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Research Article

Superstability of Generalized Multiplicative Functionals

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Let X be a set with a binary operation $\circ$ such that, for each $x, y, z \in X$, either $(x \circ y) \circ z = (x \circ z) \circ y$, or $z \circ (x \circ y) = x \circ (z \circ y)$. We show the superstability of the functional equation $g(x \circ y) = g(x)g(y)$. More explicitly, if $\varepsilon \geq 0$ and $f : X \to \mathbb{C}$ satisfies $|f(x \circ y) - f(x)f(y)| \leq \varepsilon$ for each $x, y \in X$, then $f(x \circ y) = f(x)f(y)$ for all $x, y \in X$, or $|f(x)| \leq (1 + \sqrt{1 + 4\varepsilon})/2$ for all $x \in X$. In the latter case, the constant $(1 + \sqrt{1 + 4\varepsilon})/2$ is the best possible.

1. Introduction

It seems that the stability problem of functional equations had been first raised by S. M. Ulam (cf. [1, Chapter VI]). "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism? (An ε-automorphism of G means a transformation f of G into itself such that $\rho(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon$ for all $x, y \in G$.)" D. H. Hyers [2] gave an affirmative answer to the problem: if $\varepsilon \geq 0$ and $f : E_1 \to E_2$ is a mapping between two real Banach spaces E_1 and E_2 satisfying $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in E_1$. If, in addition, the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear. This result is called *Hyers-Ulam stability* of the *additive* Cauchy equation $g(x + y) = g(x) + g(y)$. J. A. Baker [3, Theorem 1] considered stability of the multiplicative Cauchy equation $g(xy) = g(x)g(y)$: if $\varepsilon \geq 0$ and f is a complex valued function on a semigroup S such that $|f(xy) - f(x)f(y)| \leq \varepsilon$ for all $x, y \in S$, then f is multiplicative, or $|f(x)| \leq (1 + \sqrt{1 + 4\varepsilon}\ )/2$

for all $x \in S$. This result is called superstability of the functional equation $g(xy) = g(x)g(y)$. Recently, A. Najdecki [4, Theorem 1] proved the superstability of the functional equation $g(x\phi(y)) = g(x)g(y)$: if $\varepsilon \geq 0$, f is a (real or complex valued) functional from a commutative semigroup $(X, \circ)$, and ϕ is a mapping from X into itself such that $|f(x \circ \phi(y)) - f(x)f(y)| \leq \varepsilon$ for all $x, y \in X$, then $f(x \circ \phi(y)) = f(x)f(y)$ holds for all $x, y \in X$, or f is bounded.

In this paper, we show that superstability of the functional equation $g(x \circ y) = g(x)g(y)$ holds for a set X with a binary operation $\circ$ under an additional assumption.

2. Main Result

Theorem 2.1. *Let $\varepsilon \geq 0$ and X a set with a binary operation $\circ$ such that, for each $x, y, z \in X$, either*

$$(x \circ y) \circ z = (x \circ z) \circ y, \quad \text{or} \quad z \circ (x \circ y) = x \circ (z \circ y). \tag{2.1}$$

If $f : X \to \mathbb{C}$ satisfies

$$|f(x \circ y) - f(x)f(y)| \leq \varepsilon \quad (\forall x, y \in X), \tag{2.2}$$

then $f(x \circ y) = f(x)f(y)$ for all $x, y \in X$, or $|f(x)| \leq (1 + \sqrt{1 + 4\varepsilon}\,)/2$ for all $x \in X$. In the latter case, the constant $(1 + \sqrt{1 + 4\varepsilon}\,)/2$ is the best possible.

Proof. Let $f : X \to \mathbb{C}$ be a functional satisfying (2.2). Suppose that f is bounded. There exists a constant $C < \infty$ such that $|f(x)| \leq C$ for all $x \in X$. Set $M = \sup_{x \in X}|f(x)| < \infty$. By (2.2), we have, for each $x \in X$, $|f(x \circ x) - f(x)^2| \leq \varepsilon$, and therefore

$$|f(x)|^2 \leq \varepsilon + |f(x \circ x)| \leq \varepsilon + M. \tag{2.3}$$

Thus, $M^2 \leq \varepsilon + M$. Now it is easy to see that $M \leq (1 + \sqrt{1 + 4\varepsilon}\,)/2$. Consequently, if f is bounded, then $|f(x)| \leq (1 + \sqrt{1 + 4\varepsilon}\,)/2$ for all $x \in X$. The constant $(1 + \sqrt{1 + 4\varepsilon}\,)/2$ is the best possible since $g(x) = (1 + \sqrt{1 + 4\varepsilon}\,)/2$ for $x \in X$ satisfies $g(x)g(y) - g(x \circ y) = \varepsilon$ for each $x, y \in X$. It should be mentioned that the above proof is essentially due to P. Šemrl [5, Proof of Theorem 2.1 and Proposition 2.2] (cf. [6, Proposition 5.5]).

