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Research Article

Complementary Lidstone Interpolation and Boundary Value Problems

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We shall introduce and construct explicitly the complementary Lidstone interpolating polynomial $P_{2m}(t)$ of degree $2m$, which involves interpolating data at the odd-order derivatives. For $P_{2m}(t)$ we will provide explicit representation of the error function, best possible error inequalities, best possible criterion for the convergence of complementary Lidstone series, and a quadrature formula with best possible error bound. Then, these results will be used to establish existence and uniqueness criteria, and the convergence of Picard's, approximate Picard's, quasilinearization, and approximate quasilinearization iterative methods for the complementary Lidstone boundary value problems which consist of a $(2m+1)$th order differential equation and the complementary Lidstone boundary conditions.

1. Introduction

In our earlier work [1, 2] we have shown that the interpolating polynomial theory and the qualitative as well as quantitative study of boundary value problems such as existence and uniqueness of solutions, and convergence of various iterative methods are directly connected. In this paper we will extend this technique to the following *complementary Lidstone boundary value problem* involving an odd order differential equation

$$(-1)^m x^{(2m+1)}(t) = f(t, x(t)), \quad t \in (0,1), \; m \geq 1, \tag{1.1}$$

and the boundary data at the odd order derivatives

$$x(0) = \alpha_0, \quad x^{(2i-1)}(0) = \alpha_i, \quad x^{(2i-1)}(1) = \beta_i, \quad 1 \leq i \leq m. \tag{1.2}$$

Here $\mathbf{x} = (x, x', \ldots, x^{(q)})$, $0 \le q \le 2m$ but fixed, and $f : [0,1] \times \mathbb{R}^{q+1} \to \mathbb{R}$ is continuous at least in the interior of the domain of interest. Problem (1.1), (1.2) complements *Lidstone boundary value problem* (nomenclature comes from the expansion introduced by Lidstone [3] in 1929, and thoroughly characterized in terms of completely continuous functions in the works of Boas [4], Poritsky [5], Schoenberg [6–8], Whittaker [9, 10], Widder [11, 12], and others) which consists of an even-order differential equation and the boundary data at the even-order derivatives

$$(-1)^m x^{(2m)}(t) = f(t, \mathbf{x}(t)), \quad t \in (0,1), \ m \ge 1,$$
$$x^{(2i)}(0) = a_i, \quad x^{(2i)}(1) = b_i, \quad 0 \le i \le m-1. \tag{1.3}$$

Problem (1.3) has been a subject matter of numerous studies in the recent years [13–45], and others.

In Section 2, we will show that for a given function $x : C^{(2m+1)}[0,1] \to \mathbb{R}$ explicit representations of the interpolation polynomial $P_{2m}(t)$ of degree $2m$ satisfying the conditions

$$P_{2m}(0) = x(0), \quad P_{2m}^{(2i-1)}(0) = x^{(2i-1)}(0), \quad P_{2m}^{(2i-1)}(1) = x^{(2i-1)}(1), \quad 1 \le i \le m \tag{1.4}$$

and the corresponding residue term $R(t) = x(t) - P_{2m}(t)$ can be deduced rather easily from our earlier work on Lidstone polynomials [46–48]. Our method will avoid unnecessarily long procedure followed in [49] to obtain the same representations of $P_{2m}(t)$ and $R(t)$. We will also obtain error inequalities

$$\left| x^{(k)}(t) - P_{2m}^{(k)}(t) \right| \le C_{2m+1,k} \max_{0 \le t \le 1} \left| x^{(2m+1)}(t) \right|, \quad k = 0, 1, \ldots, 2m, \tag{1.5}$$

where the constants $C_{2m+1,k}$ are the best possible in the sense that in (1.5) equalities hold if and only if $x(t)$ is a certain polynomial. The best possible constant $C_{2m+1,0}$ was also obtained in [49]; whereas they left the cases $1 \le k \le 2m$ without any mention. In Section 2, we will also provide best possible criterion for the convergence of complementary Lidstone series, and a quadrature formula with best possible error bound.

If $f = 0$ then the complementary Lidstone boundary value problem (1.1), (1.2) obviously has a unique solution $x(t) = P_{2m}(t)$; if f is linear, that is, $f = \sum_{i=0}^{q} a_i(t) x^{(i)}$ then (1.1), (1.2) gives the possibility of interpolation by the solutions of the differential equation (1.1). In Sections 3–5, we will use inequalities (1.5) to establish existence and uniqueness criteria, and the convergence of Picard's, approximate Picard's, quasilinearization, and approximate quasilinearization iterative methods for the complementary Lidstone boundary value problem (1.1), (1.2). In Section 6, we will show the monotone convergence of Picard's iterative method. Since the proofs of most of the results in Sections 3–6 are similar to those of our previous work [1, 2] the details are omitted; however, through some simple examples it is shown how easily these results can be applied in practice.

2. Interpolating Polynomial

We begin with the following well-known results.

Lemma 2.1 (see [47]). *Let $y \in C^{(2m)}[0,1]$. Then,*

$$y(t) = Q_{2m-1}(t) + E(t), \tag{2.1}$$

where $Q_{2m-1}(t)$ is the Lidstone interpolating polynomial of degree $(2m-1)$,

$$Q_{2m-1}(t) = \sum_{i=0}^{m-1}\left[y^{(2i)}(0)\Lambda_i(1-t) + y^{(2i)}(1)\Lambda_i(t)\right], \tag{2.2}$$

and $E(t)$ is the residue term

$$E(t) = \int_0^1 g_m(t,s)y^{(2m)}(s)ds, \tag{2.3}$$

here

$$\Lambda_0(t) = t, \quad \Lambda_i''(t) = \Lambda_{i-1}(t), \quad \Lambda_i(0) = \Lambda_i(1) = 0, \quad i \ge 1, \tag{2.4}$$

$$\begin{gathered} g_1(t,s) = \begin{cases} (t-1)s, & s \le t, \\ (s-1)t, & t \le s, \end{cases} \\ g_i(t,s) = \int_0^1 g_1(t,t_1)g_{i-1}(t_1,s)dt_1, \quad i \ge 2. \end{gathered} \tag{2.5}$$

Recursively, it follows that

$$\begin{aligned} \Lambda_i(t) = \int_0^1 g_i(t,s)sds &= \frac{1}{6}\left[\frac{6t^{2i+1}}{(2i+1)!} - \frac{t^{2i-1}}{(2i-1)!}\right] - \sum_{k=0}^{i-2}\frac{2\left(2^{2k+3}-1\right)}{(2k+4)!}B_{2k+4}\frac{t^{2i-2k-3}}{(2i-2k-3)!} \\ &= \frac{2^{2i+1}}{(2i+1)!}B_{2i+1}\left(\frac{1+t}{2}\right), \quad i \ge 1 \end{aligned} \tag{2.6}$$

($B_{2i+1}(t)$ is the Bernoulli polynomial of degree $2i+1$, and B_{2k+4} is the $(2k+4)$th Bernoulli number $B_{2k+1} = 0$, $k = 1,2,3,\ldots$; $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$, $B_{14} = 7/6$).

Lemma 2.2 (see [47]). *The following hold:*

$$g_m(t,s) = \begin{cases} g_m^1(t,s) = -\sum_{i=0}^{m-1} \Lambda_i(t) \dfrac{(1-s)^{2m-2i-1}}{(2m-2i-1)!}, & t \le s, \\ g_m^2(t,s) = -\sum_{i=0}^{m-1} \Lambda_i(1-t) \dfrac{s^{2m-2i-1}}{(2m-2i-1)!}, & s \le t, \end{cases} \tag{2.7}$$

$$0 \le (-1)^m g_m(t,s) = |g_m(t,s)|, \tag{2.8}$$

$$\int_0^1 |g_m(t,s)| ds = (-1)^m E_{2m}(t) \le (-1)^m E_{2m}\left(\frac{1}{2}\right) = \frac{(-1)^m E_{2m}}{2^{2m}(2m)!} \tag{2.9}$$

($E_{2m}(t)$ is the Euler polynomial of degree $2m$, and E_{2m} is the $(2m)$th Euler number $E_{2m+1} = 0$, $m = 0, 1, 2, \ldots$; $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$)

$$\begin{aligned} \int_0^1 |g_m'(t,s)| ds &= (-1)^m [2E_{2m}(t) + (1-2t)E_{2m-1}(t)] \le (-1)^m E_{2m-1}(0) \\ &= (-1)^{m+1} \frac{2(2^{2m}-1)}{(2m)!} B_{2m}. \end{aligned} \tag{2.10}$$

Theorem 2.3. *Let $x \in C^{(2m+1)}[0,1]$. Then,*

$$x(t) = P_{2m}(t) + R(t), \tag{2.11}$$

where $P_{2m}(t)$ is the complementary Lidstone interpolating polynomial of degree $2m$,

$$P_{2m}(t) = x(0) + \sum_{i=1}^{m} \left[x^{(2i-1)}(0)(v_i(1) - v_i(1-t)) + x^{(2i-1)}(1)(v_i(t) - v_i(0)) \right], \tag{2.12}$$

and $R(t)$ is the residue term

$$R(t) = \int_0^1 h_m(t,s) x^{(2m+1)}(s) ds, \tag{2.13}$$

here

$$h_m(t,s) = \int_0^t g_m(\tau,s) d\tau = \begin{cases} -\sum_{i=1}^{m} (v_i(t) - v_i(0)) \dfrac{(1-s)^{2m-2i+1}}{(2m-2i+1)!}, & t \le s, \\ \dfrac{s^{2m}}{(2m)!} + \sum_{i=1}^{m} (v_i(1-t) - v_i(1)) \dfrac{s^{2m-2i+1}}{(2m-2i+1)!}, & s \le t, \end{cases} \tag{2.14}$$

$$\Lambda_i'(t) = v_i(t), \quad i \ge 0. \tag{2.15}$$

Remark 2.4. From (2.4) and (2.15) it is clear that $v_0(t) = 1$; $v_i'(t) = \Lambda_{i-1}(t)$, $i \geq 1$; $\int_0^1 v_i(s)ds = 0$, $i \geq 1$; $v_i'(0) = 0$, $i \geq 1$; $v_i'(1) = 0$, $i \geq 2$; $v_i'(t) = \int_0^t v_{i-1}(s)ds$, $i \geq 1$;

$$v_0(t) = 1, \qquad v_1(t) = \frac{t^2}{2} - \frac{1}{6}, \qquad v_2(t) = \frac{t^4}{24} - \frac{t^2}{12} + \frac{7}{360}. \tag{2.16}$$

