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*Research Article*

# An Extension of the Hilbert's Integral Inequality

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It is shown that an extension of the Hilbert's integral inequality can be established by introducing two parameters $m$ ($m \in N$) and $\lambda$ ($\lambda > 0$). The constant factors expressed by the Euler number and $\pi$ as well as by the Bernoulli number and $\pi$, respectively, are proved to be the best possible. Some important and especial results are enumerated. As applications, some equivalent forms are given.



## 1. Introduction and Lemmas

Let $f(x),\ g(x) \in L^2(0,+\infty)$. Define $(\ln(x/y))^0 = 1$, when $x = y$. If $k = 0, 1$, then

$$\iint_0^\infty \frac{(\ln(x/y))^k f(x)g(y)}{x+(-1)^k y}\,dx\,dy \le \pi^{k+1}\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \quad (1.1)$$

where the constant factor $\pi^{k+1}$ is the best possible. This is the famous Hilbert's integral inequality (see [1, 2]). Owing to the importance of the Hilbert's inequality and the Hilbert-type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) appear in a great deal of papers (see [3–11], etc.). Specially, Gao and Hsu enumerated the research articles more than 40 in the paper [6]. The purpose of the present paper is to establish the Hilbert-type inequality of the form

$$\iint_0^\infty \frac{(\ln(x/y))^n f(x)g(y)}{x^\lambda+(-1)^n y^\lambda}\,dx\,dy \le C\left\{\int_0^\infty \widetilde{\omega}(x)f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty \widetilde{\omega}(x)g^2(x)dx\right\}^{1/2}, \quad (1.2)$$

where $n$ is a nonnegative integer and $\lambda$ is a positive number. We will give the constant factor $C$ and the expression of the weigh function $\tilde{\omega}(x)$, prove the constant factor $C$ to be the best possible, and then give some especial results and discuss some equivalent forms of them. Evidently inequality (1.2) is an extension of (1.1). The new inequality established is significant in theory and applications. We will discover that the constant factor $C$ in (1.2) is very interesting. It can be expressed by $\pi$ and the Bernoulli number, when $n$ is an odd number, and it can be expressed by $\pi$ and the Euler number, when $n$ is an even number, and that $\pi$ seems to play a bridge role between two cases.

In order to prove our main results, we need the following lemmas.

**Lemma 1.1.** *Let $a$ be a positive number and $b > -1$. Then*

$$\int_0^{\infty} x^b e^{-ax} dx = \frac{\Gamma(b+1)}{a^{b+1}}. \tag{1.3}$$

*Proof.* According to the definition of $\Gamma$-function, (1.3) easily follows. This result can be also found in the paper [12, page 226, formula 1053]. □

**Lemma 1.2.** *Let $m$ be a positive integer. Then*

$$S = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2m}} = \frac{(2^{2m}-1)\pi^{2m}}{2(2m)!} B_m, \tag{1.4}$$

*where the $B_{m's}$ are the Bernoulli numbers, namely, $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, and so forth.*

*Proof.* It is known from the paper [13, page 231] that

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = \frac{2^{2m-1}\pi^{2m}}{(2m)!} B_m, \tag{1.5}$$

where the $B_{m's}$ are the Bernoulli numbers, namely, $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, and so forth. It is easy to deduce that

$$S_2 = \sum_{k=1}^{\infty} \frac{1}{(2k)^{2m}} = \frac{1}{2^{2m}} S_1. \tag{1.6}$$

Notice that $S = S_1 - S_2$. Equality (1.4) follows. □

**Lemma 1.3.** *Let $a$ be a positive number.*

(i) *If $m$ is a positive integer, then*

$$\int_0^{\infty} \frac{x^{2m-1}}{\sinh ax} dx = \frac{2^{2m-1}(2^{2m}-1)}{m} \left(\frac{\pi}{2a}\right)^{2m} B_m, \tag{1.7}$$

