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Research Article

Mixed Variational-Like Inequality for Fuzzy Mappings in Reflexive Banach Spaces

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Some existence theorems for the mixed variational-like inequality for fuzzy mappings (FMVLIP) in a reflexive Banach space are established. Further, the auxiliary principle technique is used to suggest a novel and innovative iterative algorithm for computing the approximate solution. Consequently, not only the existence of solutions of the FMVLIP is shown, but also the convergence of iterative sequences generated by the algorithm is also proven. The results proved in this paper represent an improvement of previously known results.

1. Introduction

The concept of fuzzy set theory was introduced by Zadeh [1]. The applications of the fuzzy set theory can be found in many branches of mathematical and engineering sciences including artificial intelligence, control engineering, management sciences, computer science, and operations research [2]. On the other hand, the concept of variational inequality was introduced by Hartman and Stampacchia [3] in early 1960s. These have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, economics and transportation equilibrium, and so forth. The generalized mixed variational-like inequalities, which are generalized forms of variational inequalities, have potential and significant applications in optimization theory [4, 5], structural analysis [6], and economics [4, 7]. Motivated and inspired by the recent research work going on these two different fields, Chang [8], Chang and Huang [9], Chang and Zhu [10] and Noor [11] introduced and studied the concept of variational inequalities and complementarity problems for fuzzy mappings in different contexts.

It is noted that there are many effective numerical methods for finding approximate solutions of various variational inequalities (e.g., the projection method and its variant forms, linear approximation, descent and Newton's methods), and there are very few methods for general variational-like inequalities. For example, among the most effective numerical technique is the projection method and its variant forms; however, the projection type techniques cannot be extended for constructing iterative algorithms for mixed variational-like inequalities, since it is not possible to find the projection of the solution. Thus, the development of an efficient and implementable technique for solving variational-like inequalities is one of the most interesting and important problems in variational inequality theory. These facts motivated Glowinski et al. [12] to suggest another technique, which does not depend on the projection. The technique is called the auxiliary principle technique.

Recently, the auxiliary principle technique was extended by Huang and Deng [13] to study the existence and iterative approximation of solutions of the set-valued strongly nonlinear mixed variational-like inequality, under the assumptions that the operators are bounded closed values. On the other hand, by using the concept of α-strongly mixed monotone of a fuzzy mapping on a bounded closed convex set, the auxiliary principle technique was extended by Chang et al. [14] to study the existence and iterative approximation of solutions of the mixed variational-like inequality problem for fuzzy mappings in a Hilbert space.

In this paper, the mixed variational-like inequality problem for fuzzy mapping (FMVLIP) in a reflexive Banach space is studied, and some existence theorems for the problem are proved. We also prove the existence theorem for auxiliary problem of FMVLIP. Further, by exploiting the theorem, we construct and analyze an iterative algorithm for finding the solution of the FMVLIP. Finally, we discuss the convergence analysis of iterative sequence generated by the iterative algorithm.

2. Preliminaries

Throughout this paper, we assume that E is a real Banach space with its topological dual E^*, K a nonempty convex subset of E, $\langle\cdot,\cdot\rangle$ is the generalized duality pairing between E and E^*, $CB(E^*)$ is the family of all nonempty bounded and closed subsets of E^*, and $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E^*)$ defined by

$$H(C,D)=\max\left\{\sup_{x\in C}d(x,D),\sup_{y\in D}d(C,y)\right\},\quad \forall C,D\in CB(E^*). \tag{2.1}$$

In the sequel we denote the collection of all fuzzy sets on E^* by $\mathcal{F}(E^*)=\{f:E^*\to[0,1]\}$. A mapping T from K to $\mathcal{F}(E^*)$ is called a fuzzy mapping. If $T:K\to\mathcal{F}(E^*)$ is a fuzzy mapping, then the set $T(u)$, for $u\in K$, is a fuzzy set in $\mathcal{F}(E^*)$ (in the sequel we denote $T(u)$ by T_u) and $T_u(y)$, for each $y\in E^*$, is the degree of membership of y in T_u.

A fuzzy mapping $T:K\to\mathcal{F}(E^*)$ is said to be closed, if, for each $u\in K$, the function $y\mapsto T_u(y)$ is upper semicontinuous; that is, for any given net $y_\alpha\subset E^*$ satisfying $y_\alpha\to y_0\in E^*$, we have $\limsup_\alpha T_u(y_\alpha)\le T_u(y_0)$.

For $f\in\mathcal{F}(E^*)$ and $\lambda\in[0,1]$, the set

$$(f)_\lambda=\{y\in E^*:f(y)\ge\lambda\} \tag{2.2}$$

is called a λ-cut set of f.

A closed fuzzy mapping $T : K \to \mathcal{F}(E^*)$ is said to satisfy condition $(*)$, if there exists a function $b : E \to [0,1]$ such that for each $u \in K$ the set

$$(T_u)_{b(u)} = \{y \in E^* : T_u(y) \geq b(u)\} \tag{2.3}$$

is a nonempty bounded subset of E^*.

Remark 2.1. It is worth mentioning that if T is a closed fuzzy mapping satisfying condition $(*)$, then for each $u \in E$, the set $(T_u)_{b(u)} \in CB(E^*)$. Indeed, let $\{y_\alpha\}_{\alpha\in\Gamma} \subset (T_u)_{b(u)}$ be a net and $y_\alpha \to y_0 \in E^*$, then $(T_u)(y_\alpha) \geq b(u)$ for each $\alpha \in \Gamma$. Since the fuzzy mapping T is closed, we have

$$T_u(y_0) \geq \limsup_{\alpha\in\Gamma} T_u(y_\alpha) \geq b(u). \tag{2.4}$$

This implies that $y_0 \in (T_u)_{b(u)}$, and so $(T_u)_{b(u)} \in CB(E^*)$.

Let E be a real reflexive Banach space with the dual space E^*. In this paper, we devote our study to a class of mixed variational-like inequality problem for fuzzy mappings, which is stated as follows.