Proof. In (2.1), we let $y(t) = x'(t)$ and integrate both sides from 0 to t, to obtain

$$\begin{aligned} \int_0^t x'(\tau)d\tau = x(t) - x(0) = \sum_{i=0}^{m-1}\left[x^{(2i+1)}(0)\int_0^t \Lambda_i(1-\tau)d\tau + x^{(2i+1)}(1)\int_0^t \Lambda_i(\tau)d\tau\right] \\ + \int_0^t \left(\int_0^1 g_m(\tau,s)x^{(2m+1)}(s)ds\right)d\tau. \end{aligned} \tag{2.17}$$

Now, since

$$\int_0^t \Lambda_i(\tau)d\tau = \int_0^t \Lambda_{i+1}''(\tau)d\tau = \Lambda_{i+1}'(t) - \Lambda_{i+1}'(0) = v_{i+1}(t) - v_{i+1}(0), \quad i \geq 0, \tag{2.18}$$

and, similarly

$$\int_0^t \Lambda_i(1-\tau)d\tau = \Lambda_{i+1}'(1) - \Lambda_{i+1}'(1-t) = v_{i+1}(1) - v_{i+1}(1-t), \quad i \geq 0, \tag{2.19}$$

it follows that

$$\begin{aligned} x(t) &= x(0) + \sum_{i=1}^{m}\left[x^{(2i-1)}(0)(v_i(1) - v_i(1-t)) + x^{(2i-1)}(1)(v_i(t) - v_i(0))\right] \\ &\quad + \int_0^t \left(\int_0^1 g_m(\tau,s)x^{(2m+1)}(s)ds\right)d\tau \\ &= P_{2m}(t) + R(t). \end{aligned} \tag{2.20}$$

Next since

$$R(t) = \int_0^t \left(\int_0^1 g_m(\tau,s)x^{(2m+1)}(s)ds\right)d\tau = \int_0^1 \left(\int_0^t g_m(\tau,s)d\tau\right)x^{(2m+1)}(s)ds \tag{2.21}$$

for $t \leq s$, from (2.7), we get

$$\begin{aligned}
h_m(t,s) = \int_0^t g_m(\tau,s)d\tau &= \int_0^t g_m^1(\tau,s)d\tau \\
&= -\sum_{i=0}^{m-1}\left(\int_0^t \Lambda_i(\tau)d\tau\right)\frac{(1-s)^{2m-2i-1}}{(2m-2i-1)!} \\
&= -\sum_{i=1}^{m}(v_i(t)-v_i(0))\frac{(1-s)^{2m-2i+1}}{(2m-2i+1)!}, \quad t \leq s,
\end{aligned} \tag{2.22}$$

and similarly, for $s \leq t$, we have

$$\begin{aligned}
h_m(t,s) &= \int_0^t g_m(\tau,s)d\tau = \int_0^s g_m^1(\tau,s)d\tau + \int_s^t g_m^2(\tau,s)d\tau \\
&= -\sum_{i=1}^{m}(v_i(s)-v_i(0))\frac{(1-s)^{2m-2i+1}}{(2m-2i+1)!} + \sum_{i=1}^{m}(v_i(1-t)-v_i(1-s))\frac{s^{2m-2i+1}}{(2m-2i+1)!}.
\end{aligned} \tag{2.23}$$

Finally, since (2.12) is exact for any polynomial of degree up to $2m$, we find

$$\frac{(t-s)^{2m}}{(2m)!} = \frac{(-s)^{2m}}{(2m)!} + \sum_{i=1}^{m}\left[\frac{(-s)^{2m-2i+1}}{(2m-2i+1)!}(v_i(1)-v_i(1-t)) + \frac{(1-s)^{2m-2i+1}}{(2m-2i+1)!}(v_i(t)-v_i(0))\right], \tag{2.24}$$

and hence, for $t = s$, it follows that

$$\frac{s^{2m}}{(2m)!} = \sum_{i=1}^{m}\left[\frac{(s)^{2m-2i+1}}{(2m-2i+1)!}(v_i(1)-v_i(1-s)) - \frac{(1-s)^{2m-2i+1}}{(2m-2i+1)!}(v_i(s)-v_i(0))\right]. \tag{2.25}$$

Combining (2.23) and (2.25), we obtain

$$h_m(t,s) = \int_0^t g_m(\tau,s)d\tau = \frac{s^{2m}}{(2m)!} + \sum_{i=1}^{m}(v_i(1-t)-v_i(1))\frac{s^{2m-2i+1}}{(2m-2i+1)!}, \quad s \leq t. \tag{2.26}$$

□

Theorem 2.5. *Let $x \in C^{(2m+1)}[0,1]$. Then, inequalities* (1.5) *hold with*

$$\begin{aligned}
C_{2m+1,0} &= (-1)^m \frac{4(2^{2m+2}-1)}{(2m+2)!}B_{2m+2}, \\
C_{2m+1,2k-1} &= \frac{(-1)^{m-k+1}E_{2m-2k+2}}{2^{2m-2k+2}(2m-2k+2)!}, \quad 1 \leq k \leq m, \\
C_{2m+1,2k} &= (-1)^{m-k}\frac{2(2^{2m-2k+2}-1)}{(2m-2k+2)!}B_{2m-2k+2}, \quad 1 \leq k \leq m
\end{aligned} \tag{2.27}$$

($C_{3,0} = 1/12$, $C_{3,1} = 1/8$, $C_{3,2} = 1/2$, $C_{5,0} = 1/120$. $C_{5,1} = 5/384$, $C_{5,2} = 1/24$, $C_{5,3} = 1/8$, $C_{5,4} = 1/2$).

Proof. From (2.14) and (2.8) it follows that

$$0 \le (-1)^m h_m(t,s) = |h_m(t,s)|. \tag{2.28}$$

Now, from (2.11) and (2.13), we find

$$|x(t) - P_{2m}(t)| \le \max_{0\le t\le 1}\left(\int_0^1 |h_m(t,s)|ds\right)\max_{0\le t\le 1}\left|x^{(2m+1)}(t)\right|. \tag{2.29}$$

However, from (2.9), we have

$$\int_0^1 |h_m(t,s)|ds = \int_0^1\left|\int_0^t g_m(\tau,s)d\tau\right|ds = \int_0^t\left(\int_0^1 \left|g_m(\tau,s)\right|ds\right)d\tau = \int_0^t (-1)^m E_{2m}(\tau)d\tau. \tag{2.30}$$

Thus, from $(-1)^m E_{2m}(\tau) \ge 0$, $\tau \in [0,1]$, $E'_{2m+1}(\tau) = E_{2m}(\tau)$, and $E_{2m+1}(0) + E_{2m+1}(1) = 0$, we obtain

$$\begin{aligned}\int_0^1 |h_m(t,s)|ds &\le \int_0^1 (-1)^m E'_{2m+1}(\tau)d\tau\\ &= (-1)^m[E_{2m+1}(1) - E_{2m+1}(0)] = (-1)^{m+1}2E_{2m+1}(0)\\ &= (-1)^{m+2}\frac{4(2^{2m+2}-1)}{(2m+2)!}B_{2m+2} = C_{2m+1,0}.\end{aligned} \tag{2.31}$$

Using the above estimate in (2.29), the inequality (1.5) for $k = 0$ follows.

Next, from (2.11), (2.13) and (2.14), we have

$$x^{(j)}(t) - P_{2m}^{(j)}(t) = \int_0^1 g_m^{(j-1)}(t,s)x^{(2m+1)}(s)ds, \quad 1 \le j \le 2m \tag{2.32}$$

and hence in view of (2.5) and (2.9) it follows that

$$\begin{aligned}
\left|x^{(2k-1)}(t)-P_{2m}^{(2k-1)}(t)\right| &\leq \max_{0\leq t\leq 1}\left(\int_0^1\left|g_m^{(2k-2)}(t,s)\right|ds\right)\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right| \\
&= \max_{0\leq t\leq 1}\left(\int_0^1\left|g_{m-k+1}(t,s)\right|ds\right)\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right| \\
&\leq \frac{(-1)^{m-k+1}E_{2m-2k+2}}{2^{2m-2k+2}(2m-2k+2)!}\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right| \\
&= C_{2m+1,2k-1}\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right|, \quad 1\leq k\leq m,
\end{aligned} \tag{2.33}$$

and similarly, by (2.5) and (2.10), we get

$$\begin{aligned}
\left|x^{(2k)}(t)-P_{2m}^{(2k)}(t)\right| &\leq \max_{0\leq t\leq 1}\left(\int_0^1\left|g_m^{(2k-1)}(t,s)\right|ds\right)\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right| \\
&= \max_{0\leq t\leq 1}\left(\int_0^1\left|g'_{m-k+1}(t,s)\right|ds\right)\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right| \\
&\leq (-1)^{m-k}\frac{2\left(2^{2m-2k+2}-1\right)}{(2m-2k+2)!}B_{2m-2k+2}\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right| \\
&= C_{2m+1,2k}\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right|, \quad 1\leq k\leq m.
\end{aligned} \tag{2.34}$$

□

Remark 2.6. From (2.13), (2.28), and the above considerations it is clear that

$$R(t)=\left(\int_0^1 h_m(t,s)ds\right)x^{(2m+1)}(\xi)=\left[E_{2m+1}(t)-E_{2m+1}(0)\right]x^{(2m+1)}(\xi), \quad 0<\xi<1. \tag{2.35}$$

Remark 2.7. Inequality (1.5) with the constants $C_{2m+1,k}$ given in (2.27) is the best possible, as equalities hold for the function $x(t)=E_{2m+1}(t)-E_{2m+1}(0)$ (polynomial of degree $(2m+1)$) whose complementary Lidstone interpolating polynomial $P_{2m}(t)\equiv 0$, and only for this function up to a constant factor.