*where the $B_{m's}$ are the Bernoulli numbers.*

(ii) *If $m$ is a nonnegative integer, then*

$$\int_0^\infty \frac{x^{2m}}{\cosh ax}dx = \left(\frac{\pi}{2a}\right)^{2m+1} E_m, \tag{1.8}$$

*where the $E_{m's}$ are the Euler numbers, namely,* $E_0 = 1,\ E_1 = 1,\ E_2 = 5,\ E_3 = 61,\ E_4 = 1385$, *and so forth.*

*Proof.* We prove firstly equality (1.7). Expanding the hyperbolic cosecant function $1/\sinh ax$, and then using Lemma 1.1 and noticing that $\Gamma(2m) = (2m-1)!$, we have

$$\begin{aligned}\int_0^\infty \frac{x^{2m-1}}{\sinh ax}dx &= 2\int_0^\infty \frac{x^{2m-1}e^{-ax}}{1-e^{-2ax}}dx = 2\int_0^\infty x^{2m-1}e^{-ax}\sum_{k=0}^\infty e^{-2kax}dx \\ &= 2\sum_{k=1}^\infty \int_0^\infty x^{2m-1}e^{-(2k-1)ax}dx = \frac{2(2m-1)!}{a^{2m}}\sum_{k=1}^\infty \frac{1}{(2k-1)^{2m}}.\end{aligned} \tag{1.9}$$

By Lemma 1.2, we obtain (1.7) at once. □

Next we consider (1.8). Similarly by expanding the hyperbolic secant function $1/\cosh ax$ and then using Lemma 1.1, we have

$$\begin{aligned}\int_0^\infty \frac{x^{2m}}{\cosh ax}dx &= 2\int_0^\infty \frac{x^{2m}e^{-ax}}{1+e^{-2ax}}dx = 2\int_0^\infty x^{2m}e^{-ax}\sum_{k=1}^\infty (-1)^{k+1}e^{-2(k-1)ax}dx \\ &= 2\sum_{k=1}^\infty (-1)^{k+1}\int_0^\infty x^{2m}e^{-(2k-1)ax}dx = \frac{2(2m)!}{a^{2m+1}}\sum_{k=1}^\infty \frac{(-1)^{k+1}}{(2k-1)^{2m+1}}.\end{aligned} \tag{1.10}$$

It is known from the paper [13, pp. 231] that

$$\sum_{k=1}^\infty \frac{(-1)^{k+1}}{(2k-1)^{2m+1}} = \frac{\pi^{2m+1}}{2^{2m+2}(2m)!}E_m, \tag{1.11}$$

where the $E_{m's}$ are Euler numbers, namely, $E_1 = 1,\ E_2 = 5,\ E_3 = 61,\ E_4 = 1385$, and so forth. In particular, when $m = 0$, we have $\sum_{k=1}^\infty((-1)^{k+1}/(2k-1)) = \pi/4$, hence we can define $E_0 = 1$. It follows from (1.10) and (1.11) that the equality (1.8) is true.

By the way, there is an error in the paper [12, page 260, formula 1566], namely, the integral in the paper [12] $\int_0^\infty(x^m/\sinh ax)dx = ((2^{m+1}-1)m!/2^m a^{m+1})\sum_{k=1}^\infty((-1)^{k+1}/(2k-1)^{m+1})$ is wrong. It should be $\int_0^\infty(x^m/\sinh ax)dx = ((2^{m+1}-1)m!/2^m a^{m+1})\sum_{k=1}^\infty 1/k^{2m}$.

By applying this correct result, it is easy to verify the formulas 1562–1565 in the paper [12, pp. 259]. These are omitted here.

## 2. Main Results

In this section, we will prove our assertions by using the above lemmas.