Let $T, A : K \to \mathcal{F}(E^*)$ are two closed fuzzy mappings satisfying the condition $(*)$ with functions $b, c : E \to [0,1]$, respectively. $N : E^* \times E^* \to E^*$ and $\eta : K \times K \to E$ are two single-valued mappings. Let $\varphi : E \times E \to (-\infty, +\infty]$ be a real bifunction. We shall study the following problem :

$$\text{FMVLIP}(T, A, N, \eta, \varphi) \begin{cases} \text{find } u \in K, x, y \in E^* \text{ such that } x \in (T_u)_{b(u)},\ y \in (A_u)_{c(u)}, \\ \langle N(x,y), \eta(v,u)\rangle + \varphi(u,v) - \varphi(u,u) \geq 0,\ \forall v \in K. \end{cases} \tag{2.5}$$

The problem (2.5) is called a fuzzy mixed variational-like inequality problem, and we will denote by FMVLIP(T, A, N, η, φ) the solution set of the problem (2.5).

Now, let us consider some special cases of problem (2.5).

(1) Let $T, A : K \to CB(E^*)$ be two ordinary set-valued mappings, and let N, η, φ be the mappings as in problem (2.5). Define two fuzzy mappings $\widetilde{T}(\cdot), \widetilde{V}(\cdot) : K \to \mathcal{F}(E^*)$ as follows:

$$\widetilde{T}_u = \chi_{T(u)}, \qquad \widetilde{A}_u = \chi_{A(u)}, \tag{2.6}$$

where $\chi_{T(u)}$ and $\chi_{A(u)}$ are the characteristic functions of the sets $T(u)$ and $A(u)$, respectively. It is easy to see that $\widetilde{T}$ and $\widetilde{A}$ both are closed fuzzy mappings satisfying condition $(*)$ with constant functions $b(u) = 1$ and $c(u) = 1$, for all $u \in E$, respectively. Furthermore,

$$\begin{aligned} \left(\widetilde{T}_u\right)_{b(u)} &= \chi_{T(u)_1} = \{y \in E^* : \chi_{T(u)}(y) = 1\} = T(u), \\ \left(\widetilde{A}_u\right)_{c(u)} &= \chi_{A(u)_1} = \{y \in E^* : \chi_{A(u)}(y) = 1\} = A(u). \end{aligned} \tag{2.7}$$

Thus, problem (2.5) is equivalent to the following problem:

$$\begin{aligned} &\text{find } u \in K,\ x \in T(u), y \in A(u) \text{ such that} \\ &\langle N(x,y), \eta(v,u)\rangle + \varphi(u,v) - \varphi(u,u) \geq 0,\ \forall v \in K. \end{aligned} \tag{2.8}$$

This kind of problem is called the set-valued strongly nonlinear mixed variational-like inequality, which was studied by Huang and Deng [13], when $K = \mathcal{H}$.

(2) If $E = \mathcal{H}$ is a Hilbert space, then problem (2.5) collapses to the following problem: Let $T, A : K \to \mathcal{F}(\mathcal{H})$ are two closed fuzzy mappings satisfying the condition $(*)$ with functions $b, c : \mathcal{H} \to [0,1]$, respectively. $N, \eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ are two single-valued mappings. Let $\varphi : \mathcal{H} \times \mathcal{H} \to (-\infty, +\infty]$ be a real bifunction. We consider the following problem:

$$\begin{aligned} &\text{find } u \in K, x, y \in \mathcal{H} \text{ such that } x \in (T_u)_{b(u)},\ y \in (A_u)_{c(u)}, \\ &\langle N(x,y), \eta(v,u)\rangle + \varphi(u,v) - \varphi(u,u) \geq 0,\ \forall v \in K. \end{aligned} \tag{2.9}$$

The inequality of type (2.9) was studied by Chang et al. [14] under the additional condition that K is a nonempty bounded closed subset of $\mathcal{H}$.

(3) If $E = \mathcal{H}$ is a Hilbert space and φ(u,v)=0, then problem (2.5) is equivalent to the following problem:

$$\begin{aligned} &\text{find } u \in K, x,\ y \in \mathcal{H} \text{ such that } x \in (T_u)_{b(u)},\ y \in (A_u)_{c(u)}, \\ &\langle N(x,y), \eta(v,u)\rangle \geq 0,\ \forall v \in K. \end{aligned} \tag{2.10}$$

This is also a class of special fuzzy variational-like inequalities, which has been studying by many authors.

Evidently, for appropriate and suitable choice of the fuzzy mappings T, A, mappings N, η, the bifunction φ, and the space E, one can obtain a number of the known classes of variational inequalities and variational-like inequalities as special cases from problem (2.5) (see [1, 4, 5, 7–19]).

The following basic concepts will be needed in the sequel.

Definition 2.2. Let K be a nonempty subset of a Banach space E. Let $T, A : K \to \mathcal{F}(E^*)$ be two closed fuzzy mappings satisfying the condition $(*)$ with functions $b, c : E \to [0,1]$, respectively. Let $N : E^* \times E^* \to E^*, \eta : K \times K \to K$ be mappings. Then

(i) T is said to be η-cocoercive with respect to the first argument of $N(\cdot,\cdot)$, if there exists a constant $\tau > 0$, such that

$$\langle N(x,\cdot) - N(x',\cdot), \eta(u,v)\rangle \geq \tau \left\| N(x,\cdot) - N(x',\cdot)\right\|^2 \tag{2.11}$$

for each $u, v \in K$, and for all $x \in (T_u)_{b(u)}, x' \in (T_v)_{b(v)}$;

(ii) $N(\cdot,\cdot)$ is Lipschitz continuous in the second argument with respect to the fuzzy mapping A, if there exists a constant $\alpha > 0$ such that

$$\left\| N(\cdot, y) - N(\cdot, y') \right\| \leq \alpha \|u - v\| \tag{2.12}$$

for any $u, v \in K$ and $y \in (A_u)_{c(u)}, y' \in (A_v)_{c(v)}$;