Remark 2.8. From the identity (see [47, equation (1.2.21)])

$$\sum_{k=1}^{\infty}\frac{1}{k^{2m+2}}=(-1)^m\frac{(2\pi)^{2m+2}}{2(2m+2)!}B_{2m+2}, \tag{2.36}$$

we have

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\geq\frac{(2\pi)^{2m+2}}{2(2m+2)!}\left|B_{2m+2}\right|, \tag{2.37}$$

and hence

$$|B_{2m+2}| \leq \left(\frac{\pi^2}{3}\right)\frac{(2m+2)!}{(2\pi)^{2m+2}}. \tag{2.38}$$

We also have the estimate (see [47, equation (1.2.41)])

$$|E_{2m+2}| \leq \left(\frac{2}{\pi}\right)^{2m+1}(2m+2)!. \tag{2.39}$$

Thus, from (2.27), (2.38), and (2.39), we obtain

$$\begin{gathered} C_{2m+1,0} \leq \frac{4\pi}{3}\left(\frac{1}{\pi}\right)^{2m+1}, \quad C_{2m+1,2k-1} \leq \frac{\pi}{2}\left(\frac{1}{\pi}\right)^{2m-2k+2}, \\ C_{2m+1,2k} \leq \frac{2\pi}{3}\left(\frac{1}{\pi}\right)^{2m-2k+1}, \quad 1 \leq k \leq m. \end{gathered} \tag{2.40}$$

Therefore, it follows that

$$C_{2m+1,k} \leq \frac{4\pi}{3}\left(\frac{1}{\pi}\right)^{2m+1-k}, \quad 0 \leq k \leq 2m. \tag{2.41}$$

Combining (1.5) and (2.41), we get

$$\left|x^{(k)}(t) - P_{2m}^{(k)}(t)\right| \leq \frac{4\pi}{3}\left(\frac{1}{\pi}\right)^{2m+1-k}\max_{0\leq t\leq 1}\left|x^{(2m+1)}(t)\right|, \quad k = 0,1,\dots,2m. \tag{2.42}$$

Hence, if $x \in C^{\infty}[0,1]$, for a fixed k as $m \to \infty$, $P_{2m}^{(k)}(t)$ converges absolutely and uniformly to $x^{(k)}(t)$ in $[0,1]$, provided that there exists a constant $\lambda, |\lambda| < \pi$ and an integer n such that $x^{(2m+1)}(t) = \mathcal{O}(\lambda^{2m+1-k})$ for all $m \geq n$, $t \in [0,1]$.

In particular, the function $x(t) = \cos\lambda t$, $t \in [0,1]$ satisfies the above conditions. Thus, for each fixed k, expansions

$$x^{(2k)}(t) = (-1)^k\lambda^{2k}\cos\lambda t = (-1)^k\lambda^{2k}\left[1 + \sum_{i=1}^{\infty}(-1)^i\lambda^{2i-1}\sin\lambda(v_i(t) - v_i(0))\right], \tag{2.43}$$

$$x^{(2k+1)}(t) = (-1)^{k+1}\lambda^{2k+1}\sin\lambda t = (-1)^k\lambda^{2k}\sum_{i=1}^{\infty}(-1)^i\lambda^{2i-1}\sin\lambda\Lambda_{i-1}(t) \tag{2.44}$$

converge absolutely and uniformly in $[0,1]$, provided $|\lambda| < \pi$. For $\lambda = \pm\pi$, (2.43) and (2.44), respectively, reduce to absurdities, $\cos\pi t = 1$ and $\sin\pi t = 0$. Thus, the condition $|\lambda| < \pi$ is the best possible.

Remark 2.9. If $x \in C^{(2m+1)}[a,b]$, then

$$P_{2m}(t)=x(a)+\sum_{i=1}^{m}(b-a)^{2i-1}\left[x^{(2i-1)}(a)\left(v_i(1)-v_i\left(\frac{b-t}{b-a}\right)\right)+x^{(2i-1)}(b)\left(v_i\left(\frac{t-a}{b-a}\right)-v_i(0)\right)\right], \tag{2.45}$$

$$R(t) = (b-a)^{2m}\int_a^b h_m\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)x^{(2m+1)}(s)ds. \tag{2.46}$$

Thus, in view of $\int_0^1 v_i(s)ds = 0$, $i \geq 1$ we have

$$\int_a^b P_{2m}(t)dt = (b-a)x(a) + \sum_{i=1}^{m}(b-a)^{2i}\left[x^{(2i-1)}(a)v_i(1) - x^{(2i-1)}(b)v_i(0)\right]. \tag{2.47}$$

Now, since $B_k'(t) = kB_{k-1}(t)$, $B_k(1-t) = (-1)^k B_k(t)$, $k = 1,2,\ldots$, from (2.6), we find

$$\Lambda_i'(t) = \frac{2^{2i}}{(2i)!}B_{2i}\left(\frac{1+t}{2}\right) = \frac{2^{2i}}{(2i)!}B_{2i}\left(\frac{1-t}{2}\right), \tag{2.48}$$

and hence by (2.15) it follows that

$$\begin{gathered} v_i(0) = \Lambda_i'(0) = \frac{2^{2i}}{(2i)!}B_{2i}\left(\frac{1}{2}\right) = \frac{2^{2i}}{(2i)!}\left(2^{1-2i}-1\right)B_{2i}, \\ v_i(1) = \Lambda_i'(1) = \frac{2^{2i}}{(2i)!}B_{2i}. \end{gathered} \tag{2.49}$$

Using these relations in (2.47), we obtain an approximate quadrature formula

$$\int_a^b x(t)dt \simeq (b-a)x(a) + \sum_{i=1}^{m}(b-a)^{2i}B_{2i}\frac{2^{2i}}{(2i)!}\left[x^{(2i-1)}(a) - \left(2^{1-2i}-1\right)x^{(2i-1)}(b)\right]. \tag{2.50}$$

It is to be remarked that (2.50) is different from the Euler-MacLaurin formula, but the same as in [49] obtained by using different arguments. To find the error e in (2.50), from (2.28) and

(2.46) we have

$$
\begin{aligned}
e = \int_a^b R(t)dt &= (b-a)^{2m+2}\int_0^1\left(\int_0^1 h_m(t,s)x^{(2m+1)}(a+s(b-a))ds\right)dt \\
&= (b-a)^{2m+2}\left(\int_0^1\left(\int_0^1 h_m(t,s)ds\right)dt\right)x^{(2m+1)}(\xi), \quad a<\xi<b \\
&= (b-a)^{2m+2}\left(\int_0^1 [E_{2m+1}(t)-E_{2m+1}(0)]dt\right)x^{(2m+1)}(\xi) \\
&= (b-a)^{2m+2}(-E_{2m+1}(0))x^{(2m+1)}(\xi) \\
&= \frac{2(2^{2m+2}-1)}{(2m+2)!}B_{2m+2}(b-a)^{2m+2}x^{(2m+1)}(\xi).
\end{aligned}
\tag{2.51}
$$

Thus, it immediately follows that

$$
\begin{aligned}
|e| &= \left|\int_a^b x(t)dt-(b-a)x(a)-\sum_{i=1}^m (b-a)^{2i}B_{2i}\frac{2^{2i}}{(2i)!}\left[x^{(2i-1)}(a)-\left(2^{1-2i}-1\right)x^{(2i-1)}(b)\right]\right| \\
&\le (-1)^m\frac{2(2^{2m+2}-1)}{(2m+2)!}B_{2m+2}(b-a)^{2m+2}\max_{t\in[a,b]}\left|x^{(2m+1)}(t)\right|.
\end{aligned}
\tag{2.52}
$$

From (2.52) it is clear that (2.50) is exact for any polynomial of degree at most $(2m)$. Further, in (2.52) equality holds for the function $x(t) = E_{2m+1}[(t-a)/(b-a)] - E_{2m+1}(0)$ and only for this function up to a constant factor.

We will now present two examples to illustrate the importance of (2.50) and (2.52).

Example 2.10. Consider integrating $(t^{14}+1)$ over $[0,1]$. Here, $a = 0$, $b = 1$, and $x(t) = t^{14}+1 \in C^{\infty}[0,1]$. The exact value of the integral is

$$
\int_0^1\left(t^{14}+1\right)dt = 1\frac{1}{15}. \tag{2.53}
$$

In Table 1, we list the approximates of the integral using (2.50) with different values of m, the actual errors incurred, and the error bounds deduced from (2.52).

Note that $x^{(15)}(t) \equiv 0$, hence the error $e = 0$ when $(2m+1) = 15$ or $m = 7$. Although the errors for other values of m (< 7) are large, ultimately the approximates tend to the exact value as $m \to \infty$.