Suppose that $f : X \to \mathbb{C}$ is an unbounded functional satisfying the inequality (2.2). Since f is unbounded, there exists a sequence $\{z_k\}_{k \in \mathbb{N}} \subset X$ such that $\lim_{k \to \infty}|f(z_k)| = \infty$. Take $x, y \in X$ arbitrarily. Set

$$\begin{aligned} N_1 &= \{k \in \mathbb{N} : (x \circ y) \circ z_k = (x \circ z_k) \circ y\}, \\ N_2 &= \{k \in \mathbb{N} : z_k \circ (x \circ y) = x \circ (z_k \circ y)\}. \end{aligned} \tag{2.4}$$

By (2.1), $\mathbb{N} = N_1 \cup N_2$. Thus either N_1 or N_2 is an infinite subset of $\mathbb{N}$. First we consider the case when N_1 is infinite. Take $k_1 \in N_1$ arbitrarily. Choose $k_2 \in N_1$ with $k_1 < k_2$. Since N_1 is assumed to be infinite, for each $m > 2$ there exists $k_m \in N_1$ such that $k_{m-1} < k_m$. Then $\{z_{k_m}\}_{m\in\mathbb{N}}$ is a subsequence of $\{z_k\}_{k\in\mathbb{N}}$ with $k_m \in N_1$ for every $m \in \mathbb{N}$. By the choice of $\{z_k\}_{k\in\mathbb{N}}$, we have

$$\lim_{m\to\infty} \left| f(z_{k_m}) \right| = \infty. \tag{2.5}$$

Thus we may and do assume that $f(z_{k_m}) \neq 0$ for every $m \in \mathbb{N}$. By (2.2) we have, for each $w \in X$ and $m \in \mathbb{N}$, $|f(w \circ z_{k_m}) - f(w)f(z_{k_m})| \leq \varepsilon$. According to (2.5), we have

$$\left| \frac{f(w \circ z_{k_m})}{f(z_{k_m})} - f(w) \right| \leq \frac{\varepsilon}{\left| f(z_{k_m}) \right|} \longrightarrow 0 \quad \text{as } m \to \infty. \tag{2.6}$$

Consequently, we have, for each $w \in X$,

$$f(w) = \lim_{m\to\infty} \frac{f(w \circ z_{k_m})}{f(z_{k_m})}. \tag{2.7}$$

Since $k_m \in N_1$, we have $(x \circ y) \circ z_{k_m} = (x \circ z_{k_m}) \circ y$ for every $m \in \mathbb{N}$. Applying (2.7), we have

$$\begin{aligned} f(x \circ y) &= \lim_{m\to\infty} \frac{f((x \circ y) \circ z_{k_m})}{f(z_{k_m})} \\ &= \lim_{m\to\infty} \frac{f((x \circ z_{k_m}) \circ y)}{f(z_{k_m})} \\ &= \lim_{m\to\infty} \frac{f((x \circ z_{k_m}) \circ y) - f(x \circ z_{k_m})f(y)}{f(z_{k_m})} + \lim_{m\to\infty} \frac{f(x \circ z_{k_m})f(y)}{f(z_{k_m})}. \end{aligned} \tag{2.8}$$

By (2.2) and (2.5), we have

$$\lim_{m\to\infty} \left| \frac{f((x \circ z_{k_m}) \circ y) - f(x \circ z_{k_m})f(y)}{f(z_{k_m})} \right| \leq \lim_{m\to\infty} \frac{\varepsilon}{\left| f(z_{k_m}) \right|} = 0. \tag{2.9}$$

Consequently, we have by (2.8) and (2.7)

$$f(x \circ y) = \lim_{m\to\infty} \frac{f(x \circ z_{k_m})f(y)}{f(z_{k_m})} = \lim_{m\to\infty} \frac{f(x \circ z_{k_m})}{f(z_{k_m})} f(y) = f(x)f(y). \tag{2.10}$$

