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Research Article

Almost Automorphic and Pseudo-Almost Automorphic Solutions to Semilinear Evolution Equations with Nondense Domain

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We study the existence and uniqueness of almost automorphic (resp., pseudo-almost automorphic) solutions to a first-order differential equation with linear part dominated by a Hille-Yosida type operator with nondense domain.

1. Introduction

In recent years, the theory of almost automorphic functions has been developed extensively (see, e.g., Bugajewski and N'guérékata [1], Cuevas and Lizama [2], and N'guérékata [3] and the references therein). However, literature concerning pseudo-almost automorphic functions is very new (cf. [4]). It is well known that the study of composition of two functions with special properties is important and basic for deep investigations. Recently an interesting article has appeared by Liang et al. [5] concerning the composition of pseudo-almost automorphic functions. The same authors in [6] have applied the results to obtain pseudo-almost automorphic solutions to semilinear differentail equations (see also [7]). On the other hand, in article by Blot et al. [8], the authors have obtained existence and uniqueness of pseudo-almost automorphic solutions to some classes of partial evolutions equations.

In this work, we study the existence and uniqueness of almost automorphic and pseudo-almost automorphic solutions for a class of abstract differential equations described in the form

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where A is an unbounded linear operator, assumed to be Hille-Yosida (see Definition 2.5) of negative type, having the domain $D(A)$, not necessarily dense, on some Banach space X; $f : \mathbb{R} \times X_0 \to X$ is a continuous function, where $X_0 = \overline{D(A)}$. The regularity of solutions for (1.1) in the space of pseudo-almost periodic solutions was considered in Cuevas and Pinto [9] (see [10–12]). We note that pseudo-almost automorphic functions are more general and complicated than pseudo-almost periodic functions (cf. [5]).

The existence of almost automorphic and pseudo-almost automorphic solutions for evolution equations with linear part dominated by a Hille-Yosida type operator constitutes an untreated topic and this fact is the main motivation of this paper.

2. Preliminaries

Let $(Z, \|\cdot\|)$, $(W, \|\cdot\|)$ be Banach spaces. The notations $C(\mathbb{R}; Z)$ and $BC(\mathbb{R}; Z)$ stand for the collection of all continuous functions from $\mathbb{R}$ into Z and the Banach space of all bounded continuous functions from $\mathbb{R}$ into Z endowed with the uniform convergence topology. Similar definitions as above apply for both $C(\mathbb{R} \times Z; W)$ and $BC(\mathbb{R} \times Z; W)$. We recall the following definition (cf. [7]).

Definition 2.1. (1) A continuous function $f : \mathbb{R} \to Z$ is called almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t + s_n)$ is well defined for each $t \in \mathbb{R}$, and $f(t) = \lim_{n\to\infty} g(t - s_n)$, for each $t \in \mathbb{R}$. Since the range of an almost automorphic function is relatively compact, then it is bounded. Almost automorphic functions constitute a Banach space, $AA(Z)$, when it is endowed with the supremum norm.

A continuous function $f : \mathbb{R} \times W \to Z$ is called almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all x in any bounded subset of W. $AA(\mathbb{R} \times W, Z)$ is the collection of those functions.

(2) A continuous function $f : \mathbb{R} \to Z$ (resp., $\mathbb{R} \times W \to Z$) is called pseudo-almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA(Z)$ (resp., $AA(\mathbb{R} \times W, Z)$) and ϕ is a bounded continuous function with vanishing mean value, that is,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(t)\| dt = 0, \tag{2.1}$$

(resp., $\phi(t, x)$ is a bounded continuous function with

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(t, x)\| dt = 0, \tag{2.2}$$

uniformly for x in any bounded subset of W). Denote by $PAA(\mathbb{R}, Z)$ (resp., $PAA(\mathbb{R} \times W, Z)$) the set of all such functions. In both cases above, g and ϕ are called, respectively, the principal and the ergodic terms of f.

We define

$$AA_0(\mathbb{R}, Z) := \left\{ \phi \in BC(\mathbb{R}, Z) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(t)\| dt = 0 \right\},$$
$$AA_0(\mathbb{R}\times W, Z) := \left\{ \begin{array}{l} \phi \in BC(\mathbb{R}\times W, Z) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(t,x)\| dt = 0, \\ \text{uniformly for } x \text{ in any bounded subset of } W \end{array} \right\}. \quad (2.3)$$

Remark 2.2. $(PAA(\mathbb{R}, Z), \|\cdot\|_\infty)$ is a Banach space, where $\|\cdot\|_\infty$ is the supremum norm (see [6]).