Example 2.11. Consider integrating $\sin 2t$ over $[0,\pi/2]$. Here, $a = 0$, $b = \pi/2$, and $x(t) = \sin 2t \in C^{\infty}[0,\pi/2]$. The exact value of the integral is

$$
\int_0^{\pi/2}\sin 2t\,dt = 1. \tag{2.54}
$$

Table 1

m	Approximate (2.50)	Actual error $\lvert e\rvert$	Error bound (2.52)
1	$3\frac{1}{3}$	$2\frac{4}{15}$	91
2	$-39\frac{2}{15}$	$40\frac{1}{5}$	1001
3	$453\frac{19}{45}$	$452\frac{16}{45}$	7293
4	$-3178\frac{7}{9}$	$3179\frac{38}{45}$	31031
5	$12321\frac{5}{9}$	$12320\frac{22}{45}$	62881
6	$-19111\frac{4}{15}$	$19112\frac{1}{3}$	38227
7	$1\frac{1}{15}$	0	0

Table 2

m	Approximate (2.50)	Actual error $\lvert e\rvert$	Error bound (2.52)
1	0.822467	0.177533	2.029356
2	0.957757	0.042243	2.002894
3	0.989549	0.010451	2.000310
4	0.997394	0.002606	2.000034
5	0.999349	0.000651	2.0000038
6	0.999837	0.000163	2.00000042
7	0.999959	0.000041	2.000000046

In Table 2, we list the approximates of the integral using (2.50) with different values of m, the actual errors incurred, and the error bounds deduced from (2.52).

Unlike Example 2.10, here the error decreases as m increases. In both examples, the approximates tend to the exact value as $m \to \infty$. Of course, for increasing accuracy, instead of taking large values of m, one must use composite form of formula (2.50).

3. Existence and Uniqueness

The equalities and inequalities established in Section 2 will be used here to provide necessary and sufficient conditions for the existence and uniqueness of solutions of the complementary Lidstone boundary value problem (1.1), (1.2).

Theorem 3.1. *Suppose that $M_k > 0$, $0 \le k \le q$ are given real numbers and let Q be the maximum of $|f(t, x_0, x_1, \dots, x_q)|$ on the compact set $[0,1] \times D_0$, where*

$$D_0 = \{(x_0, x_1, \dots, x_q) : |x_k| \le 2M_k,\ 0 \le k \le q\}. \tag{3.1}$$

Further, suppose that

$$QC_{2m+1,k} \le M_k, \quad \max_{t\in[0,1]} \left| P_{2m}^{(k)}(t) \right| = p_k \le M_k, \quad 0 \le k \le q, \tag{3.2}$$

then, the boundary value problem (1.1), (1.2) *has a solution in* D_0.

Proof. The set

$$B[0,1] = \left\{ x(t) \in C^{(q)}[0,1] : \left\| x^{(k)} \right\| = \max_{t\in[0,1]} \left| x^{(k)}(t) \right| \le 2M_k,\ 0 \le k \le q \right\} \tag{3.3}$$

is a closed convex subset of the Banach space $C^{(q)}[0,1]$. We define an operator $T : C^{(q)}[0,1] \to C^{(2m)}[0,1]$ as follows:

$$(Tx)(t) = P_{2m}(t) + \int_0^1 |h_m(t,s)| f(s, \mathbf{x}(s)) ds. \tag{3.4}$$

In view of Theorem 2.3 and (2.28) it is clear that any fixed point of (3.4) is a solution of the boundary value problem (1.1), (1.2). Let $x(t) \in B[0,1]$. Then, from (1.5), (3.2), and (3.4), we find

$$\left| (Tx)^{(k)}(t) \right| \le M_k + QC_{2m+1,k} = 2M_k, \quad 0 \le k \le q. \tag{3.5}$$

Thus, $TB[0,1] \subseteq B[0,1]$. Inequalities (3.5) imply that the sets $\{(Tx)^{(k)}(t) : x(t) \in B[0,1]\}$, $0 \le k \le q$ are uniformly bounded and equicontinuous in $[0,1]$. Hence, $\overline{TB}[0,1]$ that is compact follows from the Ascoli-Arzela theorem. The Schauder fixed point theorem is applicable and a fixed point of T in D_0 exists. □

Corollary 3.2. *Assume that the function* $f(t, x_0, x_1, \ldots, x_q)$ *on* $[0,1] \times \mathbb{R}^{q+1}$ *satisfies the following condition:*

$$\left| f(t, x_0, x_1, \ldots, x_q) \right| \le L + \sum_{i=0}^{q} L_i |x_i|^{\lambda_i}, \tag{3.6}$$

where L, L_i, $0 \le i \le q$ *are nonnegative constants, and* $0 \le \lambda_i < 1$, $0 \le i \le q$, *then, the boundary value problem* (1.1), (1.2) *has a solution.*

Theorem 3.3. *Suppose that the function* $f(t, x_0, x_1, \ldots, x_q)$ *on* $[0,1] \times D_1$ *satisfies the following condition:*

$$\left| f(t, x_0, x_1, \ldots, x_q) \right| \le L + \sum_{i=0}^{q} L_i |x_i|, \tag{3.7}$$

where

$$D_1 = \left\{ (x_0, x_1, \ldots, x_q) : |x_k| \le p_k + C_{2m+1,k} \frac{L + c}{1 - \theta},\ 0 \le k \le q \right\}, \tag{3.8}$$

$$c = \sum_{i=0}^{q} L_i p_i, \tag{3.9}$$

$$\theta = \sum_{i=0}^{q} C_{2m+1,i} L_i < 1, \tag{3.10}$$

then, the boundary value problem (1.1), (1.2) *has a solution in* D_1.

Theorem 3.4. *Suppose that the differential equation* (1.1) *together with the homogeneous boundary conditions*

$$x(0) = 0, \quad x^{(2i-1)}(0) = 0, \quad x^{(2i-1)}(1) = 0, \quad 1 \le i \le m \tag{3.11}$$

has a nontrivial solution $x(t)$ *and the condition* (3.7) *with* $L = 0$ *is satisfied on* $[0,1] \times D_2$, *where*

$$D_2 = \left\{ (x_0, x_1, \ldots, x_q) : |x_k| \le C_{2m+1,k} M,\ 0 \le k \le q \right\} \tag{3.12}$$

and $M = \max_{t \in [0,1]} |x^{(2m+1)}(t)|$, *then, it is necessary that* $\theta \ge 1$.

Remark 3.5. Conditions of Theorem 3.4 ensure that in (3.7) at least one of the L_i, $0 \le i \le q$ will not be zero; otherwise the solution $x(t)$ will be a polynomial of degree at most $2m$ and will not be a nontrivial solution of (1.1), (1.2). Further, $x(t) \equiv 0$ is obviously a solution of (1.1), (1.2), and if $\theta < 1$, then it is also unique.

Theorem 3.6. *Suppose that for all* $(t, x_0, x_1, \ldots, x_q), (t, \overline{x}_0, \overline{x}_1, \ldots, \overline{x}_q) \in [0,1] \times D_1$ *the function* f *satisfies the Lipschitz condition*

$$\left| f(t, x_0, x_1, \ldots, x_q) - f(t, \overline{x}_0, \overline{x}_1, \ldots, \overline{x}_q) \right| \le \sum_{i=0}^{q} L_i |x_i - \overline{x}_i|, \tag{3.13}$$

where $L = \max_{t \in [0,1]} |f(t, 0, 0, \ldots, 0)|$, *then, the boundary value problem* (1.1), (1.2) *has a unique solution in* D_1.

Example 3.7. Consider the complementary Lidstone boundary value problem

$$-x^{(3)}(t) = f\left(t, x, x', \ldots, x^{(q)}\right), \quad t \in (0,1), \tag{3.14}$$

$$x(0) = 1, \qquad x'(0) = -1, \qquad x'(1) = 1, \tag{3.15}$$

where $0 \leq q \leq 2$ is fixed. Here, $m = 1$ and the interpolating polynomial satisfying (1.4) is computed as $P_2(t) = 1 - t + t^2$ with

$$p_0 = \max_{t\in[0,1]}|P_2(t)| = P_2(0) = 1, \qquad p_1 = \max_{t\in[0,1]}\left|P_2'(t)\right| = P_2'(1) = 1, \qquad p_2 = \max_{t\in[0,1]}\left|P_2''(t)\right| = 2. \tag{3.16}$$

We illustrate Theorem 3.1 by the following two cases.

Case 1. Suppose $q = 0$ and $f(t,x) = tx^2$, then, Theorem 3.1 states that (3.14), (3.15) has a solution in the set $D_0 = \{x : |x| \leq 2M_0\}$ provided

$$M_0 \geq p_0 = 1, \qquad QC_{3,0} \leq M_0. \tag{3.17}$$

We will look for a constant M_0 that satisfies (3.17). Since

$$Q = \max_{(t,x)\in[0,1]\times D_0}\left|f(t,x)\right| = (2M_0)^2, \tag{3.18}$$

the condition $QC_{3,0} \leq M_0$ simplifies to $0 \leq M_0 \leq 3$. Coupled with another condition $M_0 \geq 1$, we see that $1 \leq M_0 \leq 3$ fulfills (3.17). Therefore, we conclude that the differential equation

$$-x^{(3)}(t) = tx^2, \quad t \in (0,1) \tag{3.19}$$

with the boundary conditions (3.15) has a solution in $D_0 = \{x : |x| \leq 2M_0\}$ where $M_0 \in [1,3]$.

