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Research Article

Derivatives of Integrating Functions for Orthonormal Polynomials with Exponential-Type Weights

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Let $w_\rho(x) := |x|^\rho \exp(-Q(x))$, $\rho > -1/2$, where $Q \in C^2 : (-\infty, \infty) \to [0, \infty)$ is an even function. In 2008 we have a relation of the orthonormal polynomial $p_n(w_\rho^2; x)$ with respect to the weight $w_\rho^2(x)$; $p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) - 2\rho_n p_n(x)/x$, where $A_n(x)$ and $B_n(x)$ are some integrating functions for orthonormal polynomials $p_n(w_\rho^2; x)$. In this paper, we get estimates of the higher derivatives of $A_n(x)$ and $B_n(x)$, which are important for estimates of the higher derivatives of $p_n(w_\rho^2; x)$.

1. Introduction and Results

Let $\mathbb{R} = (-\infty, \infty)$. Let $Q \in C^2 : \mathbb{R} \to \mathbb{R}^+ = [0, \infty)$ be an even function, and let $w(x) = \exp(-Q(x))$ be such that $\int_0^\infty x^n w^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$. For $\rho > -1/2$, we set

$$w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}. \tag{1.1}$$

Then we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree n with respect to $w_\rho^2(x)$. That is,

$$\int_{-\infty}^{\infty} p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x) dx = \delta_{mn} \quad \text{(Kronecker's delta)},$$
$$p_{n,\rho}(x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_{n,\rho} > 0. \tag{1.2}$$

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$ for $0 < x < y$. For any two sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ of nonzero real numbers (or functions), we write $b_n \lesssim c_n$ if there exists a constant $C > 0$ independent of n (or x) such that $b_n \leq Cc_n$ for n large enough. We write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote the class of polynomials of degree at most n by $\mathcal{P}_n$.

Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t, and polynomials of degree at most n. The same symbol does not necessarily denote the same constant in different occurrences.

We will be interested in the following subclass of weights from [1].

Definition 1.1. Let $Q : \mathbb{R} \to \mathbb{R}^+$ be even and satisfy the following properties.

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.

(c)

$$\lim_{x \to \infty} Q(x) = \infty. \tag{1.3}$$

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{1.4}$$

is quasi-increasing in $(0, \infty)$ with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}^+ \setminus \{0\}. \tag{1.5}$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{\left|Q'(x)\right|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}. \tag{1.6}$$

Then we write $w \in \mathcal{F}(C^2)$. If there also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{\left|Q'(x)\right|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J, \tag{1.7}$$

then we write $w \in \mathcal{F}(C^2+)$.

In the following we introduce useful notations.

(a) Mhaskar-Rahmanov-Saff (MRS) numbers a_x are defined as the positive roots of the following equations:

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1 - u^2)^{1/2}} du, \quad x > 0. \tag{1.8}$$

(b) Let

$$\eta_x = (xT(a_x))^{-2/3}, \quad x > 0. \tag{1.9}$$

(c) The function $\varphi_u(x)$ is defined as follows:

$$\varphi_u(x) = \begin{cases} \dfrac{a_{2u}^2 - x^2}{u[(a_u + x + a_u\eta_u)(a_u - x + a_u\eta_u)]^{1/2}}, & |x| \le a_u, \\ \varphi_u(a_u), & a_u < |x|. \end{cases} \tag{1.10}$$

In the rest of this paper we often denote $p_{n,\rho}(x)$ simply by $p_n(x)$. Let $\rho_n = \rho$ if n is odd, $\rho_n = 0$ otherwise and define the integrating functions $A_n(x)$ and $B_n(x)$ with respect to $p_n(x)$ as follows:

$$\begin{aligned} A_n(x) &:= 2b_n \int_{-\infty}^{\infty} p_n^2(u)\overline{Q(x,u)}w_\rho^2(u)du, \\ B_n(x) &:= 2b_n \int_{-\infty}^{\infty} p_n(u)p_{n-1}(u)\overline{Q(x,u)}w_\rho^2(u)du, \end{aligned} \tag{1.11}$$

where $\overline{Q(x,u)} = (Q'(x) - Q'(u))/(x-u)$ and $b_n = \gamma_{n-1}/\gamma_n$. Then in [2, Theorem 4.1] we have a relation of the orthonormal polynomial $p_n(x)$ with respect to the weight $w_\rho^2(x)$:

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) - 2\rho_n\frac{p_n(x)}{x}, \quad \rho_n = \begin{cases} \rho, & n \text{ is odd}, \\ 0, & n \text{ is even}, \end{cases} \tag{1.12}$$

and in [2, Theorem 4.2] we already have the estimates of the integrating functions $A_n(x)$ and $B_n(x)$ with respect to $p_n(x)$. So, in this paper we will estimate the higher derivatives of $A_n(x)$ and $B_n(x)$ for the estimates of the higher derivatives of $p_n(w_\rho^2; x)$, because the higher derivatives of $p_{n,\rho}(x)$ play an important role in approximation theory such as investigating convergence of Hermite-Fejér and Hermite interpolation based on the zeros of $p_n(w_\rho^2; x)$ (see [3, 4]).

To estimate of the higher derivatives of $A_n(x)$ and $B_n(x)$ we need further assumptions for $Q(x)$ as follows.

Definition 1.2. Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$, and let ν be a positive integer. Assume that $Q(x)$ is ν−times continuously differentiable on $\mathbb{R}$ and satisfies the followings.

(a) $Q^{(\nu+1)}(x)$ exists and $Q^{(i)}(x)$, $i = 0, 1, \ldots, \nu + 1$ are nonnegative for $x > 0$.

(b) There exist positive constants $C_i > 0$ such that for $x \in \mathbb{R} \setminus \{0\}$

$$\left|Q^{(i+1)}(x)\right| \le C_i \left|Q^{(i)}(x)\right| \frac{\left|Q'(x)\right|}{Q(x)}, \quad i = 1, \ldots \nu. \tag{1.13}$$

(c) There exist constants $0 \le \delta < 1$ and $c_1 > 0$ such that on $(0, c_1]$

$$Q^{(\nu+1)}(x) \le C\left(\frac{1}{x}\right)^{\delta}. \tag{1.14}$$

Then we write $w(x) \in \mathcal{F}_\nu(C^2+)$.

Let ν be a positive integer. Define for $m + \alpha - \nu > 0$, $m \ge 0$, $l \ge 1$, and $\alpha \ge 0$,

$$Q_{l,\alpha,m}(x) := |x|^m \left(\exp_l\left(|x|^\alpha\right) - \alpha^* \exp_l(0)\right), \tag{1.15}$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$ and define

$$Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1, \quad \alpha > 1. \tag{1.16}$$

Here we let $\exp_0(x) := x$ and for $l \ge 1$, $\exp_l(x) := \exp(\exp(\cdots(\exp(x))\cdots))$ denotes the lth iterated exponential. In particular, $\exp_l(x) = \exp(\exp_{l-1}(x))$. Then $\exp(-Q_{l,\alpha,m}(x))$ and $\exp(-Q_\alpha(x))$ are typical examples of $\mathcal{F}_\nu(C^2+)$ (see [5]).