(iii) $N(\cdot,\cdot)$ is η-strongly monotone in the first argument with respect to the fuzzy mapping T if there exists a constant $\xi > 0$ such that

$$\langle N(x, \cdot) - N(x', \cdot), \eta(u, v) \rangle \geq \xi \|u - v\|^2 \tag{2.13}$$

for any $u, v \in K$ and $x \in (T_u)_{b(u)}, x' \in (T_v)_{b(v)}$. Similarly, η-strongly monotone of $N(\cdot,\cdot)$ in the second argument with respect to the fuzzy mapping A can be defined;

(iv) T is said to be H-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$H\Big((T_u)_{b(u)}, (T_v)_{b(v)} \Big) \leq \gamma \|u - v\| \tag{2.14}$$

for any $u, v \in K$;

(v) η is Lipschitz continuous, if there exists a constant $\delta > 0$ such that

$$\left\| \eta(u, v) \right\| \leq \delta \|u - v\| \tag{2.15}$$

for any $u, v \in K$.

Definition 2.3. The bifunction $\varphi : E \times E \longrightarrow (-\infty, +\infty]$ is said to be skew-symmetric, if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0 \tag{2.16}$$

for all $u, v \in E$.

Remark 2.4. The skew-symmetric bifunctions have properties which can be considered as an analogs of monotonicity of gradient and nonnegativity of a second derivative for a convex function. As for the investigations of the skew-symmetric bifunction, we refer the reader to [20].

Definition 2.5 (see [15, 21]). Let K be a nonempty convex subset of a Banach space E. Let $\psi : K \longrightarrow (-\infty, +\infty)$ be a Fréchet differentiable function and $\eta : K \times K \longrightarrow E$. Then ψ is said to be

(i) η-convex, if

$$\psi(v) - \psi(u) \geq \langle \psi'(u), \eta(v, u) \rangle, \tag{2.17}$$

for all $u, v \in K$;

(ii) η-strongly convex, if there exists a constant $\mu > 0$ such that

$$\psi(v) - \psi(u) - \langle \psi'(u), \eta(v,u) \rangle \geq \frac{\mu}{2}\|u-v\|^2 \tag{2.18}$$

for all $u, v \in K$.

Note that if $\eta(u,v) = u - v$ for all $u, v \in K$, then ψ is said to be strongly convex.

Throughout this paper, we shall use the notations "$\rightharpoonup$" and "$\to$" for weak convergence and strong convergence, respectively.

Remark 2.6. (i) Assume that for each fixed $v \in K$ the mapping $\eta(v,\cdot) : K \to E$ is continuous from the weak topology to the weak topology. Let $v \in K$ and $f \in E^*$ be fixed, and let $g : K \to (-\infty, +\infty)$ be a functional defined by

$$g(u) = \langle f, \eta(v,u) \rangle, \quad \forall u \in K. \tag{2.19}$$

Then, it is easy to see that g is a weakly continuous functional on K.

(ii) Let $\psi : K \to (-\infty, +\infty)$ be a Fréchet differentiable function, and let $\eta : K \times K \to K$ be a mapping such that $\eta(u,v) + \eta(v,u) = 0$, for all $u, v \in K$. If ψ is an η-strongly convex functional with constant $\mu > 0$ on a convex subset K of E, then ψ' is η-strongly monotone with constant $\mu > 0$ (see [19], Proposition 2.1).

The following lemma due to Zeng et al. [19] will be needed in proving our results.

Lemma 2.7 (see [19, Lemma 2]). *Let K be a nonempty convex subset of a topological vector space X and let $\Phi : K \times K \to [-\infty, +\infty]$ be such that*

(i) *for each $v \in K, u \mapsto \Phi(v,u)$ is lower semicontinuous on each nonempty compact subset of K;*

(ii) *for each finite set $\{v_1, \ldots, v_m\} \subset K$ and for each $u = \sum_{i=1}^m \lambda_i v_i$ $(\lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1)$, $\min_{1\leq i\leq m} \Phi(v_i, u) \leq 0$;*

(iii) *there exists a nonempty compact convex subset K_0 of K such that for some $v_0 \in K_0$, there holds*

$$\Phi(v_0, u) > 0, \quad \forall u \in K \setminus K_0. \tag{2.20}$$

Then there exists $\hat{u} \in K$, such that $\Phi(v, \hat{u}) \leq 0$, for all $v \in K$.

We also need the following lemma.

Lemma 2.8 (see [22]). *Let (X,d) be a complete metric space and let $B_1, B_2 \in CB(X)$ and $r > 1$ be any real number. Then, for every $b_1 \in B_1$ there exists $b_2 \in B_2$ such that $d(b_1, b_2) \leq rH(B_1, B_2)$.*

In the sequel, we assume that N and η satisfy the following assumption.

Assumption 2.9. Let $N : E^* \times E^* \to E^*, \eta : K \times K \to E$ be two mappings satisfying the following conditions:

(a) $\eta(u,v) = \eta(u,z) + \eta(z,v)$ for each $u,v,z \in K$;

(b) for each fixed $(u,x,y) \in K \times E^* \times E^*, v \mapsto \langle N(x,y), \eta(u,v)\rangle$ is a concave function;

(c) for each fixed $v \in K$, the functional $(u,x,y) \mapsto \langle N(x,y), \eta(u,v)\rangle$ is weakly lower semicontinuous function from $K \times E^* \times E^*$ to $\mathbb{R}$, that is,

$$u_n \rightharpoonup u, x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y \text{ imply } \langle N(x,y), \eta(u,v)\rangle \leq \liminf_{n\to\infty} \langle N(x_n,y_n), \eta(u_n,v)\rangle. \quad (2.21)$$

Remark 2.10. It follows from Assumption 2.9(a) that $\eta(u,u) = 0$ and $\eta(u,v) = -\eta(v,u)$, for all $u,v \in K$.