Case 2. Suppose $q = 2$ and $f(t,x,x',x'') = t^2x + \sqrt{t}x' + (t/2)x''$, then, Theorem 3.1 states that (3.14), (3.15) has a solution in the set $D_0 = \{(x,x',x'') : |x| \leq 2M_0, |x'| \leq 2M_1, |x''| \leq 2M_2\}$ provided

$$M_k \geq p_k, \quad QC_{3,k} \leq M_k, \quad k = 0,1,2. \tag{3.20}$$

Here

$$Q = \max_{(t,x,x',x'')\in[0,1]\times D_0}\left|f(t,x,x,x'')\right| = 2M_0 + 2M_1 + M_2, \tag{3.21}$$

and the conditions $QC_{3,k} \leq M_k$, $k = 0,1,2$, reduce to

$$10M_0 - 2M_1 - M_2 \geq 0, \quad -2M_0 + 6M_1 - M_2 \geq 0, \quad -2M_0 - 2M_1 + M_2 \geq 0. \tag{3.22}$$

Pick $M_0 = 1$, $M_1 = 1$, $M_2 = 4$ which satisfy (3.22) and also $M_k \geq p_k$, $k = 0,1,2$. It follows from Theorem 3.1 that the differential equation

$$-x^{(3)}(t) = t^2x + \sqrt{t}x' + \left(\frac{t}{2}\right)x'', \quad t \in (0,1) \tag{3.23}$$

with the boundary conditions (3.15) has a solution in $D_0 = \{(x, x', x'') : |x| \leq 2, |x'| \leq 2, |x''| \leq 8\}$.

Example 3.8. Consider the complementary Lidstone boundary value problem

$$-x^{(3)}(t) = \sin t + (\sin t)x + (\cos t)x' + \frac{x''}{4}, \quad t \in (0,1) \tag{3.24}$$

with the boundary conditions (3.15). Here, $m = 1$, $q = 2$ and the interpolating polynomial $P_2(t)$ satisfying (1.4) is given in Example 3.7. To illustrate Theorem 3.3, we note that for $t \in [0,1]$ and any (x_0, x_1, x_2),

$$\left| f(t, x_0, x_1, x_2) \right| = \left| \sin t + (\sin t)x_0 + (\cos t)x_1 + \frac{x_2}{4} \right| \leq 1 + |x_0| + |x_1| + \frac{|x_2|}{4}. \tag{3.25}$$

Thus, condition (3.7) is satisfied with $L = 1$, $L_0 = 1$, $L_1 = 1$, $L_2 = 1/4$. The constants c and θ are then computed as

$$c = \sum_{i=0}^{2} L_i p_i = \frac{5}{2}, \qquad \theta = \sum_{i=0}^{2} C_{3,i} L_i = \frac{1}{3} < 1. \tag{3.26}$$

By Theorem 3.3, problem (3.24), (3.15) has a solution in

$$D_1 = \left\{ (x, x', x'') : |x| \leq \frac{23}{16}, \ \left|x'\right| \leq \frac{53}{32}, \ \left|x''\right| \leq \frac{37}{8} \right\}. \tag{3.27}$$

4. Picard's and Approximate Picard's Methods

Picard's method of successive approximations has an important characteristic, namely, it is constructive; moreover, bounds of the difference between iterates and the solution are easily available. In this section, we will provide a priori as well as posteriori estimates on the Lipschitz constants so that Picard's iterative sequence $\{x_n(t)\}$ converges to the unique solution $x^*(t)$ of the problem (1.1), (1.2).

Definition 4.1. A function $\overline{x}(t) \in C^{(2m+1)}[0,1]$ is called an *approximate solution* of (1.1), (1.2) if there exist nonnegative constants δ and ϵ such that

$$\max_{t \in [0,1]} \left| (-1)^m \overline{x}^{(2m+1)}(t) - f(t, \overline{\mathrm{x}}(t)) \right| \leq \delta, \tag{4.1}$$

$$\max_{t \in [0,1]} \left| P_{2m}^{(k)}(t) - \overline{P}_{2m}^{(k)}(t) \right| \leq \epsilon C_{2m+1,k}, \quad 0 \leq k \leq q, \tag{4.2}$$

where $P_{2m}(t)$ and $\overline{P}_{2m}(t)$ are polynomials of degree $2m$ satisfying (1.2), and

$$\overline{P}(0) = \overline{x}(0), \quad \overline{P}^{(2i-1)}(0) = \overline{x}^{(2i-1)}(0), \quad \overline{P}^{(2i-1)}(1) = \overline{x}^{(2i-1)}(1), \quad 0 \leq i \leq m, \tag{4.3}$$

respectively.

Inequality (4.1) means that there exists a continuous function $\eta(t)$ such that

$$\begin{gathered} (-1)^m \overline{x}^{(2m+1)}(t) = f(t, \overline{\mathbf{x}}(t)) + \eta(t), \\ \max_{t\in[0,1]} |\eta(t)| \leq \delta. \end{gathered} \tag{4.4}$$

Thus, from Theorem 2.3 the approximate solution $\overline{x}(t)$ can be expressed as

$$\overline{x}(t) = \overline{P}_{2m}(t) + \int_0^1 |h_m(t,s)| [f(s, \overline{\mathbf{x}}(s)) + \eta(s)] ds. \tag{4.5}$$

In what follows, we will consider the Banach space $B = C^{(q)}[0,1]$ and for $x \in C^{(q)}[0,1]$,

$$\|x\| = \max_{0\leq k\leq q} \left\{ \frac{C_{2m+1,0}}{C_{2m+1,k}} \max_{t\in[0,1]} \left| x^{(k)}(t) \right| \right\}. \tag{4.6}$$

Theorem 4.2. *With respect to the boundary value problem* (1.1), (1.2) *one assumes that there exists an approximate solution* $\overline{x}(t)$, *and*

(i) *the function* $f(t, x_0, x_1, \ldots, x_q)$ *satisfies the Lipschitz condition* (3.13) *on* $[0,1] \times D_3$, *where*

$$D_3 = \left\{ (x_0, x_1, \ldots, x_q) : \left| x_k - \overline{x}^{(k)}(t) \right| \leq N \frac{C_{2m+1,k}}{C_{2m+1,0}}, \ 0 \leq k \leq q, \ N > 0 \right\}, \tag{4.7}$$

(ii) $N_0 = (1-\theta)^{-1}(\epsilon + \delta) C_{2m+1,0} \leq N$.

Then, the following hold:

(1) *there exists a solution* $x^*(t)$ *of* (1.1), (1.2) *in* $\overline{S}(\overline{x}, N_0) = \{x \in B : \|x - \overline{x}\| \leq N_0\}$,

(2) $x^*(t)$ *is the unique solution of* (1.1), (1.2) *in* $\overline{S}(\overline{x}, N)$,

(3) *the Picard iterative sequence* $\{x_n(t)\}$, *defined by*

$$x_{n+1}(t) = P_{2m}(t) + \int_0^1 |h_m(t,s)| f(s, \mathbf{x}_n(s)) ds, \quad n = 0, 1, \ldots, \tag{4.8}$$

where $x_0(t) = \overline{x}(t)$ *converges to* $x^*(t)$ *with* $\|x^* - x_n\| \leq \theta^n N_0$, *and*

$$\|x^* - x_n\| \leq \theta(1-\theta)^{-1} \|x_n - x_{n-1}\|, \tag{4.9}$$

(4) *for any* $x_0(t) = x(t) \in \overline{S}(\overline{x}, N_0)$, $x^*(t) = \lim_{n\to\infty} x_n(t)$.

In Theorem 4.2 conclusion (3) ensures that the sequence $\{x_n(t)\}$ obtained from (4.8) converges to the solution $x^*(t)$ of the boundary value problem (1.1), (1.2). However, in practical evaluation this sequence is approximated by the computed sequence, say, $\{z_n(t)\}$. To find $z_{n+1}(t)$, the function f is approximated by f_n. Therefore, the computed sequence $\{z_n(t)\}$ satisfies the recurrence relation

$$z_{n+1}(t) = P_{2m}(t) + \int_0^1 |h_m(t,s)| f_n(s, \mathbf{z}_n(s)) ds, \quad n = 0, 1, \ldots, \tag{4.10}$$

where $z_0(t) = x_0(t) = \overline{x}(t)$.

With respect to f_n we will assume the following condition.

Condition C1. For $\mathbf{z}_n(t)$ obtained from (4.10), the following inequality holds:

$$\left| f(t, \mathbf{z}_n(t)) - f_n(t, \mathbf{z}_n(t)) \right| \le \mu \left| f(t, \mathbf{z}_n(t)) \right|, \quad n = 0, 1, \ldots, \tag{4.11}$$

where μ is a nonnegative constant.

Inequality (4.11) corresponds to the relative error in approximating the function f by f_n for the $(n+1)$th iteration.

Theorem 4.3. *With respect to the boundary value problem* (1.1), (1.2) *one assumes that there exists an approximate solution* $\overline{x}(t)$, *and Condition C1 is satisfied. Further, one assumes that*

(i) *condition (i) of Theorem 4.2,*

(ii) $\theta_1 = (1+\mu)\theta < 1$,

(iii) $N_1 = (1-\theta_1)^{-1}(\epsilon + \delta + \mu F) C_{2m+1,0} \le N$, *where* $F = \max_{t \in [0,1]} |F(t, \overline{\mathbf{x}}(t))|$,

then,

(1) *all the conclusions (1)–(4) of Theorem 4.2 hold,*

(2) *the sequence* $\{z_n(t)\}$ *obtained from* (4.10) *remains in* $\overline{S}(\overline{x}, N_1)$,

(3) *the sequence* $\{z_n(t)\}$ *converges to* $x^*(t)$, *the solution of* (1.1), (1.2) *if and only if* $\lim_{n\to\infty} a_n = 0$, *where*

$$a_n = \left\| z_{n+1}(t) - P_{2m}(t) - \int_0^1 |h_m(t,s)| f(s, \mathbf{z}_n(s)) ds \right\|, \tag{4.12}$$

and the following error estimate holds

$$\|x^* - z_{n+1}\| \le (1-\theta)^{-1} \left[\theta \|z_{n+1} - z_n\| + \mu C_{2m+1,0} \max_{t \in [0,1]} \left| f(t, \mathbf{z}_n(t)) \right| \right]. \tag{4.13}$$

In our next result we will assume the following.