In the following we improve the inequality (4.3) in [2, Theorem 4.2].

Theorem 1.3. *Let $\rho > -1/2$ and $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. Additionally assume that $Q''(x)$ is nondecreasing. Then for $|x| \le \varepsilon a_n$ with $0 < \varepsilon < 1/2$ one has*

$$|B_n(x)| < \lambda(\varepsilon, n) A_n(x), \tag{1.17}$$

where

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \lambda(\varepsilon, n) = 0. \tag{1.18}$$

In this paper our main theorem is as follows.

Theorem 1.4. *Let $\rho > -1/2$ and $w(x) = \exp(-Q(x)) \in \mathcal{F}_\nu(C^2+)$ for positive integer $\nu \ge 2$. Assume that $1 + 2\rho - \delta \ge 0$ for $\rho < 0$ and*

$$a_n \lesssim n^{1/(1+\nu-\delta)}, \tag{1.19}$$

where $0 \le \delta < 1$ is defined in (1.14).

(a) *If $Q'(x)/Q(x)$ is quasi-increasing on $[c_2, \infty)$, then one has for $|x| \le a_n(1 + \eta_n)$ and $j = 0, \ldots, \nu - 1$*

$$\left|A_n^{(j)}(x)\right| \lesssim A_n(x)\left(\frac{T(a_n)}{a_n}\right)^j, \qquad \left|B_n^{(j)}(x)\right| \lesssim A_n(x)\left(\frac{T(a_n)}{a_n}\right)^j. \tag{1.20}$$

Moreover, for any $0 < \varepsilon < 1/2$ *there exists* $\varepsilon^*(\varepsilon, n) > 0$ *such that for* $|x| \le \varepsilon a_n$ *and* $j = 1, \dots, \nu - 1$,

$$\left|A_n^{(j)}(x)\right| \le \varepsilon^*(\varepsilon, n) A_n(x) \left(\frac{n}{a_n}\right)^j, \qquad \left|B_n^{(j)}(x)\right| \le \varepsilon^*(\varepsilon, n) A_n(x) \left(\frac{n}{a_n}\right)^j, \tag{1.21}$$

with $\varepsilon^*(\varepsilon, n) \to 0$ *as* $n \to \infty$.

(b) *If* $Q^{(\nu+1)}(x)$ *is non-decreasing on* $[c_2, \infty)$*, then one has* (1.20) *and* (1.21) *for the respective ranges of* x.

(c) *If there exists a constant* $0 \le \delta < 1$ *such that* $Q^{(\nu+1)}(x) \le C(1/x)^{\delta}$ *on* $[c_2, \infty)$*, then one has* (1.20) *and* (1.21) *for the respective ranges of* x.

The examples satisfying the conditions (a), (b), or (c) of Theorem 1.4 are given in [5].

Remark 1.5. Under the assumptions of Theorem 1.4, we have from [2, Theorem 4.2] that there exists $C, n_0 > 0$ such that for $n \ge n_0$ and $|x| \le a_n(1 + L\eta_n)$,

$$\frac{A_n(x)}{2b_n} \sim \varphi_n(x)^{-1}\left(a_n^2(1 + 2L\eta_n)^2 - x^2\right)^{-1/2}, \quad |B_n(x)| \lesssim A_n(x), \tag{1.22}$$

because $w(x) = \exp(-Q(x)) \in \mathcal{F}_\nu(C^2+)$ for positive integer $\nu \ge 1$ and $1 + 2\rho - \delta \ge 0$ for $\rho < 0$.

In addition, for our future work we estimate a_t and $T(a_t)$ using $\lambda = C_1$ in (1.6) for the weight class $\mathcal{F}(C^2+)$.

Theorem 1.6. *Let* $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$*, and we assume*

$$\frac{Q''(x)}{|Q'(x)|} \le \lambda \frac{\left|Q'(x)\right|}{Q(x)}, \quad |x| \ge b > 0, \tag{1.23}$$

where $b > 0$ *is large enough.*

(a) *Assume that* $T(x)$ *is unbounded. Then for any* $\eta > 0$ *there exists* $C(\eta) > 0$ *such that for* $t \ge 1$,

$$a_t \le C(\eta) t^{\eta}. \tag{1.24}$$

(b) *Suppose that there exist constants* $\eta > 0$ *and* $C_2 > 0$ *such that* $a_t \le C_2 t^{\eta}$*. Then there exists a constant* C *depending only on* λ, η*, and* C_2 *such that for* $a_t \ge 1$*, if* $\lambda > 1$

$$T(a_t) \le C t^{2(\eta+\lambda-1)/(\lambda+1)}, \tag{1.25}$$

and if $0 < \lambda \le 1$,

$$T(a_t) \le C t^{\eta}. \tag{1.26}$$

Remark 1.7. (a) Levin and Lubinsky showed the following [1, Lemma 3.7]: there exists $C > 0$ such that for some $\varepsilon > 0$, and for large enough t,

$$T(a_t) \le Ct^{2-\varepsilon}. \tag{1.27}$$

If from (1.25) and (1.26) we set for any $0 < \eta < 2$

$$\varepsilon = \begin{cases} 2-\eta, & 0 < \lambda \le 1, \\ \dfrac{2(2-\eta)}{(\lambda+1)}, & \lambda > 1, \end{cases} \tag{1.28}$$

then we have (1.27) in Levin and Lubinsky's lemma.

(b) If $T(x)$ is unbounded, then (1.19) is trivially satisfied by (1.24).

2. Proof of Theorems

In this section we will prove the theorems of Section 1.

Lemma 2.1. *Let $\rho > -1/2$ and let $w(x) \in \mathcal{F}(C^2)$. Then uniformly for $n \ge 1$,*

(a)

$$\sup_{x\in\mathbb{R}} \left|p_{n,\rho}(x)w(x)\right| \left(|x| + \frac{a_n}{n}\right)^{\rho} \left|x^2 - a_n^2\right|^{1/4} \sim 1. \tag{2.1}$$

(b)

$$\sup_{x\in\mathbb{R}} \left|p_{n,\rho}(x)w(x)\right| \left(|x| + \frac{a_n}{n}\right)^{\rho} \sim a_n^{-1/2}(nT(a_n))^{1/6}. \tag{2.2}$$

(c) *Markov inequality. Let $0 < p \le \infty$. For any polynomial $P \in \mathcal{P}_n$*

$$\left\|(P'w)(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\mathbb{R})} \lesssim \frac{nT(a_n)^{1/2}}{a_n} \left\|(Pw)(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right\|_{L_p(\mathbb{R})}. \tag{2.3}$$

(d) *Let $\beta \in \mathbb{R}$, $0 < p \le \infty$, and $r > 1$. Then there exist positive constants L, δ, and C_2 such that for any polynomial $P \in \mathcal{P}_n$*

$$\begin{aligned} &\left\|(Pw)(x)\left(|x| + \frac{a_n}{n}\right)^{\beta}\right\|_{L_p(a_{rn}\le|x|)} \\ &\quad \lesssim \exp\left(-C_2 n^{\delta}\right) \left\|(Pw)(x)\left(|x| + \frac{a_n}{n}\right)^{\beta}\right\|_{L_p(La_n/n\le|x|\le a_n(1-L\eta_n))}. \end{aligned} \tag{2.4}$$