**Theorem 2.1.** *Let* $f$ *and* $g$ *be two real functions, and let* $m$ *be a positive integer,* $\lambda > 0$*. If* $0 < \int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty x^{1-\lambda} g^2(x)dx < +\infty$*, then*

$$\iint_0^\infty \frac{(\ln(x/y))^{2m-1} f(x)g(y)}{x^\lambda - y^\lambda} dx\,dy < C_B \left\{\int_0^\infty x^{1-\lambda} f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty x^{1-\lambda} g^2(x)dx\right\}^{1/2}, \quad (2.1)$$

*where the constant factor* $C_B$ *is defined by*

$$C_B = \frac{2^{2m-1}(2^{2m}-1)}{m}\left(\frac{\pi}{\lambda}\right)^{2m} B_m, \quad (2.2)$$

*andthe* $B_{m's}$ *are the Bernoulli numbers, namely,* $B_1 = 1/6, B_2 = 1/30,\ B_3 = 1/42,\ B_4 = 1/30,\ B_5 = 5/66$*, and so forth. And the constant factor* $C_B$ *in* (2.1) *is the best possible.*

*Proof.* We may apply the Cauchy inequality to estimate the left-hand side of (2.1) as follows:

$$\begin{aligned}
&\iint_0^\infty \frac{(\ln(x/y))^{2m-1} f(x)g(y)}{x^\lambda - y^\lambda} dx\,dy \\
&= \iint_0^\infty \left(\frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda}\right)^{1/2} \left(\frac{x}{y}\right)^{(2-\lambda)/4} f(x) \left(\frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda}\right)^{1/2} \left(\frac{y}{x}\right)^{(2-\lambda)/4} g(y)dxdy \\
&\le \left(\iint_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda}\left(\frac{x}{y}\right)^{(2-\lambda)/2} f^2(x)dx\,dy\right)^{1/2} \\
&\quad \times \left(\iint_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda}\left(\frac{y}{x}\right)^{(2-\lambda)/2} g^2(y)dx\,dy\right)^{1/2} \\
&= \left(\int_0^\infty \omega(x) f^2(x)dx\right)^{1/2} \left(\int_0^\infty \omega(x) g^2(x)dx\right)^{1/2},
\end{aligned} \quad (2.3)$$

where $\omega(x) = \int_0^\infty ((\ln(x/y))^{2m-1}/(x^\lambda - y^\lambda))(x/y)^{1-(\lambda/2)}dy$.

By using Lemma 1.3, it is easy to deduce that

$$\begin{aligned}
\omega(x) &= \int_0^{\infty} \frac{(\ln(x/y))^{2m-1}}{x^{\lambda}\left(1-(y/x)^{\lambda}\right)}\left(\frac{x}{y}\right)^{1-(\lambda/2)} dy = -x^{1-\lambda}\int_0^{\infty} u^{(\lambda/2)-1}(\ln u)^{2m-1}\frac{1}{1-u^{\lambda}}du \\
&= -x^{1-\lambda}\int_{-\infty}^{\infty}\frac{t^{2m-1}e^{(\lambda/2)t}}{1-e^{\lambda t}}dt = x^{1-\lambda}\int_{-\infty}^{\infty}\frac{t^{2m-1}}{e^{(\lambda/2)t}-e^{-(\lambda/2)t}}dt = x^{1-\lambda}\int_0^{\infty}\frac{t^{2m-1}}{\sinh((\lambda/2)t)}dt \\
&= x^{1-\lambda}\frac{(2^{2m}-1)\pi^{2m}}{2m(\lambda/2)^{2m}}B_m = C_B x^{1-\lambda},
\end{aligned} \tag{2.4}$$

where the constant factor $C_B$ is defined by (2.2).

It follows from (2.3) and (2.4) that

$$\iint_0^{\infty}\frac{(\ln(x/y))^{2m-1}f(x)g(y)}{x^{\lambda}-y^{\lambda}}dx\,dy \le C_B\left\{\int_0^{\infty}x^{1-\lambda}f^2(x)dx\right\}^{1/2}\left\{\int_0^{\infty}x^{1-\lambda}g^2(x)dx\right\}^{1/2}. \tag{2.5}$$