Lemma 2.3 (see [13]). *Let $f : \mathbb{R}\times W \to Z$ be an almost automorphic function in $t \in \mathbb{R}$ for each $x \in W$ and assume that f satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}$. Let $\phi : \mathbb{R} \to W$ be an almost automorphic function. Then the function $\Phi : \mathbb{R} \to Z$ defined by $\Phi(t) = f(t, \phi(t))$ is almost automorphic.*

Lemma 2.4 (see [5, 7]). *Let $f \in PAA(\mathbb{R}\times W, Z)$ and assume that $f(t,x)$ is uniformly continuous in any bounded subset $K \subset W$ uniformly in $t \in \mathbb{R}$. If $\phi \in PAA(\mathbb{R}, W)$, then the function $t \to f(t, \phi(t))$ belongs to $PAA(\mathbb{R}, Z)$.*

We recall some basic properties of extrapolation spaces for Hille-Yosida operators which are a natural tool in our setting. The abstract extrapolation spaces have been used from various purposes, for example, to study Volterra integro differential equations and retarded differential equations (see [14]).

Definition 2.5. Let X be a Banach space, and let A be a linear operator with domain $D(A)$. One says that $(A, D(A))$ is a Hille-Yosida operator on X if there exist $\omega \in \mathbb{R}$ and a positive constant $M \geq 1$ such that $(\omega, \infty) \subset \rho(A)$ and $\sup\{(\lambda-\omega)^n \|(\lambda - A)^{-n}\| : n \in \mathbb{N}, \lambda > \omega\} \leq M$. The infinimum of such ω is called the type of A. If the constant ω can be chosen smaller than zero, A is called of negative type.

Let $(A, D(A))$ be a Hille-Yosida operator on X, and let $X_0 = \overline{D(A)}$; $D(A_0) = \{x \in D(A) : Ax \in X_0\}$, and let $A_0 : D(A_0) \subset X_0 \to X_0$ be the operator defined by $A_0 x = Ax$. The following result is well known.

Lemma 2.6 (see [12]). *The operator A_0 is the infinitesimal generator of a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0 with $\|T_0(t)\| \leq Me^{\omega t}$ for $t \geq 0$. Moreover, $\rho(A) \subset \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)|_{X_0}$, for $\lambda \in \rho(A)$.*

For the rest of paper we assume that $(A, D(A))$ is a Hille-Yosida operator of negative type on X. This implies that $0 \in \rho(A)$, that is, $A^{-1} \in \mathcal{L}(X)$. We note that the expression $\|x\|_{-1} = \|A_0^{-1}x\|$defines a norm on X_0. The completion of $(X_0, \|\cdot\|_{-1})$, denoted by X_{-1}, is called the extrapolation space of X_0 associated with A_0. We note that X is an intermediary space between X_0 and X_{-1} and that $X_0 \hookrightarrow X \hookrightarrow X_{-1}$(see [12]). Since $A_0^{-1}T_0(t) = T_0(t)A_0^{-1}$, we have that $\|T_0(t)x\|_{-1} \leq \|T_0(t)\|_{\mathcal{L}(X_0)}\|x\|_{-1}$ which implies that $T_0(t)$ has a unique bounded linear extension $T_{-1}(t)$ to X_{-1}. The operator family $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1}, called the extrapolated semigroup of $(T_0(t))_{t\geq 0}$. In the sequel, $(A_{-1}, D(A_{-1}))$ is the generator of $(T_{-1}(t))_{t\geq 0}$.

Lemma 2.7 (see [12]). *Under the previous conditions, the following properties are verified.*

(i) $D(A_{-1}) = X_0$ *and* $\|T_{-1}(t)\|_{\mathcal{L}(X_{-1})} = \|T_0(t)\|_{\mathcal{L}(X_0)}$ *for every* $t \geq 0$.

(ii) *The operator* $A_{-1} : X_0 \to X_{-1}$ *is the unique continuous extension of* $A_0 : D(A_0) \subset (X_0, \|\cdot\|) \to (X_{-1}, \|\cdot\|_{-1})$*, and* $\lambda - A_{-1}$ *is an isometry from* $(X_0, \|\cdot\|)$ *into* $(X_{-1}, \|\cdot\|_{-1})$.