Proof. (a) follows from [2, Theorem 2.3]. (b) follows from [2, Theorem 2.4]. (c) follows form [6, Theorem 2.1(b)]. (d) follows form [6, Theorem 2.3]. □

Lemma 2.2. *Let $\rho > -1/2$ and let $w(x) \in \mathcal{F}(C^2)$. Then one has for $c > 0$,*

$$\int_{0\le u\le c} (p_n w_\rho)^2(u)du \lesssim \frac{1}{a_n}. \tag{2.5}$$

Proof. For $\rho \ge 0$, the results are immediate from Lemma 2.1(a). So we assume $-1/2 < \rho < 0$. First we see

$$\begin{aligned}\int_{0\le u\le a_n/n} (p_n w_\rho)^2(u)du &= \int_{0\le u\le a_n/n} (p_n w)^2(u)\left(|u| + \frac{a_n}{n}\right)^{2\rho} \frac{|u|^{2\rho}}{(|u| + a_n/n)^{2\rho}}du \\ &\le C\frac{1}{a_n}\int_{0\le u\le a_n/n} \frac{|u|^{2\rho}}{(|u| + a_n/n)^{2\rho}}du \\ &\le C\frac{1}{a_n}\left(\frac{n}{a_n}\right)^{2\rho}\int_{0\le u\le a_n/n} |u|^{2\rho}du \\ &\le C\frac{1}{a_n}\left(\frac{n}{a_n}\right)^{2\rho}\left(\frac{a_n}{n}\right)^{1+2\rho} \\ &\le C\frac{1}{n},\end{aligned} \tag{2.6}$$

because we know that $a_n = o(n)$ from [1, Lemma 3.5(c)]. Next we see by Lemma 2.1(a)

$$\int_{a_n/n\le u\le c} (p_n w_\rho)^2(u)du \le C\frac{1}{a_n}. \tag{2.7}$$

Therefore, we have the result. □

Lemma 2.3. *Let $\rho > -1/2$ and let $w(x) \in \mathcal{F}(C^2)$. Then*

(a) *one has*

$$\int_{0\le u\le\infty} (p_n w)^2(u)\left(|u| + \frac{a_n}{n}\right)^{2\rho} Q'(u)du \sim \frac{n}{a_n}, \tag{2.8}$$

(b) *for $x \in [0, a_n/2]$ one has*

$$Q'(x) \le C\frac{n}{a_n}\left(\frac{x}{a_n}\right)^{\Lambda-1}. \tag{2.9}$$

Proof. (a) It is from [2, Lemma 4.3(d)]. (b) It is from [1, Lemma 3.8 (3.42)]. □

Proof of Theorem 1.3. Since $B_n(x)$ is an odd function, we prove only for $0 \le x \le \varepsilon a_n$. Let $\theta := \varepsilon^{(\Lambda-1)/2\Lambda}$. Then we have the following two lemmas.

Lemma 2.4. *Uniformly for θ and n*

$$\left|\int_{|u|\le\theta a_n} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \left(\frac{1}{n\theta}+1\right)\theta^{\Lambda-1}\frac{n}{a_n^2}. \tag{2.10}$$

Proof. For $|u| \le \theta a_n$, we have by Lemma 2.1(a)

$$p_n^2(u)w_\rho^2(u) \lesssim \frac{1}{\sqrt{a_n^2-(\theta a_n)^2}}\frac{|u|^{2\rho}}{(|u|+a_n/n)^{2\rho}} \lesssim \frac{1}{a_n}\frac{|u|^{2\rho}}{(|u|+a_n/n)^{2\rho}}. \tag{2.11}$$

Since $Q''(x)$ is nondecreasing and $1-(1/2)^{(\Lambda+1)/2\Lambda} \le (\theta-\varepsilon)/\theta \le 1$, we have using Lemma 2.3(b):

$$\overline{Q(x,u)} \le \frac{Q'(\theta a_n)-Q'(x)}{\theta a_n - x} \lesssim \frac{Q'(\theta a_n)}{(\theta-\varepsilon)a_n} \lesssim \theta^{\Lambda-2}\frac{n}{a_n^2}. \tag{2.12}$$

Moreover we know that for $\rho > -1/2$,

$$\int_0^{\theta a_n}\frac{|u|^{2\rho}}{(|u|+a_n/n)^{2\rho}}dx = \int_{|u|\le a_n/n} + \int_{a_n/n\le|u|\le\theta a_n} \lesssim \frac{a_n}{n}+\theta a_n. \tag{2.13}$$

Therefore, we have

$$\left|\int_{|u|\le\theta a_n} p_n^2(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \left(\frac{1}{n\theta}+1\right)\theta^{\Lambda-1}\frac{n}{a_n^2}. \tag{2.14}$$

Consequently, we have the result using Cauchy-Schwartz inequality

$$\left|\int_{|u|\le\theta a_n} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \left(\frac{1}{n\theta}+1\right)\theta^{\Lambda-1}\frac{n}{a_n^2}. \tag{2.15}$$

□

Lemma 2.5. *Uniformly for $\theta = \varepsilon^{(\Lambda-1)/2\Lambda}$ and for n*

$$\left|\int_{\theta a_n\le|u|\le a_{2n}} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \left(\varepsilon^{(1-1/\Lambda)(\Lambda-1)}+\varepsilon^{1/\Lambda}\right)\frac{n}{a_n^2}. \tag{2.16}$$

Proof. For $\theta a_n \le |u| \le a_{2n}$, we have similarly to [2, (4.6)]

$$\begin{aligned}\left|\overline{Q(x,u)} - \overline{Q(x,-u)}\right| &= 2\left|\frac{uQ'(x) - xQ'(u)}{x^2 - u^2}\right| \\ &\lesssim \frac{a_n\left|Q'(\varepsilon a_n)\right| + \varepsilon a_n\left|Q'(u)\right|}{(\theta a_n)^2} \\ &\lesssim \varepsilon^{(1-1/\Lambda)(\Lambda-1)}\frac{n}{a_n^2} + \frac{\varepsilon^{1/\Lambda}}{a_n}\left|Q'(u)\right|\end{aligned} \tag{2.17}$$

(see Lemma 2.3(b)). Therefore, we have by Lemma 2.3(a),

$$\begin{aligned}&\left|\int_{\theta a_n \le |u| \le a_{2n}} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \\ &\quad\le \int_{\theta a_n \le |u| \le a_{2n}} \left|p_n(u)p_{n-1}(u)w_\rho^2(u)\right|\left|\overline{Q(x,u)} - \overline{Q(x,-u)}\right|du \\ &\quad\lesssim \varepsilon^{(1-1/\Lambda)(\Lambda-1)}\frac{n}{a_n^2}\int_{\theta a_n \le |u| \le a_{2n}} \left|p_n(u)p_{n-1}(u)\right|w_\rho^2(u)du \\ &\qquad+ \frac{\varepsilon^{1/\Lambda}}{a_n}\int_{\theta a_n \le |u| \le a_{2n}} \left|p_n(u)p_{n-1}(u)\right|w_\rho^2(u)\left|Q'(u)\right|du \\ &\quad\lesssim \varepsilon^{(1-1/\Lambda)(\Lambda-1)}\frac{n}{a_n^2} + \varepsilon^{1/\Lambda}\frac{n}{a_n^2}.\end{aligned} \tag{2.18}$$