If (2.5) takes the form of the equality, then there exists a pair of non-zero constants $c_1$ and $c_2$ such that

$$c_1\frac{(\ln(x/y))^{2m-1}}{x^{\lambda}-y^{\lambda}}f^2(x)\left(\frac{x}{y}\right)^{1-(\lambda/2)} = c_2\frac{(\ln(x/y))^{2m-1}}{x^{\lambda}-y^{\lambda}}g^2(y)\left(\frac{y}{x}\right)^{1-(\lambda/2)}, \quad \text{a.e. on } (0,+\infty)\times(0,+\infty). \tag{2.6}$$

Then we have

$$c_1x^{2-\lambda}f^2(x) = c_2y^{2-\lambda}g^2(y) = \widetilde{C}, \quad (\text{constant}) \quad \text{a.e. on } (0,+\infty)\times(0,+\infty) \tag{2.7}$$

Without losing the generality, we suppose that $c_1 \neq 0$, then

$$\int_0^{\infty}x^{1-\lambda}f^2(x)dx = \frac{\widetilde{C}}{c_1}\int_0^{\infty}x^{-1}dx. \tag{2.8}$$

This contradicts that $0 < \int_0^{\infty}x^{1-\lambda}f^2(x)dx < +\infty$. Hence it is impossible to take the equality in (2.5). So the inequality (2.1) is valid.

It remains only to show that $C_B$ in (2.1) is the best possible, for all $0 < \varepsilon < \lambda$. Define two functions by

$$\widetilde{f}(x) = \begin{cases} 0, & x \in (0,1), \\ x^{-(2-\lambda+\varepsilon)/2}, & x \in [1,\infty), \end{cases} \qquad \widetilde{g}(y) = \begin{cases} 0, & y \in (0,1), \\ y^{-(2-\lambda+\varepsilon)/2}, & y \in [1,\infty), \end{cases} \tag{2.9}$$

It is easy to deduce that

$$\int_0^{+\infty} x^{1-\lambda}\widetilde{f}^2(x)dx = \int_0^{+\infty} y^{1-\lambda}\widetilde{g}^2(y)dy = \frac{1}{\varepsilon}. \tag{2.10}$$

If $C_B$ in (2.1) is not the best possible, then there exists $K > 0$, such that

$$\begin{aligned} H(\lambda,m) &= \iint_0^\infty \frac{(\ln(x/y))^{2m-1}\widetilde{f}(x)\widetilde{g}(y)}{x^\lambda - y^\lambda}dx\,dy = \iint_1^\infty \frac{(\ln(x/y))^{2m-1}\widetilde{f}(x)\widetilde{g}(y)}{x^\lambda - y^\lambda}dx\,dy \\ &\le K\left(\int_1^\infty x^{1-\lambda}\widetilde{f}^2(x)dx\right)^{1/2}\left(\int_1^\infty y^{1-\lambda}\widetilde{g}^2(y)dy\right)^{1/2} = \frac{K}{\varepsilon} < \frac{C}{\varepsilon}. \end{aligned} \tag{2.11}$$