3. The Existence Theorems

Theorem 3.1. *Let E be a real reflexive Banach space with the dual space E^*, and K be a nonempty convex subset of E. Let $T, A : K \to \mathcal{F}(E^*)$ be two closed fuzzy mappings satisfying the condition $(*)$ with functions $b, c : E \to [0,1]$, respectively. Let $N : E^* \times E^* \to E^*$, and $\eta : K \times K \to E$. Let $\varphi : E \times E \to (-\infty, +\infty]$ be skew-symmetric and weakly continuous such that* $\operatorname{int}\{u \in K : \varphi(u,u) < \infty\} \neq \emptyset$ *and $\varphi(u,\cdot)$ is a proper convex, for each $u \in E$. Suppose that*

(i) *T is η-cocoercive with respect to the first argument of $N(\cdot,\cdot)$ with constant τ;*

(ii) *η is Lipschitz continuous with constant $\delta > 0$;*

(iii) *$N(\cdot,\cdot)$ is Lipschitz continuous and η-strongly monotone in the second argument with respect to A with constant $\alpha > 0$ and $\beta > 0$, respectively.*

If Assumption 2.9 is satisfied, then $\mathrm{FMVLIP}(T, A, N, \eta, \varphi) \neq \emptyset$.

Proof. For any $u, v \in K$, we define a function $\Phi : K \times K \to \mathbb{R}$ by

$$\Phi(v,u) = \langle N(x,y), \eta(u,v)\rangle + \varphi(u,u) - \varphi(u,v) \quad \forall u,v \in K, \quad (3.1)$$

where $x \in (T_u)_{b(u)}, y \in (A_u)_{c(u)}$.

Observe that, by $\varphi(\cdot,\cdot)$ is weakly continuous functional and since each fixed $v \in K$ the functional $(u,x,y) \mapsto \langle N(x,y), \eta(u,v)\rangle$ is weakly lower semicontinuous, we have the functional $u \mapsto \Phi(v,u)$ is weakly lower semicontinuous for each $v \in K$. This shows that condition (i) in Lemma 2.7 holds. Next, we claim that $\Phi(v,u)$ satisfies condition (ii) in Lemma 2.7. If it is not true, then there exist a finite set $\{v_1, v_2, \ldots, v_m\} \subset K$ and $u = \sum_{i=1}^m \varepsilon_i v_i$ ($\varepsilon_i \geq 0, \sum_{i=1}^m \varepsilon_i = 1$), such that $\Phi(v_i, u) > 0$ for all $i = 1, 2, \ldots, m$, that is,

$$\langle N(x,y), \eta(u,v_i)\rangle + \varphi(u,u) - \varphi(u,v_i) > 0 \quad \forall i = 1, 2, \ldots, m. \quad (3.2)$$

This gives

$$\sum_{i=1}^{m}\varepsilon_i\langle N(x,y),\eta(u,v_i)\rangle+\varphi(u,u)-\sum_{i=1}^{m}\varepsilon_i\varphi(u,v_i)>0. \tag{3.3}$$

Note that for each $u\in E$, $\varphi(u,\cdot)$ is a convex functional, that is $\sum_{i=1}^{m}\varepsilon_i\varphi(u,v_i)\geq\varphi(u,\sum_{i=1}^{m}\varepsilon_i v_i)=\varphi(u,u)$. Hence,

$$\sum_{i=1}^{m}\varepsilon_i\langle N(x,y),\eta(u,v_i)\rangle>0. \tag{3.4}$$

From Assumption 2.9, we obtain

$$0<\sum_{i=1}^{m}\varepsilon_i\langle N(x,y),\eta(u,v_i)\rangle\leq\left\langle N(x,y),\eta\left(u,\sum_{i=1}^{m}\varepsilon_i v_i\right)\right\rangle=\langle N(x,y),\eta(u,u)\rangle=0, \tag{3.5}$$

which is a contradiction. Thus condition (ii) in Lemma 2.7 holds. Since for each $u\in E$, $v\mapsto\varphi(u,v)$ is a proper convex weakly lower semicontinuous functional and $\operatorname{int}\{u\in K:\varphi(u,u)<\infty\}\neq\emptyset$, the element $u^*\in\operatorname{int}\{u\in K:\varphi(u,u)<\infty\}$ can be found. Moreover, by Proposition I.2.6 of Pascali and Sburlan [23, page 27], $\varphi(u^*,\cdot)$ is subdifferentiable at u^*. This means

$$\varphi(u^*,v)-\varphi(u^*,u^*)\geq\langle r,v-u^*\rangle,\quad\forall r\in\partial\varphi(u^*,\cdot),v\in E. \tag{3.6}$$

Since φ is skew-symmetric, it follows that

$$\varphi(v,v)-\varphi(v,u^*)\geq\varphi(u^*,v)-\varphi(u^*,u^*)\geq\langle r,v-u^*\rangle,\quad\forall r\in\partial\varphi(u^*,\cdot),v\in E. \tag{3.7}$$

Letting $x^*\in(T_{u^*})_{b(u^*)}, y^*\in(A_{u^*})_{c(u^*)}, w\in(T_v)_{b(v)}$, and $z\in(A_v)_{c(v)}$ be fixed, by using conditions (ii) and (iv) and equality $\eta(u,v)=-\eta(v,u)$, we have

$$\begin{aligned}\Phi(u^*,u)&=\langle N(w,z),\eta(u,u^*)\rangle+\varphi(u,u)-\varphi(u,u^*)\\&\geq\langle N(x^*,y^*)-N(w,z),\eta(u^*,u)\rangle-\langle N(x^*,y^*),\eta(u^*,u)\rangle+\langle r,u-u^*\rangle\\&=\langle N(x^*,y^*)-N(w,y^*),\eta(u^*,u)\rangle+\langle N(w,y^*)-N(w,z),\eta(u^*,u)\rangle\\&\quad-\langle N(x^*,y^*),\ \eta(u^*,u)\rangle+\langle r,u-u^*\rangle\\&\geq\tau\left\|N(x^*,y^*)-N(w,y^*)\right\|^2+\beta\|u^*-u\|^2-\delta\left\|N(x^*,y^*)\right\|\|u^*-u\|-\|r\|\|u^*-u\|\\&\geq\beta\|u^*-u\|^2-\delta\left\|N(x^*,y^*)\right\|\|u^*-u\|-\|r\|\|u^*-u\|\\&=\|u^*-u\|\left[\beta\|u^*-u\|-\delta\left\|N(x^*,y^*)\right\|-\|r\|\right].\end{aligned} \tag{3.8}$$