Here we used Lemma 2.1(b). □

Since for a constant $C > 0$

$$\left|\int_{a_{2n} \le |u|} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim O\left(e^{-n^C}\right), \tag{2.19}$$

(see [2, page 233]), there exists $\lambda(n) > 0$ such that

$$\left|\int_{a_{2n} \le |u|} p_n(u)p_{n-1}(u)w_\rho^2(u)\overline{Q(x,u)}du\right| \lesssim \lambda(n)\frac{n}{a_n^2}, \tag{2.20}$$

and $\lambda(n) \to 0$ as $n \to \infty$. We know from [2, Lemma 4.7] that $b_n = \gamma_{n-1}/\gamma_n \sim a_n$. From (1.22) we have $A_n(x)/b_n \sim n/a_n^2$ for $|x| \le \varepsilon a_n$ and from the preceding considerations and the definition of $B_n(x)$ it follows that for $|x| \le \varepsilon a_n$

$$\frac{|B_n(x)|}{b_n} \lesssim \frac{\lambda(\varepsilon,n)n}{a_n^2} \sim \frac{\lambda(\varepsilon,n)A_n(x)}{b_n}, \tag{2.21}$$

where for some positive constant $C > 0$

$$\lambda(\varepsilon, n) := C \cdot \max\left\{ \left(\frac{1}{n\theta} + 1\right)\theta^{\Lambda-1}, \varepsilon^{(1-1/\Lambda)(\Lambda-1)}, \varepsilon^{1/\Lambda}, \lambda(n) \right\}. \tag{2.22}$$

Consequently, (1.17) is proved, and we can obtain that $\lim_{\varepsilon\to 0}\lim_{n\to\infty}\lambda(\varepsilon, n) = 0$. Now, we have for $|x| \le \varepsilon a_n$

$$A_n(x) \sim \frac{n}{a_n}, \qquad |B_n(x)| < \lambda(\varepsilon, n)\frac{n}{a_n}. \tag{2.23}$$

□

Proof of Theorem 1.4. First, we see that for $1 \le j \le \nu - 1$

$$A_n^{(j)}(x) = 2b_n \int_{-\infty}^{\infty} (p_n w_\rho)^2(u) \frac{d^j}{dx^j}\overline{Q(x,u)}du. \tag{2.24}$$

We split proof of (1.20) into some lemmas as follows:

(1) Lemma 2.6 is for $0 \le x \le a_n(1+\eta_n)$, $a_{4n} \le u$, and $1 \le j \le \nu - 1$;

(2) Lemma 2.9 is for $a_n/2 \le x \le a_n(1+\eta_n)$, $0 \le u \le a_{4n}$, and $j = \nu - 1$;

(3) Lemma 2.10 is for $0 \le x \le a_n/2$, $0 \le u \le a_{4n}$, and $j = \nu - 1$;

(4) Lemma 2.11 is for $0 \le x \le a_n(1+\eta_n)$, $0 \le u \le a_{4n}$, and $1 \le j \le \nu - 2$;

on the other hand, (1.21) will be proved by Lemmas 2.13 and 2.6.

For $1 \le j \le \nu - 1$ there exists η between u and x such that

$$\begin{aligned} \frac{d^j}{dx^j}\overline{Q(x,u)} &= \frac{j!}{(x-u)^{j+1}}\left(\sum_{k=0}^{j}(-1)^k\frac{Q^{(j+1-k)}(x)}{(j-k)!}(x-u)^{j-k} + (-1)^{j+1}Q'(u)\right) \\ &= \frac{Q^{(j+1)}(x) - Q^{(j+1)}(\eta)}{x-u}. \end{aligned} \tag{2.25}$$

Then for $x \ge 0$ and $u \ge 0$, since $Q^{(j+1)}(u)$ is increasing for $1 \le j \le \nu - 1$, we have

$$0 \le \frac{d^j}{dx^j}\overline{Q(x,u)} \le \frac{Q^{(j+1)}(x) - Q^{(j+1)}(u)}{x-u}. \tag{2.26}$$

If $u < 0$ and $x > 0$, then since $|Q^{(j+1)}(\eta)| \le Q^{(j+1)}(-u)$ for $\eta < 0$,

$$\begin{aligned} \left|\frac{d^j}{dx^j}\overline{Q(x,u)}\right| &= \left|\frac{Q^{(j+1)}(x) - Q^{(j+1)}(\eta)}{x-u}\right| \\ &\le \frac{Q^{(j+1)}(x) + Q^{(j+1)}(-u)}{x+(-u)} \\ &\le \frac{Q^{(j+1)}(x) - Q^{(j+1)}(-u)}{x-(-u)} + 2\frac{Q^{(j+1)}(-u) - Q^{(j+1)}(0)}{-u-0}. \end{aligned} \tag{2.27}$$

So, for this case we can prove the result similarly to the case $x, u > 0$. For the other cases, we can prove it by the symmetry of Q, similarly. Therefore, we assume that u and x are nonnegative, and we will prove this theorem only for nonnegative x and u. Moreover, for simplicity, we let $c_1 = c_2$ without loss of generality, because we know by (1.13) that $Q^{(\nu+1)}(u)$ is bounded for any u between c_1 and c_2.

On the other hand, if $Q^{(j+2)}(u)$ is increasing, then

$$\frac{Q^{(j+1)}(u) - Q^{(j+1)}(x)}{u - x} \tag{2.28}$$

is also increasing for u because there exists a point ξ between x and u such that

$$\begin{aligned} \frac{d}{du}\left(\frac{Q^{(j+1)}(u) - Q^{(j+1)}(x)}{u - x}\right) &= \frac{Q^{(j+2)}(u) - \left(Q^{(j+1)}(u) - Q^{(j+1)}(x)\right)/(u - x)}{u - x} \\ &= \frac{Q^{(j+2)}(u) - Q^{(j+2)}(\xi)}{u - x} \geq 0. \end{aligned} \tag{2.29}$$

Moreover, if $Q^{(\nu+1)}(t) \leq C(1/t)^{\delta}$ for t between x and u, then we see

$$\frac{Q^{(\nu)}(u) - Q^{(\nu)}(x)}{u - x} = \frac{1}{u - x}\int_x^u Q^{(\nu+1)}(t)dt \leq \frac{C}{u - x}\left(u^{1-\delta} - x^{1-\delta}\right) \leq C\left(\frac{1}{u}\right)^{\delta}. \tag{2.30}$$

To complete the proof of Theorem 1.4 we prove a series of lemmas.