On the other hand, we have

$$\begin{aligned} H(\lambda,m) &= \iint_0^\infty \frac{(\ln(x/y))^{2m-1}\widetilde{f}(x)\widetilde{g}(y)}{x^\lambda - y^\lambda}dx\,dy \\ &= \iint_1^\infty \frac{\left\{x^{-(2-\lambda+\varepsilon)/2}\right\}\left\{(\ln(x/y))^{2m-1}y^{-(2-\lambda+\varepsilon)/2}\right\}}{x^\lambda - y^\lambda}dx\,dy \\ &= \int_1^\infty\left\{\int_1^\infty \frac{(\ln(x/y))^{2m-1}y^{-(2-\lambda+\varepsilon)/2}}{x^\lambda\left(1-(y/x)^\lambda\right)}dy\right\}\left\{x^{-(2-\lambda+\varepsilon)/2}\right\}dx \\ &= \int_1^\infty\left\{\int_{1/x}^\infty \frac{(\ln(1/u))^{2m-1}u^{-(2-\lambda+\varepsilon)/2}}{1-u^\lambda}du\right\}\left\{x^{-1-\varepsilon}\right\}dx \\ &= \int_1^\infty\left\{\int_{1/x}^1 \frac{(\ln(1/u))^{2m-1}u^{-(2-\lambda+\varepsilon)/2}}{1-u^\lambda}du\right\}\left\{x^{-1-\varepsilon}\right\}dx \\ &\quad+ \int_1^\infty\left\{\int_1^\infty \frac{(\ln(1/u))^{2m-1}u^{-(2-\lambda+\varepsilon)/2}}{1-u^\lambda}du\right\}\left\{x^{-1-\varepsilon}\right\}dx \\ &= \int_0^1\left\{\int_{1/u}^\infty x^{-1-\varepsilon}dx\right\}\frac{(\ln(1/u))^{2m-1}u^{-(2-\lambda+\varepsilon)/2}}{1-u^\lambda}du \\ &\quad+ \int_1^\infty\left\{\int_1^\infty \frac{(\ln(1/u))^{2m-1}u^{-(2-\lambda+\varepsilon)/2}}{1-u^\lambda}du\right\}\left\{x^{-1-\varepsilon}\right\}dx \\ &= \frac{1}{\varepsilon}\int_0^1 \frac{(\ln(1/u))^{2m-1}u^{-(2-\lambda-\varepsilon)/2}}{1-u^\lambda}du + \frac{1}{\varepsilon}\int_1^\infty \frac{(\ln(1/u))^{2m-1}u^{-(2-\lambda+\varepsilon)/2}}{1-u^\lambda}du. \end{aligned} \tag{2.12}$$

When $\varepsilon$ is sufficiently small, we obtain from (2.12) that

$$\begin{aligned}
H(\lambda,m) &= \frac{1}{\varepsilon}\left(\int_0^1 \frac{(\ln 1/u)^{2m-1}u^{-(2-\lambda)/2}}{1-u^\lambda}du + \circ_1(1)\right) + \frac{1}{\varepsilon}\left(\int_1^\infty \frac{(\ln 1/u)^{2m-1}u^{-(2-\lambda)/2}}{1-u^\lambda}du + \circ_2(1)\right) \\
&= \frac{1}{\varepsilon}\left(\int_0^\infty \frac{(\ln 1/u)^{2m-1}u^{-(2-\lambda)/2}}{1-u^\lambda}du + \circ(1)\right) \\
&= \frac{1}{\varepsilon}\left(-\int_0^\infty \frac{(\ln u)^{2m-1}u^{(\lambda/2)-1}}{1-u^\lambda}du + \circ(1)\right) \quad (\varepsilon \longrightarrow 0).
\end{aligned} \tag{2.13}$$

Noticing the proof of (2.4), we have

$$H(\lambda,m) = \frac{1}{\varepsilon}\left(\int_0^\infty \frac{t^{2m-1}}{\sinh((\lambda/2)t)}dt + \circ(1)\right) = \frac{C_B}{\varepsilon} + \circ(1) \quad (\varepsilon \longrightarrow 0). \tag{2.14}$$

Evidently, inequality (2.14) is in contradiction with that in (2.11). Therefore, the constant factor $C_B$ in (2.1) is the best possible. Thus the proof of the theorem is completed. □

Based on Theorem 2.1, we may build some important and interesting inequalities.

In particular, when $\lambda = m = 1$, we have $C_B = \pi^2$, the inequality (2.1) can be reduced to (1.1).

It shows that Theorem 2.1 is an extension of (1.1).