(iii) *If* $\lambda \in \rho(A_0)$*, then* $(\lambda - A_{-1})^{-1}$ *exists and* $(\lambda - A_{-1})^{-1} \in \mathcal{L}(X_{-1})$*. In particular,* $\lambda \in \rho(A_{-1})$ *and* $R(\lambda, A_{-1})|_{X_0} = R(\lambda, A_0)$.

(iv) *The space* $X_0 = \overline{D(A)}$ *is dense in* $(X_{-1}, \|\cdot\|_{-1})$*. Thus, the extrapolation space* X_{-1} *is also the completion of* $(X, \|\cdot\|_{-1})$ *and* $X \hookrightarrow X_{-1}$*. Moreover,* A_{-1} *is an extension of* A *to* X_{-1}*. In particular, if* $\lambda \in \rho(A)$*, then* $R(\lambda, A_{-1})|_X = R(\lambda, A)$ *and* $R(\lambda, A_{-1})X = D(A)$.

Lemma 2.8 (see [12]). *Let* $f \in BC(\mathbb{R}; X)$*. Then the following properties are valid.*

(i) $T_{-1} * f(t) = \int_{-\infty}^{t} T_{-1}(t-s) f(s) ds \in X_0$*, for every* $t \in \mathbb{R}$.

(ii) $\|T_{-1} * f(t)\| \leq C e^{wt} \int_{-\infty}^{t} e^{-ws} \|f(s)\| ds$ *where* $C > 0$ *is independent of* t *and* f.

(iii) *The linear operator* $\Gamma : BC(\mathbb{R}, X) \to BC(\mathbb{R}, X_0)$ *defined by* $\Gamma(f)(t) = T_{-1} * f(t)$ *is continuous.*

(iv) $\lim_{t\to 0} \|T_{-1} * f(t) - \int_{-\infty}^{0} T_{-1}(-s) f(s) ds\| = 0$*, for every* $t \in \mathbb{R}$.

(v) $x(t) = T_{-1} * f(t)$ *is the unique bounded mild solution in* X_0 *of* $x'(t) = Ax(t) + f(t)$, $t \in \mathbb{R}$.

3. Existence Results

3.1. Almost Automorphic Solutions

The following property of convolution is needed to establish our result.

Lemma 3.1. *If* $f : \mathbb{R} \to Z$ *is an almost automorphic function and* Γf *is given by*

$$(\Gamma f)(t) := \int_{-\infty}^{t} T_{-1}(t-s) f(s) ds, \quad t \in \mathbb{R}, \tag{3.1}$$

then $\Gamma f \in AA(X_0)$.

Proof. Let $(s'_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. There exist a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$, and a continuous functions $g \in BC(\mathbb{R}, X)$ such that $f(t + s_n)$ converges to $g(t)$ and $g(t - s_n)$ converges to $f(t)$ for each $t \in \mathbb{R}$. Since

$$(\Gamma f)(t + s_n) := \int_{-\infty}^{t} T_{-1}(t-s) f(s + s_n) ds, \quad t \in \mathbb{R}, n \in \mathbb{N}. \tag{3.2}$$

Using the Lebesgue dominated convergence theorem, it follows that $\Gamma f(t + s_n)$ converges to $z(t) = \int_{-\infty}^{t} T_{-1}(t-s) g(s) ds$ for each $t \in \mathbb{R}$. Proceeding as previously, one can prove that $z(t - s_n)$ converges to $\Gamma f(t)$, for each $t \in \mathbb{R}$. This completes the proof. □

Theorem 3.2. *Assume that* $f : \mathbb{R} \times X_0 \to X$*is an almost automorphic function in* $t \in \mathbb{R}$ *for each* $x \in X_0$ *and assume that satisfies a L-Lipschitz condition in* $x \in X_0$ *uniformly in* $t \in \mathbb{R}$*. If* $CL < -\omega$, *where* $C > 0$ *is the constant in Lemma 2.8, then* (1.1) *has a unique almost automorphic mild solution which is given by*

$$y(t) = \int_{-\infty}^{t} T_{-1}(t-s) f(s, y(s)) ds, \quad t \in \mathbb{R}. \tag{3.3}$$