Lemma 2.6. *Let $0 \leq x \leq a_n(1 + \eta_n)$ and $1 \leq j \leq \nu - 1$:*

$$\int_{a_{4n}<u} (p_n w_\rho)^2(u) \frac{d^j}{dx^j}\overline{Q(x,u)}du \lesssim \left(\frac{T(a_n)}{a_n}\right)^j \frac{A_n(x)}{a_n}. \tag{2.31}$$

Proof. Since

$$\frac{A_n^{(j)}(x)}{2b_n} = \left(\int_{0\leq u\leq a_{4n}} + \int_{a_{4n}<u}\right)(p_n w_\rho)^2(u)\frac{d^j}{dx^j}\overline{Q(x,u)}du, \tag{2.32}$$

we have to estimate

$$\int_{a_{4n}<u} (p_n w_\rho)^2(u)\frac{d^j}{dx^j}\overline{Q(x,u)}du =: \int_{a_{4n}<u}. \tag{2.33}$$

First, we see for $x > 0$ large enough,

$$Q^{(j+1)}(x)e^{-Q(x)} \tag{2.34}$$

is decreasing because

$$\left(Q^{(j+1)}(x)e^{-Q(x)}\right)' = \left(Q^{(j+2)}(x) - Q^{(j+1)}(x)Q'(x)\right)e^{-Q(x)}, \tag{2.35}$$

and so from our assumption,

$$\begin{aligned} Q^{(j+2)}(x) - Q^{(j+1)}(x)Q'(x) &\le CQ^{(j+1)}(x)\frac{Q'(x)}{Q(x)} - Q^{(j+1)}(x)Q'(x) \\ &= Q^{(j+1)}(x)Q'(x)\left(\frac{C}{Q(x)} - 1\right) < 0, \end{aligned} \tag{2.36}$$

if $C < Q(x)$. We use this fact. Let $2\rho = \beta + i$ where $\beta < 0$, and let i be a nonnegative integer, and let $P(u) = p_n^2(u)u^i$. Let $u > 0$. Then since

$$\frac{Q^{(j+1)}(a_{4n})}{Q(a_{4n})} \le C\left(\frac{T(a_n)}{a_n}\right)^j, \tag{2.37}$$

by (1.13), we have for some ξ between x and u

$$\begin{aligned} \int_{a_{4n}<u} &= \int_{a_{4n}<u} (p_n w_\rho)^2(u) Q^{(j+2)}(\xi)du \\ &\le \int_{a_{4n}<u} (p_n w_\rho)^2(u) Q^{(j+2)}(u)du \\ &\le C\frac{Q^{(j+1)}(a_{4n})w(a_{4n})}{Q(a_{4n})} a_{4n}^\beta \int_{a_{4n}<u} P(u)w(u)Q'(u)du \quad (\text{by}(2.34)) \\ &\le \left(\frac{T(a_n)}{a_n}\right)^j w(a_{4n})a_{4n}^\beta \int_{a_{4n}}^\infty - P(u)\frac{d}{du}w(u)du, \\ \int_{a_{4n}}^\infty P(u)\frac{d}{du}w(u)du &= (Pw)(a_{4n}) - \int_{a_{4n}}^\infty P'(t)w(u)du. \end{aligned} \tag{2.38}$$

Applying Lemma 2.1(d) with L_∞, L_1-norm and Lemma 2.1(c),

$$\begin{aligned} |(Pw)(a_{4n})| &\le \exp(-C_2 n^\alpha)\|(Pw)(x)\|_{L_\infty(La_n/n\le|x|\le a_n(1-L\eta_n))}, \\ \int_{a_{4n}}^\infty \left|P'(u)w(u)\right|du &\le \exp(-C_2 n^\alpha)\left\|\left(P'w\right)(x)\right\|_{L_1(La_n/n\le|x|\le a_n(1-L\eta_n))} \\ &\le \exp(-C_2 n^\alpha)\frac{nT(a_n)^{1/2}}{a_n}\|(Pw)(x)\|_{L_1(La_n/n\le|x|\le a_n(1-L\eta_n))}. \end{aligned} \tag{2.39}$$

Therefore,

$$\int_{a_{4n}}^{\infty}\left|P(u)w(u)Q'(u)\right|du \leq \exp(-C_2n^{\alpha})\|(Pw)(x)\|_{L_\infty(La_n/n\leq|x|\leq a_n(1-L\eta_n))} + \exp(-C_2n^{\alpha})\frac{nT(a_n)^{1/2}}{a_n}\|(Pw)(x)\|_{L_1(La_n/n\leq|x|\leq a_n(1-L\eta_n))}. \tag{2.40}$$

Consequently we have

$$\begin{aligned}
&\left(\frac{T(a_n)}{a_n}\right)^j w(a_{4n})a_{4n}^{\beta}\int_{a_{4n}<u}\left|P(u)w(u)Q'(u)\right|du \\
&\leq \left(\frac{T(a_n)}{a_n}\right)^j \exp(-C_2n^{\alpha})\left\|p_n^2w_\rho^2\right\|_{L_\infty(La_n/n\leq|x|\leq a_n(1-L\eta_n))} \\
&\quad + \left(\frac{T(a_n)}{a_n}\right)^j \frac{nT(a_n)^{1/2}}{a_n}\exp(-C_2n^{\alpha})\left\|p_n^2w_\rho^2\right\|_{L_1(La_n/n\leq|x|\leq a_n(1-L\eta_n))} \\
&\leq O\left(e^{-n^{d_3}}\right)\left(\frac{T(a_n)}{a_n}\right)^j \\
&\lesssim \left(\frac{T(a_n)}{a_n}\right)^j \frac{A_n(x)}{a_n}.
\end{aligned} \tag{2.41}$$

□

Lemma 2.7. *If $Q'(x)/Q(x)$ is quasi-increasing on $[c_1,\infty)$ or if $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1,\infty)$, then one has*

$$\frac{Q^{(\nu)}(x)-Q^{(\nu)}(u)}{x-u} \lesssim \begin{cases} 1+\left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\dfrac{n}{a_n^2}, & 0\leq u\leq c_1,\ c_1\leq x\leq \dfrac{a_n}{2}, \\ 1+\left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\dfrac{n}{a_n^2}, & c_1\leq u\leq 2c_1,\ 0\leq x\leq c_1, \\ \left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\dfrac{n}{a_n^2}, & 2c_1\leq u\leq \dfrac{a_n}{3},\ 0\leq x\leq c_1, \\ \left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\dfrac{n}{a_n^2}, & c_1\leq u\leq \dfrac{a_n}{3},\ c_1\leq x\leq \dfrac{a_n}{2}. \end{cases} \tag{2.42}$$

Proof. Case (a-1). $0\leq u\leq c_1$ and $c_1\leq x\leq a_n/2$. Let

$$\frac{Q^{(\nu)}(u)-Q^{(\nu)}(x)}{u-x} \leq \frac{Q^{(\nu)}(u)-Q^{(\nu)}(c_1)}{u-c_1} + \frac{Q^{(\nu)}(c_1)-Q^{(\nu)}(x)}{c_1-x} =: Q_1(u)+Q_2(x). \tag{2.43}$$