**Corollary 2.2.** *If* $0 < \int_0^\infty f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty g^2(x)dx < +\infty$, *then*

$$\iint_0^\infty \frac{(\ln x/y)^3 f(x)g(y)}{x-y}dx\,dy < 2\pi^4\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{2.15}$$

*where the constant factor* $2\pi^4$ *is the best possible.*

**Corollary 2.3.** *If* $0 < \int_0^\infty x^{-1}f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty x^{-1}g^2(x)dx < +\infty$, *then*

$$\iint_0^\infty \frac{(\ln x/y) f(x)g(y)}{x^2-y^2}dx\,dy < \frac{\pi^2}{4}\left\{\int_0^\infty x^{-1}f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty x^{-1}g^2(x)dx\right\}^{1/2}, \tag{2.16}$$

*where the constant factor* $\pi^2/4$ *is the best possible.*

**Corollary 2.4.** *If* $0 < \int_0^\infty \sqrt{x}f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty \sqrt{x}g^2(x)dx < +\infty$, *then*

$$\iint_0^\infty \frac{(\ln x/y) f(x)g(y)}{\sqrt{x}-\sqrt{y}}dx\,dy < 4\pi^2\left\{\int_0^\infty \sqrt{x}f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty \sqrt{x}g^2(x)dx\right\}^{1/2}, \tag{2.17}$$

*where the constant factor* $4\pi^2$ *is the best possible.*

**Corollary 2.5.** *Let $m$ be a positive integer. If $0 < \int_0^\infty f^2(x)dx < +\infty$ and $0 < \int_0^\infty g^2(x)dx < +\infty$, then*

$$\iint_0^\infty \frac{(\ln x/y)^{2m-1} f(x)g(y)}{x-y}dx\,dy < \widetilde{C}_B\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{2.18}$$

*where the constant factor $\widetilde{C}_B$ is defined by*

$$\widetilde{C}_B = \frac{2^{2m-1}(2^{2m}-1)\pi^{2m}}{m}B_m, \tag{2.19}$$

*and the $B_{m's}$ are the Bernoulli numbers. And the constant factor $\widetilde{C}_B$ in* (2.18) *is the best possible.*

Similarly, we can establish also a great deal of new inequalities. They are omitted here.

**Theorem 2.6.** *Let $f$ and $g$ be two real functions, and let $m$ be a nonnegative integer and $\lambda > 0$. If $0 < \int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$ and $0 < \int_0^\infty x^{1-\lambda} g^2(x)dx < +\infty$, then*

$$\iint_0^\infty \frac{(\ln(x/y))^{2m} f(x)g(y)}{x^\lambda + y^\lambda}dx\,dy < C_E\left\{\int_0^\infty x^{1-\lambda} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty x^{1-\lambda} g^2(x)dx\right\}^{1/2}, \tag{2.20}$$

*where the constant factor $C_E$ is defined by*

$$C_E = \left(\frac{\pi}{\lambda}\right)^{2m+1} E_m, \tag{2.21}$$

*where $E_0 = 1$ and the $E_{m's}$ are the Euler numbers,namely, $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, and so forth. And the constant factor $C_E$ in* (2.20) *is the best possible.*

*Proof.* By applying Cauchy's inequality to estimate the left-hand side of (2.20), we have

$$\iint_0^\infty \frac{(\ln(x/y))^{2m} f(x)g(y)}{x^\lambda + y^\lambda}dx\,dy \le \left(\int_0^\infty \overline{\omega}(x) f^2(x)dx\right)^{1/2}\left(\int_0^\infty \overline{\omega}(x) g^2(x)dx\right)^{1/2}, \tag{2.22}$$

where $\overline{\omega}(x) = \int_0^\infty ((\ln(x/y))^{2m}/(x^\lambda + y^\lambda))(x/y)^{1-(\lambda/2)}dy$.

By proper substitution of variable, and then by Lemma 1.3, it is easy to deduce that

$$\begin{aligned}
\overline{\omega}(x) &= \int_0^\infty \frac{(\ln(x/y))^{2m}}{x^\lambda\left(1+(y/x)^\lambda\right)}\left(\frac{x}{y}\right)^{1-(\lambda/2)}dy = x^{1-\lambda}\int_0^\infty u^{(\lambda/2)-1}(\ln u)^{2m}\frac{1}{1+u^\lambda}du \\
&= x^{1-\lambda}\int_{-\infty}^\infty \frac{t^{2m}e^{(\lambda/2)t}}{1+e^{\lambda t}}dt = x^{1-\lambda}\int_0^\infty \frac{t^{2m}}{\cosh((\lambda/2)t)}dt \\
&= x^{1-\lambda}\left(\frac{\pi}{\lambda}\right)^{2m+1}E_m = C_E x^{1-\lambda},
\end{aligned} \tag{2.23}$$

where the constant factor $C_E$ is defined by (2.21).