Condition C2. For $\mathbf{z}_n(t)$ obtained from (4.10), the following inequality is satisfied:

$$\left|f(t,\mathbf{z}_n(t)) - f_n(t,\mathbf{z}_n(t))\right| \leq \nu, \quad n = 0,1,\ldots, \tag{4.14}$$

where ν is a nonnegative constant.

Inequality (4.14) corresponds to the absolute error in approximating the function f by f_n for the $(n+1)$th iteration.

Theorem 4.4. *With respect to the boundary value problem* (1.1), (1.2) *one assumes that there exists an approximate solution* $\overline{x}(t)$*, and Condition C2 is satisfied. Further, one assumes that*

(i) *condition (i) of Theorem 4.2,*

(ii) $N_2 = (1-\theta)^{-1}(\epsilon + \delta + \nu)C_{2m+1,0} \leq N$,

then,

(1) *all the conclusions (1)–(4) of Theorem 4.2 hold,*

(2) *the sequence* $\{z_n(t)\}$ *obtained from* (4.10) *remains in* $\overline{S}(\overline{x}, N_2)$,

(3) *the sequence* $\{z_n(t)\}$ *converges to* $x^*(t)$*, the solution of* (1.1), (1.2) *if and only if* $\lim_{n\to\infty} a_n = 0$*, and the following error estimate holds:*

$$\|x^* - z_{n+1}\| \leq (1-\theta)^{-1}\left[\theta\|z_{n+1} - z_n\| + \nu C_{2m+1,0}\right]. \tag{4.15}$$

Example 4.5. Consider the complementary Lidstone boundary value problem

$$-x^{(3)}(t) = 1 + x + x' + \frac{x''}{4}, \quad t \in (0,1) \tag{4.16}$$

with the boundary conditions (3.15). Pick $P_2(t) = 1 - t + t^2$ to be an approximate solution of (4.16), (3.15), that is, let $\overline{x}(t) = P_2(t)$. Then, from (4.2) we get $\epsilon = 0$. Further, from (4.1) we have

$$\begin{aligned}
&\max_{t\in[0,1]}\left|-\overline{x}^{(3)}(t) - f\left(t,\overline{x}(t),\overline{x}'(t),\overline{x}''(t)\right)\right| \\
&\quad = \max_{t\in[0,1]}\left|f\left(t,\overline{x}(t),\overline{x}'(t),\overline{x}''(t)\right)\right| \\
&\quad = \max_{t\in[0,1]}\left|1 + \overline{x}(t) + \overline{x}'(t) + \frac{\overline{x}''(t)}{4}\right| \\
&\quad = \max_{t\in[0,1]}\left|\frac{3}{2} + t + t^2\right| = \frac{7}{2} \equiv \delta.
\end{aligned} \tag{4.17}$$

To illustrate Theorem 4.2, we note that the Lipschitz condition (3.13) is satisfied *globally* with $L_0 = 1$, $L_1 = 1$, $L_2 = 1/4$, and the constants θ and N_0 are computed directly as

$$\theta = \sum_{i=0}^{2} C_{3,i} L_i = \frac{1}{3}, \qquad N_0 = (1-\theta)^{-1}(\epsilon+\delta)C_{3,0} = \frac{21}{4} \leq N. \tag{4.18}$$

By Theorem 4.2, it follows that

(1) there exists a solution $x^*(t)$ of (4.16), (3.15) in $\overline{S}(P_2, N_0)$,

(2) $x^*(t)$ is the unique solution of (4.16), (3.15) in $\overline{S}(P_2, N)$,

(3) the Picard iterative sequence $\{x_n(t)\}$ defined by

$$\begin{gathered} -x_{n+1}^{(3)}(t) = 1 + x_n(t) + x_n'(t) + \frac{x_n''(t)}{4}, \quad n = 0, 1, \ldots, \\ x_{n+1}(0) = 1, \quad x_{n+1}'(0) = -1, \quad x_{n+1}'(1) = 1, \end{gathered} \tag{4.19}$$

where $x_0(t) = P_2(t)$ converges to $x^*(t)$ with

$$\|x^* - x_n\| \leq \left(\frac{1}{3}\right)^n \frac{21}{4}, \qquad \|x^* - x_n\| \leq \frac{1}{2}\|x_n - x_{n-1}\|. \tag{4.20}$$

Suppose that we require the accuracy $\|x^* - x_n\| \leq 10^{-5}$, then from above we just set

$$\left(\frac{1}{3}\right)^n \frac{21}{4} \leq 10^{-5} \tag{4.21}$$

to get $n \geq 12$. Thus, $x_{12}(t)$ will fulfill the required accuracy.

Finally, we will illustrate how to obtain $x_1(t)$ from (4.19). First, we integrate

$$-x_1^{(3)}(t) = 1 + x_0(t) + x_0'(t) + \frac{x_0''(t)}{4} = \frac{3}{2} + t + t^2 \tag{4.22}$$

from 0 to t to get

$$-x_1''(t) + x_1''(0) = \frac{3t}{2} + \frac{t^2}{2} + \frac{t^3}{3}. \tag{4.23}$$

Next, integrating (4.23) from 0 to t as well as from t to 1, respectively, gives

$$-x_1'(t) + x_1'(0) + t x_1''(0) = \frac{3t^2}{4} + \frac{t^3}{6} + \frac{t^4}{12}, \tag{4.24}$$

$$-x_1'(1) + x_1'(t) + (1-t)x_1''(0) = 1 - \frac{3t^2}{4} - \frac{t^3}{6} - \frac{t^4}{12}. \tag{4.25}$$

Adding (4.24) and (4.25) yields $x_1''(0) = 3$. Now, integrate (4.24) (or (4.25)) from 0 to t gives

$$x_1(t) = 1 - t + \frac{3t^2}{2} - \frac{t^3}{4} - \frac{t^4}{24} - \frac{t^5}{60}. \tag{4.26}$$

A similar method can be used to obtain $x_n(t)$, $n \ge 2$.

5. Quasilinearization and Approximate Quasilinearization

Newton's method when applied to differential equations has been labeled as quasilinearization. This quasilinear iterative scheme for (1.1), (1.2) is defined as

$$(-1)^m x_{n+1}^{(2m+1)}(t) = f(t, \mathbf{x}_n(t)) + \beta(t) \sum_{i=0}^{q} \left(x_{n+1}^{(i)}(t) - x_n^{(i)}(t) \right) \frac{\partial}{\partial x_n^{(i)}(t)} f(t, \mathbf{x}_n(t)), \tag{5.1}$$

$$x_{n+1}(0) = \alpha_0, \quad x_{n+1}^{(2i-1)}(0) = \alpha_i, \quad x_{n+1}^{(2i-1)}(1) = \beta_i, \quad 0 \le i \le m, \; n = 0, 1, \ldots, \tag{5.2}$$

where $x_0(t) = \overline{x}(t)$ is an approximate solution of (1.1), (1.2).

In the following results once again we will consider the Banach space $C^{(q)}[0,1]$ and for $x \in C^{(q)}[0,1]$ the norm $\|x\|$ is as in (4.6).

Theorem 5.1. *With respect to the boundary value problem* (1.1), (1.2) *one assumes that there exists an approximate solution* $\overline{x}(t)$, *and*

(i) *the function* $f(t, x_0, x_1, \ldots, x_q)$ *is continuously differentiable with respect to all* x_i, $0 \le i \le q$ *on* $[0,1] \times D_3$,

(ii) *there exist nonnegative constants* L_i, $0 \le i \le q$ *such that for all* $(t, x_0, x_1, \ldots, x_q) \in [0,1] \times D_3$,

$$\left| \frac{\partial}{\partial x_i} f(t, x_0, x_1, \ldots, x_q) \right| \le L_i, \tag{5.3}$$

(iii) *the function* $\beta(t)$ *is continuous on* $[0,1]$, $\beta = \max_{t \in [0,1]} |\beta(t)|$, *and* $\theta_\beta = (1 + 2\beta)\theta < 1$,

(iv) $N_3 = (1 - \theta_\beta)^{-1} (\epsilon + \delta) C_{2m+1,0} \le N$.

