to single indices j and k , respectively, after suitable modifications, (2.1) reduces to inequality (1.4) obtained by Dragomir in [4].

(ii) Furthermore, if $x_j \in X \setminus \{0\}$ for $j \in \{1, \dots, n\}$ and $\alpha_k = 1/\|x_k\|$, $k \in \{1, \dots, n\}$ with $n \geq 2$, the inequality reduces further to inequality (1.1) obtained by Pečarić and Rajić in [1].

(iii) Further to (ii), if $n = 2$, writing $x_1 = x$ and $x_2 = -y$, we have

$$\frac{\|x - y\| - \|\|x\| - \|y\|\|}{\min\{\|x\|, \|y\|\}} \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x - y\| + \|\|x\| - \|y\|\|}{\max\{\|x\|, \|y\|\}}, \tag{2.6}$$

which holds for any nonzero vectors $x, y \in X$.

The first inequality in (2.6) was obtained by Mercer in [5].

The second inequality in (2.6) has been obtained by Maligranda in [6]. It provides a refinement of the *Massera-Schäffer inequality* [7]:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\max\{\|x\|, \|y\|\}}, \tag{2.7}$$

which, in turn, is a refinement of the *Dunkl-Williams inequality* [8]:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x - y\|}{\|x\| + \|y\|}. \tag{2.8}$$

Theorem 2.3. Let $(X, \| \cdot \|)$ be a normed linear space over the real or complex number field \mathbb{K} . If $\alpha_{j_1, \dots, j_n} \in \mathbb{K}$ and $x_{j_1, \dots, j_n} \in X \setminus \{0\}$ for $j_1, \dots, j_n \in \{1, \dots, n\}$ with $n \geq 2$, then

$$\begin{aligned} & \max_{\substack{k_i \in \{1, \dots, n\} \\ i=1, \dots, n}} \left\{ \frac{1}{\|x_{k_1, \dots, k_n}\|} \left[\left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| - \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\| - \|x_{k_1, \dots, k_n}\| \right] \right\} \\ & \leq \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \tag{2.9} \\ & \leq \min_{\substack{k_i \in \{1, \dots, n\} \\ i=1, \dots, n}} \left\{ \frac{1}{\|x_{k_1, \dots, k_n}\|} \left[\left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\| - \|x_{k_1, \dots, k_n}\| \right] \right\}. \end{aligned}$$

This follows immediately from Theorem 2.1 by requiring $x_{j_1, \dots, j_n} \neq 0$ for $j_i = 1, \dots, n$, and letting $\alpha_{k_1, \dots, k_n} = 1/\|x_{k_1, \dots, k_n}\|$ for $k_i = 1, \dots, n; n \geq 2$.

A somewhat surprising consequence of Theorem 2.3 is the following version.

Theorem 2.4. Let $(X, \| \cdot \|)$ be a normed linear space over the real or complex number field \mathbb{K} . If $x_{j_1, \dots, j_n} \in X \setminus \{0\}$ for $j_1, \dots, j_n \in \{1, \dots, n\}$ with $n \geq 2$, then

$$\begin{aligned} & \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + \left(n^n - \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \right) \min_{\substack{j_i=1, \dots, n \\ i=1, \dots, n}} \|x_{j_1, \dots, j_n}\| \\ & \leq \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\| \tag{2.10} \\ & \leq \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + \left(n^n - \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \right) \max_{\substack{j_i=1, \dots, n \\ i=1, \dots, n}} \|x_{j_1, \dots, j_n}\|. \end{aligned}$$

Proof. Letting $\|x_{i_1, \dots, i_n}\| = \max_{j_i=1, \dots, n, i=1, \dots, n} \|x_{j_1, \dots, j_n}\|$ and by using the second inequality in (2.9), we have

$$\begin{aligned} \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| & \leq \frac{1}{\|x_{i_1, \dots, i_n}\|} \left(\left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\| - \|x_{i_1, \dots, i_n}\| \right) \\ & = \frac{1}{\|x_{i_1, \dots, i_n}\|} \left(\left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + n^n \|x_{i_1, \dots, i_n}\| - \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\| \right). \tag{2.11} \end{aligned}$$

Hence

$$\|x_{i_1, \dots, i_n}\| \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \leq \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + n^n \|x_{i_1, \dots, i_n}\| - \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\|. \tag{2.12}$$

Then it follows that

$$\begin{aligned} \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\| &\leq \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + \left(n^n - \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \right) \|x_{i_1, \dots, i_n}\| \\ &= \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + \left(n^n - \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \right) \max_{\substack{j_1=1, \dots, n \\ i=1, \dots, n}} \|x_{j_1, \dots, j_n}\|. \end{aligned} \quad (2.13)$$

On the other hand, letting $\|x_{k_1, \dots, k_n}\| = \min_{j_1=1, \dots, n, i=1, \dots, n} \|x_{j_1, \dots, j_n}\|$ and by using the first inequality in (2.9), we have

$$\begin{aligned} \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| &\geq \frac{1}{\|x_{k_1, \dots, k_n}\|} \left(\left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| - \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n (\|x_{j_1, \dots, j_n}\| - \|x_{k_1, \dots, k_n}\|) \right) \\ &= \frac{1}{\|x_{k_1, \dots, k_n}\|} \left(\left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + n^n \|x_{k_1, \dots, k_n}\| - \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\| \right). \end{aligned} \quad (2.14)$$

Hence

$$\|x_{k_1, \dots, k_n}\| \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \geq \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + n^n \|x_{k_1, \dots, k_n}\| - \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\|, \quad (2.15)$$

from which we get

$$\begin{aligned} \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\| &\geq \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + \left(n^n - \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \right) \|x_{k_1, \dots, k_n}\| \\ &= \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\| + \left(n^n - \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\| \right) \min_{\substack{j_1=1, \dots, n \\ i=1, \dots, n}} \|x_{j_1, \dots, j_n}\|. \end{aligned} \quad (2.16)$$

This completes the proof. \square

Remark 2.5. In case the multi-indices j_1, \dots, j_n and k_1, \dots, k_n reduce to single indices j and k , respectively, after suitable modifications, (2.10) reduces to inequality (1.2) obtained in [2] by Kato et al.

Theorem 2.6. Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . If $x_{j_1, \dots, j_n} \in X \setminus \{0\}$ for $j_1, \dots, j_n \in \{1, \dots, n\}$ with $n \geq 2$ and $p \geq 1$, then

$$\begin{aligned} & \min_{\substack{1 \leq j_i \leq n \\ i=1, \dots, n}} \{ \|x_{j_1, \dots, j_n}\| \} \left[\sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\|^{p-1} - \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\|^p \right] \\ & \leq \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\|^p - n^{n(1-p)} \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n x_{j_1, \dots, j_n} \right\|^p \\ & \leq \max_{\substack{1 \leq j_i \leq n \\ i=1, \dots, n}} \{ \|x_{j_1, \dots, j_n}\| \} \left[\sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \|x_{j_1, \dots, j_n}\|^{p-1} - \left\| \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{x_{j_1, \dots, j_n}}{\|x_{j_1, \dots, j_n}\|} \right\|^p \right]. \end{aligned} \quad (2.17)$$

This follows much in the line as the proofs of Theorem 2.1 and Theorem 2.4, and so it is omitted here.

Remark 2.7. In case the multi-index j_1, \dots, j_n reduces to a single index j , after suitable modifications, (2.17) reduces to inequality (1.3) obtained by Dragomir in [3].

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