It follows from (2.22) and (2.23) that

$$\iint_0^\infty \frac{(\ln(x/y))^{2m} f(x)g(y)}{x^\lambda + y^\lambda} dx\,dy \le C_E \left\{\int_0^\infty x^{1-\lambda} f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty x^{1-\lambda} g^2(x)dx\right\}^{1/2}. \quad (2.24)$$

The proof of the rest is similar to that of Theorem 2.1, it is omitted here. □

In particular, when $\lambda = 1$ and $m = 0$, we have $C_E = \pi$, inequality (2.20) can be reduced to (1.1). It shows that Theorem 2.6 is also an extension of (1.1).

**Corollary 2.7.** *If* $0 < \int_0^\infty f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty g^2(x)dx < +\infty$, *then*

$$\iint_0^\infty \frac{(\ln(x/y))^2 f(x)g(y)}{x + y} dx\,dy < \pi^3 \left\{\int_0^\infty f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \quad (2.25)$$

*where the constant factor* $\pi^3$ *is the best possible.*

**Corollary 2.8.** *If* $0 < \int_0^\infty x^{-1} f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty x^{-1} g^2(x)dx < +\infty$, *then*

$$\iint_0^\infty \frac{(\ln(x/y))^2 f(x)g(y)}{x^2 + y^2} dx\,dy < \left(\frac{\pi}{2}\right)^3 \left\{\int_0^\infty x^{-1} f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty x^{-1} g^2(x)dx\right\}^{1/2}, \quad (2.26)$$

*where the constant factor* $(\pi/2)^3$ *is the best possible.*

**Corollary 2.9.** *If* $0 < \int_0^\infty \sqrt{x} f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty \sqrt{x} g^2(x)dx < +\infty$, *then*

$$\iint_0^\infty \frac{(\ln(x/y))^2 f(x)g(y)}{\sqrt{x} + \sqrt{y}} dx\,dy < (2\pi)^3 \left\{\int_0^\infty \sqrt{x} f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty \sqrt{x} g^2(x)dx\right\}^{1/2}, \quad (2.27)$$

*where the constant factor* $(2\pi)^3$ *is the best possible.*

**Corollary 2.10.** *Let* $m$ *be a nonnegative integer. If* $0 < \int_0^\infty f^2(x)dx < +\infty$ *and* $0 < \int_0^\infty g^2(x)dx < +\infty$, *then*

$$\iint_0^\infty \frac{(\ln(x/y))^{2m} f(x)g(y)}{x + y} dx\,dy < \left(\pi^{2m+1} E_m\right) \left\{\int_0^\infty f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \quad (2.28)$$

*where* $E_0 = 1$ *and the* $E_{m's}$ *are the Euler numbers. And the constant factor* $\pi^{2m+1}E_m$ *in* (2.28) *is the best possible.*

Similarly, we can establish also a great deal of new inequalities. They are omitted here.

## 3. Some Equivalent Forms

As applications, we will build some new inequalities.