Define $\theta = (1/\beta)[\delta\|N(x^*, y^*) + \|r\|]$ and $K_0 = \{u \in K : \|u - u^*\| \leq \theta\}$. Then K_0 is a weakly compact convex subset of K. Furthermore, it is easy to see that $\Phi(u^*, u) > 0$ for all $u \in K \setminus K_0$. Thus, condition (iii) of Lemma 2.7 is satisfied. By Lemma 2.7, there exists $\widehat{u} \in K$ such that $\Phi(v, \widehat{u}) \leq 0$ for all $v \in K$, this means that

$$\langle N(\widehat{x}, \widehat{y}), \eta(v, \widehat{u})\rangle + \varphi(\widehat{u}, v) - \varphi(\widehat{u}, \widehat{u}) \geq 0 \quad \forall v \in K, \tag{3.9}$$

where $\widehat{x} \in (T_{\widehat{u}})_{b(\widehat{u})}, \widehat{y} \in (A_{\widehat{u}})_{c(\widehat{u})}$. Hence, $\widehat{u} \in K,\ \widehat{x} \in (T_{\widehat{u}})_{b(\widehat{u})},\ \widehat{y} \in (A_{\widehat{u}})_{c(\widehat{u})}$ is a solution of the fuzzy variational like inequality (2.5), that is, FMVLIP$(T, A, N, \eta, \varphi) \neq \emptyset$. This completes the proof. □

Remark 3.2. If all assumptions to Theorem 3.1 hold and $N(\cdot,\cdot)$ is η-strongly monotone in the first argument with respect to T with constant $\xi > 0$, then the solution of problem (2.5) is unique up to the element $u \in K$. Indeed, supposing that $(\widehat{u}, \widehat{x}, \widehat{y})$ and $(\widetilde{u}, \widetilde{x}, \widetilde{y})$ are elements in FMVLIP(T, A, N, η, φ), we have

$$\langle N(\widehat{x}, \widehat{y}), \eta(v, \widehat{u})\rangle \geq \varphi(\widehat{u}, \widehat{u}) - \varphi(\widehat{u}, v), \quad \forall v \in K, \tag{3.10}$$

$$\langle N(\widetilde{x}, \widetilde{y}), \eta(v, \widetilde{u})\rangle \geq \varphi(\widetilde{u}, \widetilde{u}) - \varphi(\widetilde{u}, v), \quad \forall v \in K. \tag{3.11}$$

Taking $v = \widetilde{u}$ in (3.10) and $v = \widehat{u}$ in (3.11) and adding two inequalities, since φ is skew-symmetric, we obtain

$$\langle N(\widehat{x}, \widehat{y}), \eta(\widetilde{u}, \widehat{u})\rangle + \langle N(\widetilde{x}, \widetilde{y}), \eta(\widehat{u}, \widetilde{u})\rangle \geq 0. \tag{3.12}$$

Using this one, in view of Remark 2.10, we have

$$\langle N(\widetilde{x}, \widetilde{y}) - N(\widehat{x}, \widehat{y}), \eta(\widehat{u}, \widetilde{u})\rangle \geq 0. \tag{3.13}$$

Since $N(\cdot,\cdot)$ is η-strongly monotone in the first argument with respect to T with the constant ξ and η-strongly monotone in the second argument with respect to A with constant β, we obtain

$$(\beta + \xi)\|\widehat{u} - \widetilde{u}\|^2 \leq \langle N(\widetilde{x}, \widetilde{y}) - N(\widehat{x}, \widetilde{y}), \eta(\widetilde{u}, \widehat{u})\rangle + \langle N(\widehat{x}, \widetilde{y})\rangle - N(\widehat{x}, \widehat{y}), \eta(\widetilde{u}, \widehat{u})\rangle \leq 0. \tag{3.14}$$

Since $\beta, \xi > 0$, we must have $\widehat{u} = \widetilde{u}$.

4. Convergence Analysis

4.1. Auxiliary Problem and Algorithm

In this section, we extend the auxiliary principle technique to study the fuzzy mixed variational-like inequality problem (2.5) in a reflexive Banach space E. First, we give the existence theorem for the auxiliary problem for the problem (2.5). Consequently, we construct the iterative algorithm for solving the problem of type (2.5).

Let $\eta : K \times K \to E$ be a mapping, let $\psi : K \to (-\infty, +\infty]$ be a given Fréchet differentiable η-convex functional, and let $\rho > 0$ be a given positive real number. Given $u \in K, x \in (T_u)_{b(u)}, y \in (A_u)_{c(u)}$, we consider the following problem $P(u, x, y)$: find $w \in K$ such that

$$\langle \rho N(x, y) + \psi'(w) - \psi'(u), \eta(v, w) \rangle + \rho\varphi(w, v) - \rho\varphi(w, w) \geq 0, \quad \forall v \in K. \tag{4.1}$$

The problem $P(u, x, y)$ is called the auxiliary problem for fuzzy mixed variational-like inequality problem (2.5).