Proof. Let y be a function in $AA(X_0)$, from Lemma 2.3 the function $g(\cdot) := f(\cdot, y_\cdot)$ is in $AA(X)$. From Lemma 2.8 and taking into account Lemma 3.1, the equation

$$x'(t) = Ax(t) + g(t), \quad t \in \mathbb{R} \tag{3.4}$$

has a unique solution x in $AA(X_0)$, which is given by

$$x(t) = \Gamma_0 u(t) := \int_{-\infty}^{t} T_{-1}(t-s) f(s, y(s)) ds, \quad t \in \mathbb{R}. \tag{3.5}$$

It suffices now to show that the operator Γ_0 has a unique fixed point in $AA(X_0)$. For this, let u and v be in $AA(X_0)$, and we can infer that

$$\|\Gamma_0 u - \Gamma_0 v\|_\infty \le \frac{CL}{-\omega} \|u - v\|_\infty. \tag{3.6}$$

This proves that Γ_0 is a contraction, so by the Banach fixed point theorem there exists a unique $y \in AA(X_0)$ such that $\Gamma_0 y = y$. This completes the proof of the theorem. □

3.2. Pseudo-Almost Automorphic Solutions

To prove our next result, we need the following result.

Lemma 3.3. *Let* $f \in PAA(\mathbb{R}, X)$, *and let* Γf *be the function defined in Lemma 3.1. Then* $\Gamma f \in PAA(\mathbb{R}, X_0)$.

Proof. It is clear that $\Gamma f \in BC(\mathbb{R}, X_0)$. If $f = g + \Phi$, where $g \in AA(X)$ and $\Phi \in AA_0(\mathbb{R}, X)$. From Lemma 3.1 $\Gamma g \in AA(X_0)$. To complete the proof, we show that $\Gamma\Phi \in AA_0(\mathbb{R}, X_0)$. For $T > 0$ we see that

$$\int_{-T}^{T} e^{wt} \int_{-\infty}^{t} e^{-ws} \|\Phi(s)\| ds\, dt \le \frac{1}{-w} \int_{-\infty}^{-T} e^{-w(T+s)} \|\Phi(s)\| ds + \frac{1}{-w} \int_{-T}^{T} \|\Phi(s)\| ds. \tag{3.7}$$

The preceding estimates imply that

$$\frac{1}{2T} \int_{-T}^{T} \|\Gamma\Phi(t)\| dt \le \frac{C\|\Phi\|_\infty}{2Tw^2} + \frac{C}{-2Tw} \int_{-T}^{T} \|\Phi(t)\| dt. \tag{3.8}$$

The proof is now completed. □

Now, we are ready to state and prove the following result.

Theorem 3.4. *Assume that* $f : \mathbb{R} \times X_0 \to X$*is a pseudo-almost automorphic function and that there exists a bounded integrable function* $L_f : \mathbb{R} \to [0, \infty)$ *satisfying*

$$\|f(t,x) - f(t,y)\| \leq L_f(t)\|x - y\|, \quad t \in \mathbb{R},\ x,\ y \in X_0. \tag{3.9}$$

Then (1.1) *has a unique pseudo-almost automorphic (mild) solution.*

Proof. Let y be a function in $PAA(\mathbb{R}, X_0)$, from Lemma 2.4 the function $t \to f(t, y(t))$ belongs to $PAA(\mathbb{R}, X)$. From Lemmas 2.8 and 3.3, (3.4) has a unique solution in $PAA(\mathbb{R}, X_0)$ which is given by (3.5). Let u and v be in $PAA(\mathbb{R}, X_0)$, then we have

$$\begin{aligned}
\|\Gamma_0 u(t) - \Gamma_0 v(t)\| &\leq C\int_{-\infty}^{t} e^{w(t-s)} L_f(s)ds\|u - v\|_\infty \\
&\leq C\int_{-\infty}^{t} L_f(s)ds\|u - v\|_\infty \\
&\leq C\|L_f\|_1\|u - v\|_\infty,
\end{aligned} \tag{3.10}$$

hence,

$$\begin{aligned}
\left\|\left(\Gamma_0^2 u\right)(t) - \left(\Gamma_0^2 v\right)(t)\right\| &\leq C^2\left(\int_{-\infty}^{t} L_f(s)\left(\int_{-\infty}^{s} L_f(\tau)d\tau\right)ds\right)\|u - v\|_\infty \\
&\leq \frac{C^2}{2}\left(\int_{-\infty}^{t} L_f(\tau)d\tau\right)^2\|u - v\|_\infty \\
&\leq \frac{\left(C\|L_f\|_1\right)^2}{2}\|u - v\|_\infty.
\end{aligned} \tag{3.11}$$