**Theorem 3.1.** *Let f be a real function, and let m be a positive integer, let* $\lambda > 0$*.*

*If* $0 < \int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$*, then*

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda} f(x)dx\right\}^2 dy < (C_B)^2 \int_0^\infty x^{1-\lambda} f^2(x)dx, \tag{3.1}$$

*where* $C_B$ *is defined by* (2.2) *and the constant factor* $(C_B)^2$ *in* (3.1) *is the best possible. And the inequality* (3.1) *is equivalent to* (2.1)*.*

*Proof.* First, we assume that inequality (2.1) is valid. Set a real function $g(y)$ as

$$g(y) = y^{\lambda-1}\int_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda} f(x)dx, \quad y \in (0, +\infty). \tag{3.2}$$

By using (2.1), we have

$$\begin{aligned}
&\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda} f(x)dx\right\}^2 dy \\
&= \iint_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda} f(x)g(y)dx\,dy \\
&< C_B\left\{\int_0^\infty x^{1-\lambda} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty y^{1-\lambda} g^2(y)dy\right\}^{1/2} \\
&= C_B\left\{\int_0^\infty x^{1-\lambda} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda} f(x)dx\right)^2 dy\right\}^{1/2}.
\end{aligned} \tag{3.3}$$

It follows from (3.3) that inequality (3.1) is valid after some simplifications.

On the other hand, assume that inequality (3.1) keeps valid, by applying in turn Cauchy's inequality and (3.1), we have

$$\begin{aligned}
&\iint_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda} f(x)g(y)dx\,dy \\
&= \int_0^\infty y^{(\lambda-1)/2}\left\{\int_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda} f(x)dx\right\} y^{(1-\lambda)/2} g(y)dy \\
&\le \left\{\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{(\ln(x/y))^{2m-1}}{x^\lambda - y^\lambda} f(x)dx\right)^2 dy\right\}^{1/2} \left\{\int_0^\infty y^{1-\lambda} g^2(y)dy\right\}^{1/2} \\
&< \left\{(C_B)^2\int_0^\infty x^{1-\lambda} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty y^{1-\lambda} g^2(y)dy\right\}^{1/2} \\
&= C_B\left\{\int_0^\infty x^{1-\lambda} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty y^{1-\lambda} g^2(y)dy\right\}^{1/2}.
\end{aligned} \tag{3.4}$$

Therefore the inequality (3.1) is equivalent to (2.1).

If the constant factor $(C_B)^2$ in (3.1) is not the best possible, then it is known from (3.4) that the constant factor $C_B$ in (2.1) is also not the best possible. This is a contradiction. The theorem is proved. □

**Corollary 3.2.** *Let f be a real function. If* $0 < \int_0^\infty f^2(x)dx < +\infty$, *then*

$$\int_0^\infty \left\{\int_0^\infty \frac{(\ln(x/y))^3}{x-y} f(x)dx\right\}^2 dy < 4\pi^8 \int_0^\infty f^2(x)dx, \tag{3.5}$$

*where the constant factor* $4\pi^8$ *is the best possible. And the inequality* (3.5) *is equivalent to* (2.15).

Its proof is similar to the one of Theorem 3.1. Hence it is omitted.

Similarly, we can establish also some new inequalities which are, respectively, equivalent to inequalities (2.16), (2.17), and (2.18). They are omitted here.

**Theorem 3.3.** *Let f be a real function,and let m be a nonnegative integer,* $\lambda > 0$.
*If* $0 < \int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$, *then*

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{(\ln(x/y))^{2m}}{x^\lambda + y^\lambda} f(x)dx\right\}^2 dy < (C_E)^2 \int_0^\infty x^{1-\lambda} f^2(x)dx, \tag{3.6}$$

*where* $C_E$ *is defined by* (2.19) *and the constant factor* $(C_E)^2$ *in* (3.6) *is the best possible. Inequality* (3.6) *is equivalent to* (2.20).

**Corollary 3.4.** *If* $0 < \int_0^\infty f^2(x)dx < +\infty$, *then*

$$\int_0^\infty \left\{ \int_0^\infty \frac{(\ln(x/y))^2}{x+y} f(x)dx \right\}^2 dy < \pi^6 \int_0^\infty f^2(x)dx, \tag{3.7}$$

*where the constant factor* $\pi^6$ *in* (3.7) *is the best possible. And inequality* (3.7) *is equivalent to* (2.25).

Similarly, we can establish also some new inequalities which are, respectively, equivalent to inequalities (2.26), (2.27), and (2.28). These are omitted here.

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