Then, the following hold:

(1) *the sequence* $\{x_n(t)\}$ *generated by the iterative scheme* (5.1), (5.2) *remains in* $\overline{S}(\overline{x}, N_3)$,

(2) *the sequence* $\{x_n(t)\}$ *converges to the unique solution* $x^*(t)$ *of the boundary value problem* (1.1), (1.2),

(3) *a bound on the error is given by*

$$\|x_n - x^*\| \le \left(\frac{(1+\beta)\theta}{1 - \beta\theta} \right)^n (1 - \theta_\beta)^{-1} (\epsilon + \delta) C_{2m+1,0}. \tag{5.4}$$

Theorem 5.2. *Let in Theorem 5.1 the function $\beta(t) \equiv 1$. Further, let $f(t, x_0, x_1, \ldots, x_q)$ be twice continuously differentiable with respect to all x_i, $0 \le i \le q$ on $[0,1] \times D_3$, and*

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} f(t, x_0, x_1, \ldots, x_q) \right| \le L_i L_j K, \quad 0 \le i,\ j \le q. \tag{5.5}$$

Then,

$$\|x_{n+1} - x_n\| \le \alpha \|x_n - x_{n-1}\|^2 \le \frac{1}{\alpha} (\alpha \|x_1 - x_0\|)^{2^n} \le \frac{1}{\alpha} \left\{ \frac{1}{2} K(\epsilon + \delta) \left(\frac{\theta}{1-\theta} \right)^2 \right\}^{2^n}, \tag{5.6}$$

where $\alpha = K\theta^2 / [2(1-\theta) C_{2m+1,0}]$. Thus, the convergence is quadratic if

$$\frac{1}{2} K(\epsilon + \delta) \left(\frac{\theta}{1-\theta} \right)^2 < 1. \tag{5.7}$$

Conclusion (3) of Theorem 5.1 ensures that the sequence $\{x_n(t)\}$ generated from the scheme (5.1), (5.2) converges linearly to the unique solution $x^*(t)$ of the boundary value problem (1.1), (1.2). Theorem 5.2 provides sufficient conditions for its quadratic convergence. However, in practical evaluation this sequence is approximated by the computed sequence, say, $\{z_n(t)\}$ which satisfies the recurrence relation

$$\begin{gathered} (-1)^m z_{n+1}^{(2m+1)}(t) = f_n(t, \mathbf{z}_n(t)) + \beta(t) \sum_{i=0}^{q} \left(z_{n+1}^{(i)}(t) - z_n^{(i)}(t) \right) \frac{\partial}{\partial z_n^{(i)}(t)} f_n(t, \mathbf{z}_n(t)), \\ z_{n+1}(0) = \alpha_0, \quad z_{n+1}^{(2i-1)}(0) = \alpha_i, \quad z_{n+1}^{(2i-1)}(1) = \beta_i, \quad 0 \le i \le m,\ n = 0, 1, \ldots, \end{gathered} \tag{5.8}$$

where $z_0(t) = x_0(t) = \overline{x}(t)$.

With respect to f_n we will assume the following condition.

Condition C3. $f_n(t, x_0, x_1, \ldots, x_q)$ is continuously differentiable with respect to all x_i, $0 \le i \le q$ on $[0,1] \times D_3$ with

$$\left| \frac{\partial}{\partial x_i} f_n(t, x_0, x_1, \ldots, x_q) \right| \le L_i \tag{5.9}$$

and Condition C1 is satisfied.

Theorem 5.3. *With respect to the boundary value problem* (1.1), (1.2) *one assumes that there exists an approximate solution $\overline{x}(t)$, and the Condition C3 is satisfied. Further, one assumes*

(i) *conditions (i) and (ii) of Theorem 5.1,*

(ii) $\theta_{\beta,\mu} = (1 + 2\beta + \mu)\theta < 1$,

(iii) $N_4 = (1 - \theta_{\beta,\mu})^{-1} (\epsilon + \delta + \mu F) C_{2m+1,0} \le N$, *then,*

(1) *all conclusions (1)–(3) of Theorem 5.1 hold,*

(2) *the sequence* $\{z_n(t)\}$ *generated by the iterative scheme* (5.8), *remains in* $\overline{S}(\overline{x}, N_4)$,

(3) *the sequence* $\{z_n(t)\}$ *converges to* $x^*(t)$, *the unique solution of* (1.1), (1.2) *if and only if* $\lim_{n\to\infty} a_n = 0$, *and the following error estimate holds:*

$$\|x^* - z_{n+1}\| \le (1-\theta)^{-1}\left[(1+\beta)\theta\|z_{n+1} - z_n\| + \mu C_{2m+1,0}\max_{t\in[0,1]}\left|f(t, \mathbf{z}_n(t))\right|\right]. \tag{5.10}$$

Theorem 5.4. *Let the conditions of Theorem 5.3 be satisfied. Further, let* $f_n = f_0$ *for all* $n = 1, 2, \ldots$ *and* $f_0(t, x_0, x_1, \ldots, x_q)$ *be twice continuously differentiable with respect to all* x_i, $0 \le i \le q$ *on* $[0,1] \times D_3$, *and*

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} f_0(t, x_0, x_1, \ldots, x_q)\right| \le L_i L_j K, \quad 0 \le i,\ j \le q. \tag{5.11}$$

Then,

$$\|z_{n+1} - z_n\| \le \alpha\|z_n - z_{n-1}\|^2 \le \frac{1}{\alpha}(\alpha\|z_1 - z_0\|)^{2^n} \le \frac{1}{\alpha}\left[\frac{1}{2}K(\epsilon + \delta + \mu F)\left(\frac{\theta}{1-\theta}\right)^2\right]^{2^n}, \tag{5.12}$$

where α *is the same as in Theorem 5.2.*

Example 5.5. Consider the complementary Lidstone boundary value problem

$$-x^{(3)}(t) = t + x^2, \quad t \in (0,1) \tag{5.13}$$

again with the boundary conditions (3.15). First, we will illustrate Theorem 5.1. Pick $\overline{x}(t) = 0$ and $\beta(t) = 1$ (so $\beta = 1$). Clearly, $f(t, x) = t + x^2$ is continuously differentiable with respect to x for all (t, x). For $x \in D_3 = \{x : |x| \le N\}$, we have

$$\left|\frac{\partial}{\partial x} f(t, x)\right| = |2x| \le 2N \equiv L_0. \tag{5.14}$$

Thus,

$$\theta = C_{3,0}L_0 = \frac{N}{6}, \qquad \theta_\beta = (1 + 2\beta)\theta = \frac{N}{2}. \tag{5.15}$$

Let $N < 2$ so that $\theta_\beta < 1$. Next, from (4.1) we have $\max_{t\in[0,1]}|f(t, 0)| = 1 \equiv \delta$. Also, from (4.2) we find

$$\max_{t\in[0,1]}\left|P_2(t) - \overline{P}_2(t)\right| = \max_{t\in[0,1]}|P_2(t)| = 1 \le \epsilon C_{3,0} = \frac{\epsilon}{12}, \tag{5.16}$$

and so we take $\epsilon = 12$. Now,

$$N_3 = \left(1-\theta_\beta\right)^{-1}(\epsilon+\delta)C_{3,0} = \frac{13}{6N} \le N \tag{5.17}$$

yields $N \ge \sqrt{13/6} \simeq 1.633$. Coupled with $N < 2$ (so that $\theta_\beta < 1$), we should impose

$$\sqrt{\frac{13}{6}} \le N < 2. \tag{5.18}$$

The corresponding range of N_3 will then be

$$\frac{13}{12} < N_3 \le \sqrt{\frac{13}{6}}. \tag{5.19}$$

The conditions of Theorem 5.1 are satisfied and so

(1) the sequence $\{x_n(t)\}$ generated by

$$\begin{gathered} -x_{n+1}^{(3)}(t) = t + x_n^2(t) + 2[x_{n+1}(t) - x_n(t)]x_n(t), \quad n = 0,1,\ldots, \\ x_{n+1}(0) = 1, \qquad x'_{n+1}(0) = -1, \qquad x'_{n+1}(1) = 1, \end{gathered} \tag{5.20}$$

where $x_0(t) = 0$ remains in $\overline{S}(0, N_3)$, that is, $\max_{t\in[0,1]}|x_n(t)| \le N_3$,

(2) the sequence $\{x_n(t)\}$ converges to the unique solution $x^*(t)$ of (5.13), (3.15) with

$$\max_{t\in[0,1]}|x^*(t) - x_n(t)| \le \left(\frac{2N}{6-N}\right)^n \frac{13}{6(2-N)}. \tag{5.21}$$

Next, we will illustrate Theorem 5.2. For $x \in D_3 = \{x : |x| \le N\}$, we have

$$\left|\frac{\partial^2}{\partial x^2} f(t,x)\right| = 2 \le L_0^2 K = (2N)^2 K. \tag{5.22}$$

Hence, we may take $K = 1/(2N^2)$. From Theorem 5.2, we have

$$\max_{t\in[0,1]}|x_{n+1}(t) - x_n(t)| \le \frac{1}{\alpha}\left[\frac{1}{2}K(\epsilon+\delta)\left(\frac{\theta}{1-\theta}\right)^2\right]^{2^n} = 2(6-N)\left[\frac{13}{4(6-N)^2}\right]^{2^n}. \tag{5.23}$$

The convergence is quadratic if

$$\frac{1}{2}K(\epsilon+\delta)\left(\frac{\theta}{1-\theta}\right)^2 < 1 \tag{5.24}$$

which is the same as

$$\frac{13}{4} < (6 - N)^2 \tag{5.25}$$

and is satisfied if $N > 7.803$ or $N < 4.197$. Combining with (5.18), we conclude that the convergence of the scheme (5.20) is quadratic if

$$\sqrt{\frac{13}{6}} \leq N < 2. \tag{5.26}$$

6. Monotone Convergence

It is well recognized that the method of upper and lower solutions, together with uniformly monotone convergent technique offers effective tools in proving and constructing multiple solutions of nonlinear problems. The upper and lower solutions generate an interval in a suitable partially ordered space, and serve as upper and lower bounds for solutions which can be improved by uniformly monotone convergent iterative procedures. Obviously, from the computational point of view monotone convergence has superiority over ordinary convergence. We will discuss this fruitful technique for the boundary value problem (1.1), (1.2) with $q = 1$.