In general, we get

$$\|(\Gamma_0^n u)(t) - (\Gamma_0^n v)(t)\| \leq \frac{\left(C\|L_f\|_1\right)^n}{n!}\|u - v\|_\infty. \tag{3.12}$$

Hence, since $(C\|L_f\|_1)^n/n! < 1$ for n sufficiently large, by the contraction principle Γ_0 has a unique fixed point $u \in PAA(\mathbb{R}, X_0)$. This completes the proof. □

A different Lipschitz condition is considered in the following result.

Theorem 3.5. *Let* $f : \mathbb{R} \times X_0 \to X$ *be a pseudo-almost automorphic function. Assume that* f *verifies the Lipschitz condition* (3.9) *with* L_f *a bounded continuous function. Let* $\mu(t) = \int_{-\infty}^{t} e^{w(t-s)} L_f(s)ds$. *If there is a constant* $\alpha > 0$ *such that* $C\mu(t) \leq \alpha < 1$ *for all* $t \in \mathbb{R}$ *where* $C > 0$ *is the constant in Lemma 2.8, then* (1.1) *has a unique pseudo-almost automorphic (mild) solution.*

Proof. We define the map Γ_0 on $PAA(\mathbb{R}, X_0)$ by (3.5). By Lemmas 2.4 and 3.3, Γ_0 is well defined. On the other hand, we can estimate

$$\|\Gamma_0 u(t) - \Gamma_0 v(t)\| \leq C\int_{-\infty}^{t} e^{w(t-s)} L_f(s)\|u(s) - v(s)\| ds \leq C\mu(t)\|u - v\|_{\infty}, \tag{3.13}$$

Therefore Γ_0 is a contraction. □

The following consequence is now immediate.

Corollary 3.6. *Let $f : \mathbb{R}\times X_0 \to X$ be a pseudo-almost automorphic function. Assume that f verifies the uniform Lipschitz condition:*

$$\|f(t,x) - f(t,y)\| \leq k\|x - y\|, \quad t \in \mathbb{R},\ x, y \in X_0. \tag{3.14}$$

If $Ck/ - \omega < 1$, where $C > 0$ is the constant in Lemma 2.8, then (1.1) *has a unique pseudo-almost automorphic (mild) solution.*

3.3. Application

In this section, we consider a simple application of our abstract results. We study the existence and uniqueness of pseudo-almost automorphic solutions for the following partial differential equation:

$$\begin{aligned} \partial_t u(t,x) &= \partial_x^2 u(t,x) - u(t,x) + \alpha u(t,x) \sin\frac{1}{\cos^2 t + \cos^2 \pi t} \\ &\quad + \alpha \max_{k\in\mathbb{Z}}\left\{\exp\left(-\left(t \pm k^2\right)^2\right)\right\} \sin u(t,x), \quad t \in \mathbb{R},\ x \in [0,\pi], \end{aligned} \tag{3.15}$$

with boundary initial conditions

$$u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}. \tag{3.16}$$

Let $X = C([0,\pi];\mathbb{R})$, and let the operator Abe defined on X by $Au = u'' - u$, with domain

$$D(A) = \left\{u \in X : u'' \in X,\ u(0) = u(\pi) = 0\right\}. \tag{3.17}$$

It is well known that A is a Hille-Yosida operator of type-1 with domain nondense (cf. [15]). Equation (3.15) can be rewritten as an abstract system of the form (1.1), where $u(t)(s) = u(t,s)$,

$$f(t,\phi)(s) = \alpha\phi(s) \sin\frac{1}{\cos^2 t + \cos^2 \pi t} + \alpha \max_{k\in\mathbb{Z}}\left\{\exp\left(-\left(t \pm k^2\right)^2\right)\right\} \sin\phi(s), \tag{3.18}$$

for all $\phi \in X$, $t \in \mathbb{R}$, $s \in [0, \pi]$ and $\alpha \in \mathbb{R}$. By [5, Example 2.5], f is a pseudo-almost automorphic function. If we assume that $|\alpha| < -\omega/2C$, then, by Corollary 3.6, (3.15) has a unique pseudo-almost automorphic mild solution.

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