Theorem 4.1. *If the conditions of Theorem 3.1 hold and for each fixed $v \in K, w \mapsto \eta(v, w)$ is continuous from the weak topology to the weak topology. If the function ψ is η-strongly convex with constant μ and the functional $w \mapsto \langle \psi'(w), \eta(u, w) \rangle$ is weakly upper semicontinuous on K for each $u \in K$, then the auxiliary problem $P(u, x, y)$ has a unique solution.*

Proof. Let $\rho > 0$ and $u \in K, x \in (T_u)_{b(u)}, y \in (A_u)_{c(u)}$ be fixed. Define a functional $\Omega : K \times K \to [-\infty, +\infty]$ by

$$\Omega(v, w) = \langle \psi'(u) - \psi'(w) - \rho N(x, y), \eta(v, w) \rangle + \rho\varphi(w, w) - \rho\varphi(w, v) \quad \forall v, w \in K. \tag{4.2}$$

Note that, for each fixed $v \in K$, the functional $w \mapsto \langle \psi'(w), \eta(v, w) \rangle$ is weakly upper semicontinuous on K, $w \mapsto \eta(v, w)$ is continuous from the weak topology to the weak topology, and $\varphi(\cdot, \cdot)$ is weakly continuous. Thus, it is easy to see that for each fixed $v \in K$ the function $w \mapsto \Omega(v, w)$ is weakly lower semicontinuous continuous on each weakly compact subset of K, and so condition (i) in Lemma 2.7 is satisfied. We claim that condition (ii) in Lemma 2.7 holds. If this is false, then there exist a finite set $\{v_1, v_2, \ldots, v_m\} \subset K$ and a $w = \sum_{i=1}^m \varepsilon_i v_i$ with $\varepsilon_i \geq 0$ and $\sum_{i=1}^m \varepsilon_i = 1$, such that

$$\Omega(v_i, w) = \langle \psi'(u) - \psi'(w) - \rho N(x, y), \eta(v_i, w) \rangle + \rho\varphi(w, w) - \rho\varphi(w, v_i) > 0 \quad \forall i = 1, 2, \ldots, m. \tag{4.3}$$

By Assumption 2.9, in light of Remark 2.10, together with the convexity of $\varphi(w, \cdot)$, we have

$$\begin{aligned} 0 &< \sum_{i=1}^m \varepsilon_i \left[\langle \psi'(u) - \psi'(w) - \rho(N(x, y)), \eta(v_i, w) \rangle + \rho\varphi(w, w) - \rho\varphi(w, v_i) \right] \\ &\leq \langle \psi'(u) - \psi'(w) - \rho(N(x, y)), \eta(w, w) \rangle + \rho\varphi(w, w) - \rho\sum_{i=1}^m \varepsilon_i \varphi(w, v_i) \\ &= 0, \end{aligned} \tag{4.4}$$

which is a contradiction. Thus, condition (ii) in Lemma 2.7 is satisfied. Note that the η-strong convexity of ψ implies that ψ' is η-strongly monotone with constant $\mu > 0$; see Remark 2.6(ii). By using the similar argument as in the proof of Theorem 3.1, we can readily prove that condition (iii) of Lemma 2.7 is also satisfied. By Lemma 2.7 there exists a point $w \in K$, such that $\Omega(v, w) \leq 0$ for all $v \in K$. This implies that w is a solution to the problem $P(u, x, y)$.

Now we prove that the solution of problem $P(u, x, y)$ is unique. Let w_1 and w_2 be two solutions of problem (4.1). Then,

$$\langle \rho N(x, y) + \psi'(w_1) - \psi'(u), \eta(v, w_1) \rangle \geq \rho\varphi(w_1, w_1) - \rho\varphi(w_1, v), \quad \forall v \in K, \tag{4.5}$$

$$\langle \rho N(x, y) + \psi'(w_2) - \psi'(u), \eta(v, w_2) \rangle \geq \rho\varphi(w_2, w_2) - \rho\varphi(w_2, v), \quad \forall v \in K. \tag{4.6}$$

Taking $v = w_2$ in (4.5) and $v = w_1$ in (4.6), and adding these two inequalities, since $\eta(w_2, w_1) + \eta(w_1, w_2) = 0$ and $\varphi(\cdot, \cdot)$ is skew-symmetric, we obtain

$$\langle \psi'(w_2) - \psi'(w_1), \eta(w_1, w_2) \rangle \geq 0. \tag{4.7}$$

Thus, by ψ' is η-strongly monotone, we have

$$\mu \|w_1 - w_2\|^2 \leq \langle \psi'(w_1) - \psi'(w_2), \eta(w_1, w_2) \leq 0, \tag{4.8}$$

This implies that $w_1 = w_2$, and the proof is completed. □

By virtue of Theorem 4.1, we now construct an iterative algorithm for solving the fuzzy mixed variational-like inequalities problem (2.5) in a reflexive Banach space E.

Let $\rho > 0$ be fixed. For given $u_0 \in E, x_0 \in (T_{u_0})_{b(u_0)}, y_0 \in (A_{u_0})_{c(u_0)}$, from Theorem 4.1, there is $u_1 \in K$ such that

$$\langle \rho N(x_0, y_0) + \psi'(u_1) - \psi'(u_0), \eta(v, u_1) \rangle + \rho\varphi(u_1, v) - \rho\varphi(u_1, u_1) \geq 0, \quad \forall v \in K. \tag{4.9}$$

Since $x_0 \in (T_{u_0})_{b(u_0)} \in CB(E^*)$, $y_0 \in (A_{u_0})_{c(u_0)} \in CB(E^*)$, by Lemma 2.8, there exist $x_1 \in (T_{u_1})_{b(u_1)}$ and $y_1 \in (A_{u_1})_{c(u_1)}$ such that

$$\begin{aligned} \|x_0 - x_1\| &\leq (1+1) H\Big((T_{u_0})_{b(u_0)}, (T_{u_1})_{b(u_1)} \Big), \\ \|y_0 - y_1\| &\leq (1+1) H\Big((A_{u_0})_{c(u_0)}, (A_{u_1})_{c(u_1)} \Big). \end{aligned} \tag{4.10}$$

Again by Theorem 4.1, there is $u_2 \in K$ such that

$$\langle \rho N(x_1, y_1) + \psi'(u_2) - \psi'(u_1), \eta(v, u_2) \rangle + \rho\varphi(u_2, v) - \rho\varphi(u_2, u_2) \geq 0, \quad \forall v \in K. \tag{4.11}$$

Since $x_1 \in (T_{u_1})_{b(u_1)} \in CB(E^*), y_1 \in (A_{u_1})_{c(u_1)} \in CB(E^*)$, by Lemma 2.8, there exist $x_2 \in (T_{u_2})_{b(u_2)}$ and $y_2 \in (A_{u_2})_{c(u_2)}$ such that

$$\|x_1 - x_2\| \leq \left(1 + \frac{1}{2}\right) H\left((T_{u_1})_{b(u_1)}, (T_{u_2})_{b(u_2)}\right),$$
$$\|y_1 - y_2\| \leq \left(1 + \frac{1}{2}\right) H\left((A_{u_1})_{c(u_1)}, (A_{u_2})_{c(u_2)}\right). \tag{4.12}$$

Continuing in this way, we can obtain the iterative algorithm for solving problem (2.5) as follows.