Definition 6.1. A function $\mu(t) \in C^{(2m+1)}[0,1]$ is said to be a *lower solution* of (1.1), (1.2) with $q = 1$ provided

$$\begin{gathered}(-1)^m \mu^{(2m+1)}(t) \leq f\left(t, \mu(t), \mu'(t)\right), \quad t \in [0,1], \\ \left[\mu(0) - \alpha_0\right] \leq 0, \quad (-1)^{i-1}\left[\mu^{(2i-1)}(0) - \alpha_i\right] \leq 0, \quad (-1)^{i-1}\left[\mu^{(2i-1)}(1) - \beta_i\right] \leq 0, \quad 1 \leq i \leq m.\end{gathered} \tag{6.1}$$

Similarly, a function $\nu(t) \in C^{(2m+1)}[0,1]$ is said to be an *upper solution* of (1.1), (1.2) with $q = 1$ if

$$\begin{gathered}(-1)^m \nu^{(2m+1)}(t) \geq f(t, \nu(t), \nu'(t)), \quad t \in [0,1], \\ [\nu(0) - \alpha_0] \geq 0, \quad (-1)^{i-1}\left[\nu^{(2i-1)}(0) - \alpha_i\right] \geq 0, \quad (-1)^{i-1}\left[\nu^{(2i-1)}(1) - \beta_i\right] \geq 0, \quad 1 \leq i \leq m.\end{gathered} \tag{6.2}$$

Lemma 6.2. *Let $\mu(t)$ and $\nu(t)$ be lower and upper solutions of* (1.1), (1.2) *with $q = 1$, and let $P_{2m,\mu}(t)$ and $P_{2m,\nu}(t)$ be the polynomials of degree $2m$ satisfying*

$$P_{2m,\mu}(0) = \mu(0), \quad P_{2m,\mu}^{(2i-1)}(0) = \mu^{(2i-1)}(0), \quad P_{2m,\mu}^{(2i-1)}(1) = \mu^{(2i-1)}(1), \quad 1 \leq i \leq m, \tag{6.3}$$

and

$$P_{2m,\nu}(0) = \mu(0), \quad P_{2m,\nu}^{(2i-1)}(0) = \nu^{(2i-1)}(0), \quad P_{2m,\nu}^{(2i-1)}(1) = \nu^{(2i-1)}(1), \quad 1 \le i \le m, \tag{6.4}$$

respectively. Then, for all $t \in [0,1]$, $P_{2m,\mu}^{(k)}(t) \le P_{2m}^{(k)}(t) \le P_{2m,\nu}^{(k)}(t)$, $k = 0,1$.

Proof. From (2.5), (2.6), and (2.8) it is clear that $(-1)^i\Lambda_i(t) \ge 0$, $(-1)^i\Lambda_i(1-t) \ge 0$, $i \ge 0$ and this in turn from (2.18) and (2.19) implies that $(-1)^i(v_{i+1}(t)-v_{i+1}(0)) \ge 0$, $(-1)^i(v_{i+1}(1)-v_{i+1}(1-t)) \ge 0$, $(-1)^i v'_{i+1}(t) = (-1)^i\Lambda_i(t) \ge 0$, $(-1)^i v'_{i+1}(1-t) = (-1)^i\Lambda_i(1-t) \ge 0$, $i \ge 0$. Now, since

$$\begin{aligned} P_{2m,\mu}(t) &= \mu(0) + \sum_{i=1}^{m}\Big[\mu^{(2i-1)}(0)(v_i(1) - v_i(1-t)) + \mu^{(2i-1)}(1)(v_i(t) - v_i(0))\Big], \\ P'_{2m,\mu}(t) &= \sum_{i=1}^{m}\Big[\mu^{(2i-1)}(0)\Lambda_{i-1}(1-t) + \mu^{(2i-1)}(1)\Lambda_{i-1}(t)\Big], \end{aligned} \tag{6.5}$$

it follows that

$$\begin{aligned} P_{2m,\mu}(t) &= \mu(0) + \sum_{i=1}^{m}\Big[(-1)^{i-1}\mu^{(2i-1)}(0)(-1)^{i-1}(v_i(1) - v_i(1-t)) \\ &\qquad + (-1)^{i-1}\mu^{(2i-1)}(1)(-1)^{i-1}(v_i(t) - v_i(0))\Big] \\ &\le \alpha_0 + \sum_{i=1}^{m}\Big[(-1)^{i-1}\alpha_i(-1)^{i-1}(v_i(1) - v_i(1-t)) + (-1)^{i-1}\beta_i(-1)^{i-1}(v_i(t) - v_i(0))\Big] \\ &= P_{2m}(t). \end{aligned} \tag{6.6}$$

Similarly, we have $P'_{2m,\mu}(t) \le P'_{2m}(t)$. The proof of $P_{2m}^{(k)}(t) \le P_{2m,\nu}^{(k)}(t)$, $k = 0,1$ is similar. □

In the following result for $x(t) \in C^1[0,1]$ we will consider the norm $\|x\| = \max\{\max_{t\in[0,1]}|x(t)|, \max_{t\in[0,1]}|x'(t)|\}$ and introduce a partial ordering $\preceq$ as follows. For $x, y \in C^1[0,1]$ we say that $x \preceq y$ if and only if $x(t) \le y(t)$ and $x'(t) \le y'(t)$ for all $t \in [0,1]$.

Theorem 6.3. *With respect to the boundary value problem* (1.1), (1.2) *with* $q = 1$ *one assumes that* $f(t, x_0, y_0)$ *is nondecreasing in* x_0 *and* y_0*. Further, let there exist lower and upper solutions* $\mu_0(t), \nu_0(t)$ *such that* $\mu_0 \preceq \nu_0$*. Then, the sequences* $\{\mu_n(t)\}, \{\nu_n(t)\}$ *where* $\mu_n(t)$ *and* $\nu_n(t)$ *are defined by the iterative schemes*

$$\begin{aligned} \mu_{n+1}(t) &= P_{2m}(t) + \int_0^1 |g_m(t,s)| f(s, \mu_n(s), \mu'_n(s))\,ds, \quad n = 0,1,\ldots, \\ \nu_{n+1}(t) &= P_{2m}(t) + \int_0^1 |g_m(t,s)| f(s, \nu_n(s), \nu'_n(s))\,ds, \quad n = 0,1,\ldots \end{aligned} \tag{6.7}$$

are well defined, and $\{\mu_n(t)\}$ *converges to an element* $\mu(t) \in C^1[0,1]$, $\{\nu_n(t)\}$ *converges to an element* $\nu(t) \in C^1[0,1]$ *(with the convergence being in the norm of* $C^1[0,1]$*). Further,* $\mu_0 \preceq \mu_1 \preceq \cdots \preceq \mu_n \preceq \cdots \preceq \mu \preceq \nu \preceq \cdots \preceq \nu_n \preceq \cdots \preceq \nu_1 \preceq \nu_0$, $\mu(t), \nu(t)$ *are solutions of* (1.1), (1.2) *with* $q = 1$*, and each solution* $z(t)$ *of this problem which is such that* $z \in [\mu_0, \nu_0]$ *satisfies* $\mu \preceq z \preceq \nu$.

Example 6.4. Consider the complementary Lidstone boundary value problem

$$\begin{gathered} -x^{(3)}(t) = 1 + x + x', \quad t \in (0,1), \\ x(0) = 1, \qquad x'(0) = -1, \qquad x'(1) = -1. \end{gathered} \tag{6.8}$$

Here, $m = 1, q = 1$ and the function $f(t, x_0, y_0) = 1 + x_0 + y_0$ is nondecreasing in x_0 and y_0. We find that (6.8) has a lower solution

$$\mu_0(t) = 1 - t \tag{6.9}$$

and an upper solution

$$\nu_0(t) = 1 + 8t^2 - \frac{17}{3}t^3 \tag{6.10}$$

such that

$$\mu_0(t) \le \nu_0(t), \quad \mu_0'(t) \le v_0'(t), \quad t \in [0,1]. \tag{6.11}$$

Hence, $\mu_0 \preceq \nu_0$ and the conditions of Theorem 6.3 are satisfied. The iterative schemes

$$\begin{gathered} -\mu_{n+1}^{(3)}(t) = 1 + \mu_n + \mu_n', \quad n = 0, 1, \ldots, \\ \mu_{n+1}(0) = 1, \qquad \mu_{n+1}'(0) = -1, \qquad \mu_{n+1}'(1) = -1, \end{gathered} \tag{6.12}$$

$$\begin{gathered} -\nu_{n+1}^{(3)}(t) = 1 + \nu_n + \nu_n', \quad n = 0, 1, \ldots, \\ \nu_{n+1}(0) = 1, \qquad \nu_{n+1}'(0) = -1, \qquad \nu_{n+1}'(1) = -1 \end{gathered} \tag{6.13}$$

will converge respectively to some $\mu \in C^1[0,1]$ and $\nu \in C^1[0,1]$. Moreover,

$$\mu_0 \preceq \mu_1 \preceq \cdots \preceq \mu_n \preceq \cdots \preceq \mu \preceq \nu \preceq \cdots \preceq \nu_n \preceq \cdots \preceq \nu_1 \preceq \nu_0, \tag{6.14}$$

and $\mu(t), \nu(t)$ are solutions of (6.8). Any solution $z(t)$ of (6.8) which is such that $z \in [\mu_0, \nu_0]$ fulfills $\mu \preceq z \preceq \nu$. As an illustration, by direct computation (as in Example 4.5), we find

$$\begin{gathered}
\mu_1(t) = 1 - t + \frac{t^2}{6} - \frac{t^3}{6} + \frac{t^4}{24}, \\
\mu_2(t) = 1 - t - \frac{29t^2}{160} + \frac{t^3}{6} - \frac{t^4}{36} - \frac{t^5}{180} + \frac{t^7}{5040}, \\
\dots, \\
\nu_1(t) = 1 - t - \frac{79t^2}{60} + \frac{t^3}{3} + \frac{2t^4}{3} - \frac{3t^5}{20} - \frac{17t^6}{360}, \\
\nu_2(t) = 1 - t - \frac{83t^2}{40320} + \frac{t^3}{6} - \frac{109t^4}{720} - \frac{19t^5}{3600} + \frac{t^6}{40} - \frac{t^7}{2520} - \frac{13t^8}{10080} - \frac{17t^9}{181440}. \\
\dots.
\end{gathered} \tag{6.15}$$

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