Next we consider the case when N_2 is infinite. By a quite similar argument as in the case when N_1 is infinite, we see that there exists a subsequence $\{z_{k_n}\}_{n\in\mathbb{N}} \subset \{z_k\}_{k\in\mathbb{N}}$ such that $k_n \in N_2$ for every $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \left| f(z_{k_n}) \right| = \infty. \tag{2.11}$$

In the same way as in the proof of (2.7), we have

$$f(w) = \lim_{n\to\infty} \frac{f(z_{k_n} \circ w)}{f(z_{k_n})}, \tag{2.12}$$

for every $w \in X$. According to (2.2) and (2.11), we have

$$\lim_{n\to\infty} \left| \frac{f(x \circ (z_{k_n} \circ y)) - f(x) f(z_{k_n} \circ y)}{f(z_{k_n})} \right| \leq \lim_{n\to\infty} \frac{\varepsilon}{|f(z_{k_n})|} = 0. \tag{2.13}$$

Since $z_{k_n} \circ (x \circ y) = x \circ (z_{k_n} \circ y)$ for every $n \in \mathbb{N}$, (2.11) and (2.12) show that

$$\begin{aligned} f(x \circ y) &= \lim_{n\to\infty} \frac{f(z_{k_n} \circ (x \circ y))}{f(z_{k_n})} \\ &= \lim_{n\to\infty} \frac{f(x \circ (z_{k_n} \circ y))}{f(z_{k_n})} \\ &= \lim_{n\to\infty} \frac{f(x \circ (z_{k_n} \circ y)) - f(x) f(z_{k_n} \circ y)}{f(z_{k_n})} + \lim_{n\to\infty} \frac{f(x) f(z_{k_n} \circ y)}{f(z_{k_n})} \\ &= \lim_{n\to\infty} \frac{f(x) f(z_{k_n} \circ y)}{f(z_{k_n})} \\ &= f(x) \lim_{n\to\infty} \frac{f(z_{k_n} \circ y)}{f(z_{k_n})} \\ &= f(x) f(y). \end{aligned} \tag{2.14}$$

Consequently, if f is unbounded, then $f(x \circ y) = f(x)f(y)$ for all $x, y \in X$. □

Remark 2.2. Let ϕ be a mapping from a commutative semigroup X into itself. We define the binary operation $\circ$ by $x \circ y = x\phi(y)$ for each $x, y \in X$. Then $\circ$ satisfies (2.1) since

$$(x \circ y) \circ z = x\phi(y)\phi(z) = x\phi(z)\phi(y) = (x \circ z) \circ y, \tag{2.15}$$

for all $x, y, z \in X$. Therefore, Theorem 2.1 is a generalization of Najdecki [4, Theorem 1] and Baker [3, Theorem 1].

Remark 2.3. Let X be a set, and $f : X \to \mathbb{C}$. Suppose that X has a binary operation $\circ$ such that, for each $x, y, z \in X$, either

$$f((x \circ y) \circ z) = f((x \circ z) \circ y), \quad \text{or} \quad f(z \circ (x \circ y)) = f(x \circ (z \circ y)). \tag{2.16}$$

If f satisfies (2.2) for some $\varepsilon \geq 0$, then by quite similar arguments to the proof of Theorem 2.1, we can prove that $f(x \circ y) = f(x)f(y)$ for all $x, y \in X$, or $|f(x)| \leq (1 + \sqrt{1+4\varepsilon})/2$ for all $x \in X$. Thus, Theorem 2.1 is still true under the weaker condition (2.16) instead of (2.2). This was pointed out by the referee of this paper. The condition (2.16) is related to that introduced by Kannappan [7].

Example 2.4. Let φ and ψ be mappings from a semigroup X into itself with the following properties.

(a) $\varphi(xy) = \varphi(x)\varphi(y)$ for every $x, y \in X$.

(b) $\psi(X) \subset \{x \in X : \varphi(x) = x\}$.

(c) $\psi(x)\psi(y) = \psi(y)\psi(x)$ for every $x, y \in X$.