Algorithm 4.2. Let $\rho > 0$ be fixed. For given $u_0 \in K, x_0 \in (T_{u_0})_{b(u_0)}, y_0 \in (A_{u_0})_{c(u_0)}$ there exist the sequences $\{u_n\} \subset K$ and $\{x_n\}, \{y_n\} \subset E^*$ such that

$$\langle \rho N(x_n, y_n) + \psi'(u_{n+1}) - \psi'(u_n), \eta(v, u_{n+1}) \rangle + \rho\varphi(u_{n+1}, v) - \rho\varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in K,$$
$$x_n \in (T_{u_n})_{b(u_n)}, \quad \|x_n - x_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) H\left((T_{u_n})_{b(u_n)}, (T_{u_{n+1}})_{b(u_{n+1})}\right), \quad \forall n \in \mathbb{N},$$
$$y_n \in (A_{u_n})_{c(u_n)}, \quad \|y_n - y_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) H\left((A_{u_n})_{c(u_n)}, (A_{u_{n+1}})_{c(u_{n+1})}\right), \quad \forall n \in \mathbb{N}. \tag{4.13}$$

4.2. Convergence Theorems

Now, we shall prove that the sequences $\{u_n\} \subset K$ and $\{x_n\}, \{y_n\} \subset E^*$ generated by Algorithm 4.2 converge strongly to a solution of problem (2.5).

Theorem 4.3. *Suppose that conditions of Theorem 4.1 hold, and the mapping T, A are Lipschitzian continuous fuzzy mappings with Lipschitzian constant γ and ζ, respectively. If $\rho \in (0, 2\tau\mu\beta/\delta^2(\tau\alpha^2 + \beta))$, then the iterative sequences $\{u_n\}, \{x_n\},$ and $\{y_n\}$ obtained from Algorithm 4.2 converge strongly to a solution of problem* (2.5).

Proof. Let $(\hat{u}, \hat{x}, \hat{y}) \in \mathrm{FMVLIP}(T, A, N, \eta, \varphi)$. Define a function $\Delta : K \to (-\infty, +\infty]$ by

$$\Delta(u) = \psi(\hat{u}) - \psi(u) - \langle \psi'(u), \eta(\hat{u}, u) \rangle. \tag{4.14}$$

By the η-strong convexity of ψ, we have

$$\Delta(u) = \psi(\hat{u}) - \psi(u) - \langle \psi'(u), \eta(\hat{u}, u) \rangle \geq \frac{\mu}{2}\|u - \hat{u}\|^2. \tag{4.15}$$

Note that $\eta(u,v)=-\eta(v,u)$ for all $u,v\in K$ and $\varphi(\cdot,\cdot)$ is skew-symmetric. Since $u_{n+1}\in K$ and $(\widehat{u},\widehat{x},\widehat{y})\in \text{FMVLIP}(T,A,N,\eta,\varphi)$, from the η-strong convexity of ψ, and Algorithm 4.2 with $v=\widehat{u}$ it follows that

$$\begin{aligned}
\Delta(u_n)-\Delta(u_{n+1}) &= \psi(u_{n+1})-\psi(u_n)-\langle \psi'(u_n),\eta(u_{n+1},u_n)\rangle+\langle \psi'(u_{n+1})-\psi'(u_n),\eta(\widehat{u},u_{n+1})\rangle\\
&\ge \frac{\mu}{2}\|u_n-u_{n+1}\|^2+\rho\langle N(x_n,y_n),\eta(u_{n+1},\widehat{u})\rangle+\rho\left[\varphi(u_{n+1},u_{n+1})-\varphi(u_{n+1},\widehat{u})\right]\\
&\ge \frac{\mu}{2}\|u_n-u_{n+1}\|^2+\rho\langle N(x_n,y_n)-N(\widehat{x},\widehat{y}),\eta(u_{n+1},\widehat{u})\rangle\\
&\quad+\rho\left[\langle N(\widehat{x},\widehat{y}),\eta(u_{n+1},\widehat{u})\rangle+\varphi(\widehat{u},u_{n+1})-\varphi(\widehat{u},\widehat{u})\right]\\
&\ge \frac{\mu}{2}\|u_n-u_{n+1}\|^2+\rho\langle N(x_n,y_n)-N(\widehat{x},\widehat{y}),\eta(u_{n+1},\widehat{u})\rangle\\
&= \frac{\mu}{2}\|u_n-u_{n+1}\|^2+Q,
\end{aligned} \tag{4.16}$$

where $Q=\rho\langle N(x_n,y_n)-N(\widehat{x},\widehat{y}),\eta(u_{n+1},\widehat{u})\rangle$.