Then we have $Q_1(u) \lesssim 1$ from (2.30). Then if $Q'(x)/Q(x)$ is quasi-increasing on $[c_1, \infty)$, there exists a point $\xi \in [c_1, x]$ such that by (1.13)

$$
\begin{aligned}
Q_2(x) &= \left|\frac{Q^{(\nu+1)}(\xi)}{Q''(\xi)}\right|\left|\frac{Q'(x)-Q'(c_1)}{x-c_1}\right| \\
&\lesssim \left(\frac{Q'(\xi)}{Q(\xi)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)-Q'(c_1)}{a_n/2-c_1}\right| \\
&\lesssim \left(\frac{Q'(a_n/2)}{Q(a_n/2)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)}{a_n}\right| \\
&\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}.
\end{aligned}
\tag{2.44}
$$

If $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1, \infty)$, there exists a point $\xi \in [c_1, x]$ such that by (2.28) and (1.13)

$$
\begin{aligned}
Q_2(x) &\le \frac{Q^{(\nu)}(a_n/2)-Q^{(\nu)}(c_1)}{a_n/2-c_1} \\
&\lesssim \frac{Q^{(\nu)}(a_n/2)}{Q'(a_n/2)}\frac{Q'(a_n/2)-Q'(c_1)}{a_n/2-c_1} \\
&\lesssim \left(\frac{Q'(a_n/2)}{Q(a_n/2)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)}{a_n}\right| \\
&\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}.
\end{aligned}
\tag{2.45}
$$

Case (a-2). For $c_1 \le u \le 2c_1$ and $0 \le x \le c_1$, we have similarly to Case (a-1),

$$
Q_2(x) \lesssim 1, \qquad Q_1(u) \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}. \tag{2.46}
$$

Case (b). $2c_1 \le u \le a_n/3$ and $0 \le x \le c_1$. Using the method of Case (a-1), and similarly to Case (a-2),

$$
\frac{Q^{(\nu)}(u)-Q^{(\nu)}(x)}{u-x} \sim \frac{Q^{(\nu)}(u)-Q^{(\nu)}(c_1)}{u-c_1} \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}. \tag{2.47}
$$

Case (c). $c_1 \le u \le a_n/3$ and $c_1 \le x \le a_n/2$. We can prove similarly to $Q_1(u)$ and $Q_2(x)$ of Case (a-1). If $Q'(x)/Q(x)$ is quasi-increasing on $[c_1, \infty)$, there exists a point $\xi \in [c_1, x]$ such that by (1.13)

$$\begin{aligned}\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x - u} &= \left|\frac{Q^{(\nu+1)}(\xi)}{Q''(\xi)}\right|\left|\frac{Q'(x) - Q'(u)}{x - u}\right| \\ &\lesssim \left(\frac{Q'(a_n/2)}{Q(a_n/2)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)}{a_n}\right| \\ &\sim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}.\end{aligned} \tag{2.48}$$

If $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1, \infty)$, there exists a point $\xi \in [c_1, x]$ such that by (1.13)

$$\begin{aligned}\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x - u} &\le \frac{Q^{(\nu)}(a_n/2) - Q^{(\nu)}(u)}{a_n/2 - u} \\ &\lesssim \left(\frac{|Q'(a_n/2)|}{Q(a_n/2)}\right)^{\nu-1}\left|\frac{Q'(a_n/2)}{a_n}\right| \\ &\sim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{n}{a_n^2}.\end{aligned} \tag{2.49}$$

□

Lemma 2.8. *One has*

$$\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x - u} \lesssim \begin{cases} \dfrac{1}{u^\delta}, & 0 \le u \le c_1,\ 0 \le x \le c_1, \\ \left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)}, & 0 \le u \le a_{4n},\ \dfrac{a_n}{2} \le x \le a_n(1+\eta_n), \\ \left(\dfrac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)} & \dfrac{a_n}{3} \le u \le a_{4n},\ 0 \le x \le \dfrac{a_n}{2}. \end{cases} \tag{2.50}$$

Proof. Case (a). $0 \le u \le c_1$ and $0 \le x \le c_1$. From (2.30) and (1.14)

$$\frac{Q^{(\nu)}(u) - Q^{(\nu)}(x)}{u - x} \le C\left(\frac{1}{u}\right)^\delta. \tag{2.51}$$

Case (b-1). $0 \le u \le a_n/3$ and $a_n/2 \le x \le a_n(1 + \eta_n)$. Since by [1, page 64, Lemma 3.2(a)]

$$\frac{Q'(a_n/2)}{Q'(a_n/3)} \ge \left(\frac{3}{2}\right)^{\Lambda-1}, \tag{2.52}$$

we have

$$Q'(x) - Q'(u) \geq Q'(x)\left(1 - \frac{Q'(a_n/3)}{Q'(a_n/2)}\right) \geq Q'(x)\left(1 - \left(\frac{2}{3}\right)^{\Lambda-1}\right). \tag{2.53}$$

Therefore, since for this case

$$\left(Q'(x) - Q'(u)\right) \sim Q'(x), \tag{2.54}$$

we have

$$\begin{aligned}\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} &= \frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{Q'(x) - Q'(u)}\overline{Q(x,u)} \\ &\lesssim \left|\frac{Q^{(\nu)}(x)}{Q'(x)}\right|\overline{Q(x,u)} \\ &\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)}.\end{aligned} \tag{2.55}$$

Case (b-2). $a_n/3 \leq u \leq a_{4n}$ and $a_n/2 \leq x \leq a_n(1+\eta_n)$. There exists a point ξ between x and u such that by (1.13)

$$\begin{aligned}\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} &= \frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{Q'(x) - Q'(u)}\overline{Q(x,u)} \\ &\lesssim \left|\frac{Q^{(\nu+1)}(\xi)}{Q''(\xi)}\right|\overline{Q(x,u)} \\ &\lesssim \left(\frac{T(\xi)}{\xi}\right)^{\nu-1}\overline{Q(x,u)} \\ &\lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)}.\end{aligned} \tag{2.56}$$

Case (c). $a_n/3 \leq u \leq a_{4n}$ and $0 \leq x \leq a_n/4$. By the same method as Case (b), we have

$$\frac{Q^{(\nu)}(x) - Q^{(\nu)}(u)}{x-u} \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\overline{Q(x,u)}. \tag{2.57}$$

□

Lemma 2.9. *Let* $a_n/2 \leq x \leq a_n(1+\eta_n)$. *Then*

$$\int_{0\leq u\leq a_{4n}} (p_n w_\rho)^2(u)\frac{d^{\nu-1}}{dx^{\nu-1}}\overline{Q(x,u)}du \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1}\frac{A_n(x)}{a_n}. \tag{2.58}$$

Proof. It is trivial from (2.26) and Lemma 2.8. □

Lemma 2.10. *Let $0 \le x \le a_n/2$.*

(a) *If $0 \le x \le c_1$, then*

$$\int_{0\le u\le c_1} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)}du \lesssim \frac{1}{a_n}. \tag{2.59}$$

Moreover, one knows that

$$\frac{1}{a_n} \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1} \frac{A_n(x)}{a_n}. \tag{2.60}$$

(b) *If $Q'(x)/Q(x)$ is quasi-increasing on $[c_1,\infty)$, or if $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1,\infty)$, then*

$$\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)}du \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1} \frac{A_n(x)}{a_n}. \tag{2.61}$$

(c) *If there exists a constant $0 \le \delta < 1$ such that $Q^{(\nu+1)}(x) \le C(1/x)^\delta$ on $(0,\infty)$, then one has* (2.61).