If we define $x \circ y = \varphi(x)\psi(y)$ for each $x, y \in X$, then we have $(x \circ y) \circ z = (x \circ z) \circ y$ for every $x, y, z \in X$. In fact, if $x, y, z \in X$, then we have

$$\begin{aligned}
(x \circ y) \circ z &= \varphi(x \circ y)\psi(z) \\
&= \varphi(\varphi(x)\psi(y))\psi(z) \\
&\overset{\text{by (a)}}{=} \varphi^2(x)\varphi(\psi(y))\psi(z) \\
&\overset{\text{by (b)}}{=} \varphi^2(x)\psi(y)\psi(z) \\
&\overset{\text{by (c)}}{=} \varphi^2(x)\psi(z)\psi(y) \\
&\overset{\text{by (b)}}{=} \varphi^2(x)\varphi(\psi(z))\psi(y) \\
&\overset{\text{by (a)}}{=} \varphi(\varphi(x)\psi(z))\psi(y) \\
&= \varphi(x \circ z)\psi(y) \\
&= (x \circ z) \circ y
\end{aligned} \tag{2.17}$$

as claimed.

Let φ be a ring homomorphism from $\mathbb{C}$ into itself, that is, $\varphi(z + w) = \varphi(z) + \varphi(w)$ and $\varphi(zw) = \varphi(z)\varphi(w)$ for each $z, w \in \mathbb{C}$. It is well known that there exist infinitely many such homomorphisms on $\mathbb{C}$ (cf. [8, 9]). If φ is not identically 0, then we see that $\varphi(q) = q$ for every $q \in \mathbb{Q}$, the field of all rational real numbers. Thus, if we consider the case when $X = \mathbb{C}$, φ a nonzero ring homomorphism, and $\psi : X \to \mathbb{Q}$, then (X, φ, ψ) satisfies the conditions (a), (b), and (c).

If we define $x * y = y \circ x$ for each $x, y \in X$, then $z * (x * y) = x * (z * y)$ holds for every $x, y, z \in X$. In fact,

$$z * (x * y) = (x * y) \circ z = (y \circ x) \circ z = (y \circ z) \circ x = x * (z * y). \tag{2.18}$$

Example 2.5. Let $X = \mathbb{C} \times \{0, 1\}$, and, let $\varphi, \psi : \mathbb{C} \to \mathbb{C}$. We define the binary operation $\circ$ by

$$(x, a) \circ (y, b) = \begin{cases} (x\psi(y), 0), & \text{if } a = b = 0, \\ (\varphi(x)y, 1), & \text{if } a = b = 1, \\ (0, 0), & \text{if } a \neq b, \end{cases} \tag{2.19}$$

for each $(x, a), (y, b) \in X$. Then $\circ$ satisfies the condition (2.1). In fact, let $(x, a), (y, b), (z, c) \in X$.

(a) If $a = b = c = 0$, then we have

$$((x, a) \circ (y, b)) \circ (z, c) = (x\psi(y)\psi(z), 0) = ((x, a) \circ (z, c)) \circ (y, b). \tag{2.20}$$

(b) If $a = b = c = 1$, then

$$\begin{aligned} (z, c) \circ ((x, a) \circ (y, b)) &= (\varphi(z)\varphi(x)y, 1) \\ &= (x, a) \circ ((z, c) \circ (y, b)). \end{aligned} \tag{2.21}$$

(c) If $a = b = 0$ and $c = 1$, then

$$((x, a) \circ (y, b)) \circ (z, c) = (0, 0) = ((x, a) \circ (z, c)) \circ (y, b). \tag{2.22}$$

(d) If $a = b = 1$ and $c = 0$, then

$$\begin{aligned} (z, c) \circ ((x, a) \circ (y, b)) &= (0, 0) = (x, a) \circ ((z, c) \circ (y, b)), \\ ((x, a) \circ (y, b)) \circ (z, c) &= (0, 0) = ((x, a) \circ (z, c)) \circ (y, b). \end{aligned} \tag{2.23}$$

(e) If $a \neq b$, then we have

$$((x, a) \circ (y, b)) \circ (z, c) = (0, 0) = ((x, a) \circ (z, c)) \circ (y, b). \tag{2.24}$$

Therefore, $\circ$ satisfies the condition (2.1). On the other hand, if $a = b = c = 0$, then

$$\begin{aligned} (z, c) \circ ((x, a) \circ (y, b)) &= (z\psi(x\psi(y)), 0), \\ (x, a) \circ ((z, c) \circ (y, b)) &= (x\psi(z\psi(y)), 0). \end{aligned} \tag{2.25}$$

Thus, $(z, c) \circ ((x, a) \circ (y, b)) \neq (x, a) \circ ((z, c) \circ (y, b))$ in general. In the same way, we see that if $a = b = c = 1$, then $((x, a) \circ (y, b)) \circ (z, c) = ((x, a) \circ (z, c)) \circ (y, b)$ need not to be true.

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