Consider

$$\begin{aligned}
Q &= \rho\langle N(x_n,y_n)-N(\widehat{x},\widehat{y}),\eta(u_{n+1},\widehat{u})\rangle\\
&= \rho\langle N(x_n,y_n)-N(\widehat{x},\widehat{y}),\eta(u_{n+1},u_n)\rangle+\rho\langle N(x_n,y_n)-N(\widehat{x},\widehat{y}),\eta(u_n,\widehat{u})\rangle\\
&= \rho\langle N(x_n,y_n)-N(\widehat{x},y_n),\eta(u_n,\widehat{u})\rangle+\rho\langle N(\widehat{x},y_n)-N(\widehat{x},\widehat{y}),\eta(u_n,\widehat{u})\rangle\\
&\quad+\rho\langle N(x_n,y_n)-N(\widehat{x},y_n),\eta(u_{n+1},u_n)\rangle+\rho\langle N(\widehat{x},y_n)-N(\widehat{x},\widehat{y}),\eta(u_{n+1},u_n)\rangle\\
&\ge \rho\tau\left\|N(x_n,y_n)-N(\widehat{x},y_n)\right\|^2+\rho\beta\|u_n-\widehat{u}\|^2-\rho\delta\left\|N(x_n,y_n)-N(\widehat{x},y_n)\right\|\|u_{n+1}-u_n\|\\
&\quad-\rho\alpha\delta\|u_n-\widehat{u}\|\|u_{n+1}-u_n\|\\
&= \rho\left[\tau\left\|N(x_n,y_n)-N(\widehat{x},y_n)\right\|^2-\delta\left\|N(x_n,y_n)-N(\widehat{x},y_n)\right\|\|u_{n+1}-u_n\|\right]\\
&\quad-\rho\alpha\delta\|u_n-\widehat{u}\|\|u_{n+1}-u_n\|+\rho\beta\|u_n-\widehat{u}\|^2\\
&\ge \rho\left[-\frac{\delta^2}{4\tau}\right]\|u_{n+1}-u_n\|^2-\rho\alpha\delta\|u_n-\widehat{u}\|\|u_{n+1}-u_n\|+\rho\beta\|u_n-\widehat{u}\|^2.
\end{aligned} \tag{4.17}$$

Therefore, we have

$$\begin{aligned}
\Delta(u_n)-\Delta(u_{n+1}) &\ge \frac{1}{2}\left(\mu-\frac{\rho\delta^2}{2\tau}\right)\|u_{n+1}-u_n\|^2-\rho\alpha\delta\|u_n-\widehat{u}\|\|u_{n+1}-u_n\|+\rho\beta\|u_n-\widehat{u}\|^2\\
&\ge \left[\rho\beta-\frac{\rho^2\alpha^2\delta^2}{2(\mu-\rho\delta^2/2\tau)}\right]\|u_n-\widehat{u}\|^2.
\end{aligned} \tag{4.18}$$

Since $\rho \in (0, 2\tau\mu\beta/\delta^2(\tau\alpha^2+\beta))$, the inequality (4.18) implies that the sequence $\{\Delta(u_n)\}$ is strictly decreasing (unless $u_n = \hat{u}$) and is nonnegative by (4.15). Hence it converges to some number. Thus, the difference of two consecutive terms of the sequence $\{\Delta(u_n)\}$ goes to zero, and so the sequence $\{u_n\}$ converges strongly to $\hat{u}$. Further, from Algorithm 4.2, we have

$$\begin{aligned} \|x_n - x_{n+1}\| &\le \left(1+\frac{1}{n+1}\right) H\left((T_{u_n})_{b(u_n)}, (T_{u_{n+1}})_{b(u_{n+1})}\right) \le \gamma\|u_n - u_{n+1}\|, \\ \|y_n - y_{n+1}\| &\le \left(1+\frac{1}{n+1}\right) H\left((A_{u_n})_{c(u_n)}, (A_{u_{n+1}})_{c(u_{n+1})}\right) \le \zeta\|u_n - u_{n+1}\|. \end{aligned} \tag{4.19}$$

These imply that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in E^*, since $\{u_n\}$ is a convergence sequence. Thus, we can assume that $x_n \to x$ and $y_n \to y$ (as $n \to \infty$). Noting $x_n \in (T_{u_n})_{b(u_n)}$ and $y_n \in (A_{u_n})_{c(u_n)}$, we have

$$\begin{aligned} d\left(x, (T_{\hat{u}})_{b(\hat{u})}\right) &\le \|x - x_n\| + d\left(x_n, (T_{u_n})_{b(u_n)}\right) + H\left((T_{u_n})_{b(u_n)}, (T_{\hat{u}})_{b(\hat{u})}\right) \\ &\le \|x - x_n\| + 0 + \gamma\|u_n - u\| \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{4.20}$$

Hence we must have $x \in (T_{\hat{u}})_{b(\hat{u})}$. Similarly, we can obtain $y \in (A_{\hat{u}})_{b(\hat{u})}$. Finally, we will show that $(\hat{u}, x, y) \in \mathrm{FMVLIP}(T, A, N, \eta, \varphi)$. In regarded of Assumption 2.9(c), for each fixed $v \in K$ we have that the functional $(u, x, y) \mapsto \langle N(x, y), \eta(v, u)\rangle$ is an upper semicontinuous functional; this together with the weak continuity of the function $\varphi(\cdot,\cdot)$, we obtain

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \left[\langle \rho N(x_n, y_n) + \psi'(u_{n+1}) - \psi'(u_n), \eta(v, u_{n+1})\rangle + \rho\varphi(u_{n+1}, v) - \rho\varphi(u_{n+1}, u_{n+1})\right] \\ &\le \rho\left[\langle N(x, y), \eta(v, \hat{u})\rangle + \varphi(\hat{u}, v) - \varphi(\hat{u}, \hat{u})\right]. \end{aligned} \tag{4.21}$$

This implies that $(\hat{u}, x, y) \in \mathrm{FMVLIP}(T, A, N, \eta, \varphi)$, and the proof is completed. □

Remark 4.4. (i) Theorems 3.1 and 4.3 are the extension of the results by Chang et al. [14], from Hilbert setting to a general reflexive Banach space, but it is worth noting that the bounded condition of the convex set K is not imposed here.

(ii) Since every set-valued mapping is the fuzzy mapping, hence, all results obtained in this paper are still hold for any set-valued mappings T, A.

Thus, our results can be view as a refinement and improvement of the previously known results for variational inequalities.

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