Proof. (a) For $0 \le x \le c_1$ we have from Lemmas 2.8, 2.1(a), and 2.2

$$\begin{aligned}\int_{0\le u\le c_1} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)}du &\lesssim \int_{0\le u\le c_1} (p_n w_\rho)^2(u) u^{-\delta} du \\ &\lesssim \begin{cases} \dfrac{1}{a_n}, & \rho \ge 0, \\ \dfrac{1}{a_n}\left(\dfrac{n}{a_n}\right)^{2\rho} \displaystyle\int_{0\le u\le c_1} u^{2\rho-\delta} \lesssim \dfrac{1}{a_n}\left(\dfrac{n}{a_n}\right)^{2\rho}, & \rho < 0 \end{cases} \\ &\lesssim \frac{1}{a_n},\end{aligned} \tag{2.62}$$

because $1+2\rho-\delta \ge 0$ for $\rho < 0$. On the other hand, from (1.19) we see $a_n^\nu \le n^{\nu/(1+\nu-\delta)} \le n$, and from (1.22) we see $A_n(x) \sim n/a_n$ for $0 \le x \le c_1$. So we have

$$\frac{1}{a_n} \lesssim \frac{n}{a_n^{\nu+1}} \lesssim \frac{n}{a_n^2}\left(\frac{T(a_n)}{a_n}\right)^{\nu-1} \sim \frac{A_n(x)}{a_n}\left(\frac{T(a_n)}{a_n}\right)^{\nu-1}. \tag{2.63}$$

(b) For $0 \le x \le c_1$, we have from (a), Lemmas 2.7, and 2.8

$$\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)} du \lesssim \int_{0\le u\le c_1} + \int_{c_1\le u\le 2c_1} + \int_{2c_1\le u\le a_n/3} + \int_{a_n/3\le u\le a_{4n}} \lesssim \left(\frac{T(a_n)}{a_n}\right)^{\nu-1} \frac{A_n(x)}{a_n}. \tag{2.64}$$

Similarly, for $c_1 \le x \le a_n/2$ we have from Lemmas 2.7 and 2.8

$$\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)} du \lesssim \int_{0\le u\le c_1} + \int_{c_1\le u\le a_n/3} + \int_{a_n/3\le u\le a_{4n}} \lesssim \frac{A_n(x)}{a_n}\left(\frac{T(a_n)}{a_n}\right)^{\nu-1}. \tag{2.65}$$

(c) Then by (2.26) and Lemma 2.1(a)

$$\begin{aligned}\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^{\nu-1}}{dx^{\nu-1}} \overline{Q(x,u)} du &\lesssim \int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) u^{-\delta} du \\ &\lesssim \int_{0\le u\le a_{4n}} \frac{u^{2\rho-\delta}}{(u+a_n/n)^{2\rho}\sqrt{u^2-a_n^2}} du \\ &\lesssim \int_{0<u<a_n/n} + \int_{a_n/n\le u\le a_n/2} + \int_{a_n/2\le u\le a_{4n}} \\ &\lesssim \frac{1}{a_n^\delta} \\ &\lesssim \frac{n}{a_n^2}\left(\frac{1}{a_n}\right)^{\nu-1} \\ &\lesssim \frac{A_n(x)}{a_n}\left(\frac{T(a_n)}{a_n}\right)^{\nu-1}.\end{aligned} \tag{2.66}$$

Here, we use the fact $1/a_n^\delta < n/a_n^{\nu+1}$ from (1.19) for the last inequality. □

Lemma 2.11. *Let $0 \le x \le a_n(1+\eta_n)$. Then for $1 \le j \le \nu-2$,*

$$\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^j}{dx^j} \overline{Q(x,u)} du \lesssim \frac{A_n(x)}{a_n}\left(\frac{T(a_n)}{a_n}\right)^j. \tag{2.67}$$

Proof. By the same reason as the proof of Lemma 2.10 when $Q^{(\nu+1)}(x)$ is nondecreasing on $[c_1, \infty)$, it is proved. □

To prove (1.21) we need some lemmas.

Lemma 2.12. *Let $0 < \varepsilon < 1$ and $|x| \le \varepsilon a_n$.*

(a) *For some $C > 0$ one has*

$$\frac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)} \le C \frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)} \frac{n}{a_n}. \tag{2.68}$$

(b) *For any $0 < \varepsilon < 1$, there exists $\varepsilon_1(\varepsilon, n) > 0$ such that for $2\varepsilon a_n \le u$*

$$\frac{d^j}{dx^j}\overline{Q(x,u)} \le \varepsilon_1(\varepsilon, n) \frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j}, \tag{2.69}$$

and $\varepsilon_1(\varepsilon, n) \to 0$ as $n \to \infty$.

Proof. (a) It follows from Lemma 2.3(b). (b) By (2.25), Lemma 2.3(b), and (a), we have

$$\begin{aligned}
\frac{d^j}{dx^j}\overline{Q(x,u)} &\le C\sum_{k=0}^{j-1} \frac{Q^{(j+1-k)}(x)}{(\varepsilon a_n)^{k+1}} + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j} \\
&\le C\sum_{k=0}^{j-1} \frac{Q'(\varepsilon a_n)}{(\varepsilon a_n)^{k+1}}\left(\frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)}\frac{n}{a_n}\right)^{j-k} + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j} \\
&\le C\sum_{k=0}^{j-1} \frac{\varepsilon^{\Lambda-1}}{(\varepsilon a_n)^{k+1}}\frac{n}{a_n}\left(\frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)}\frac{n}{a_n}\right)^{j-k} + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j} \\
&\le C\frac{n}{a_n^2}\left(\frac{n}{a_n}\right)^j \sum_{k=0}^{j-1} \varepsilon^{\Lambda-k-2}\left(\frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)}\right)^{j-k}\left(\frac{1}{n}\right)^k + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j} \\
&\le \varepsilon_1(\varepsilon, n)\frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j + \frac{\overline{Q(x,u)}}{(\varepsilon a_n)^j},
\end{aligned} \tag{2.70}$$

where we let

$$\varepsilon_1(\varepsilon, n) := C\sum_{k=0}^{j-1} \varepsilon^{\Lambda-k-2}\left(\frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)}\right)^{j-k}\left(\frac{1}{n}\right)^k \longrightarrow 0 \tag{2.71}$$

as $n \to \infty$. Therefore, this lemma is proved. □

Lemma 2.13. *Suppose that the one of the three conditions (a), (b), and (c) in Theorem 1.4 is satisfied. Then for any $0 < \varepsilon < 1/2$, there exists $\varepsilon_2(\varepsilon, n) > 0$ such that for $|x| \le \varepsilon a_n$ and $j = 1, \dots, \nu - 1$,*

$$\int_{0 \le u \le a_{4n}} (p_n w_\rho)^2(u) \frac{d^j}{dx^j}\overline{Q(x,u)}du \lesssim \varepsilon_2(\varepsilon, n)\frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j, \tag{2.72}$$

with $\varepsilon_2(\varepsilon, n) \to 0$ as $n \to \infty$.

Proof. First, we consider the case of which (c) in Theorem 1.4 is satisfied. Then the lemma follows from (2.66) with $\varepsilon_2(\varepsilon, n) := (1/n)^{\nu-1}$. Now, we consider the other cases. If we consider only for $|x| \le \varepsilon a_n$ and $|u| \le 2\varepsilon a_n$ in proving Lemmas 2.7 and 2.8, then we know that for $|x| \le \varepsilon a_n$ and $j = 1, \ldots, \nu - 1$

$$\frac{d^j}{dx^j}\overline{Q(x,u)} \lesssim \begin{cases} 1 + u^{-\delta} + \left(\dfrac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)}\right)^j \dfrac{Q'(\varepsilon a_n)}{\varepsilon a_n}, & 0 \le u \le 2c_1, \\ \left(\dfrac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)}\right)^j \dfrac{Q'(\varepsilon a_n)}{\varepsilon a_n}, & 2c_1 \le u \le \dfrac{\varepsilon}{2} a_n, \\ \left(\dfrac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)}\right)^j \dfrac{Q'(\varepsilon a_n)}{\varepsilon a_n}, & \dfrac{\varepsilon}{2} a_n \le u \le 2\varepsilon a_n. \end{cases} \tag{2.73}$$

Then we have by Lemma 2.12(a)

$$\begin{aligned} \int_{0\le u\le 2\varepsilon a_n} (p_n w_\rho)^2(u) \frac{d^j}{dx^j}\overline{Q(x,u)}du &\lesssim \int_{0\le u\le 2c_1} + \int_{2c_1\le u\le 2\varepsilon a_n} \\ &\lesssim \frac{1}{a_n^\delta} + \left(\frac{Q'(\varepsilon a_n)}{Q(\varepsilon a_n)}\right)^j \frac{Q'(\varepsilon a_n)}{\varepsilon a_n} \\ &\lesssim \left(\frac{a_n^{2+j-\delta}}{n^{1+j}} + \varepsilon^{\Lambda-2}\left(\frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)}\right)^j\right) \frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j, \end{aligned} \tag{2.74}$$

and we can see that

$$\varepsilon_3(\varepsilon, n) := \frac{a_n^{2+j-\delta}}{n^{1+j}} + \varepsilon^{\Lambda-2}\left(\frac{\varepsilon^{\Lambda-1}}{Q(\varepsilon a_n)}\right)^j \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.75}$$

Finally, we estimate $\int_{2\varepsilon a_n \le u \le a_{4n}}$. By Lemma 2.12(b) we have

$$\begin{aligned} &\int_{2\varepsilon a_n\le u\le a_{4n}} (p_n w_\rho)^2(u) \frac{d^j}{dx^j}\overline{Q(x,u)}du \\ &\le \int_{2\varepsilon a_n\le u\le a_{4n}} \left(\varepsilon_1(\varepsilon, n)\frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j + \frac{1}{(\varepsilon a_n)^j}\overline{Q(x,u)}\right)(p_n w_\rho)^2(u)du \\ &\le \left(\varepsilon_1(\varepsilon, n) + \frac{1}{(\varepsilon n)^j}\right)\frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j. \end{aligned} \tag{2.76}$$

Therefore, if we let $\varepsilon_2(\varepsilon, n) := \varepsilon_3(\varepsilon, n) + \varepsilon_1(\varepsilon, n) + 1/(\varepsilon n)^j$, then

$$\int_{0\le u\le a_{4n}} (p_n w_\rho)^2(u)\frac{d^j}{dx^j}\overline{Q(x,u)}du \lesssim \varepsilon_2(\varepsilon, n)\frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j, \tag{2.77}$$

and $\varepsilon_2(\varepsilon, n) \to 0$ as $n \to \infty$ by (2.75) and (2.76). □

From the proof of Lemma 2.6, we have the following. There exists $\varepsilon_4(n) > 0$ satisfying $\varepsilon_4(n) \to 0$ as $n \to \infty$ such that

$$\int_{u\ge a_{4n}} (p_n w_\rho)^2(u)\frac{d^j}{dx^j}\overline{Q(x,u)}du \le \varepsilon_4(n)\frac{A_n(x)}{a_n}\left(\frac{n}{a_n}\right)^j. \tag{2.78}$$

Therefore, from Lemmas 2.6, 2.9, 2.10, and 2.11 we obtain the estimate for $A_n^{(j)}(x)$ in (1.20), and from Lemma 2.13 and (2.78) we have the estimate for $A_n^{(j)}(x)$ in (1.21). Using Cauchy-Schwarz Inequality we also have the estimate for $B_n^{(j)}(x)$ in (1.20) and (1.21). Consequently, we proved Theorem 1.4, completely. □

Proof of Theorem 1.6. (a) (1.24) follows from [1, (3.45)] easily.

(b) Suppose that (1.23) is satisfied on $|x| \ge D$ for some $D > 0$ large enough. Let $x > D$. From (1.23) we have for large $x > D$

$$\ln\left(\frac{Q'(x)}{Q'(D)}\right) \le \ln\left(\frac{Q(x)}{Q(D)}\right)^\lambda, \tag{2.79}$$

and we have for large $x > D$

$$\frac{Q'(x)}{Q'(D)} \le \left(\frac{Q(x)}{Q(D)}\right)^\lambda. \tag{2.80}$$

Case $\lambda > 1$. Then we can see by [1, Lemma 3.4 (3.18)] and (2.80)

$$T(a_t) = \frac{a_t Q'(a_t)}{Q(a_t)} \le \frac{Q'(D)}{Q(D)^\lambda} a_t Q(a_t)^{\lambda-1} \le C a_t \left(\frac{t}{\sqrt{T(a_t)}}\right)^{\lambda-1}. \tag{2.81}$$

Therefore from the assumption $a_t \le C_2 t^\eta$ we have for any $\eta > 0$

$$T(a_t) \le C(\lambda, \eta) t^{2(\eta+\lambda-1)/(\lambda+1)}. \tag{2.82}$$

Case $0 < \lambda \le 1$. Then we have by (2.80)

$$T(x) = \frac{xQ'(x)}{Q(x)} \le x\frac{Q'(D)}{Q(D)^\lambda} Q(x)^{\lambda-1} \le x\frac{Q'(D)}{Q(D)}. \tag{2.83}$$

Therefore, from the assumption $a_t \leq C_2 t^\eta$ we have for any $\eta > 0$

$$T(a_t) \leq C(\lambda, \eta) t^\eta. \tag{2.84}$$

□

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