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Research Article

Two Sharp Inequalities for Power Mean, Geometric Mean, and Harmonic Mean

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For $p \in R$, the power mean of order p of two positive numbers a and b is defined by $M_p(a,b) = ((a^p+b^p)/2)^{1/p}, p \neq 0$, and $M_p(a,b) = \sqrt{ab}$, $p = 0$. In this paper, we establish two sharp inequalities as follows: $(2/3)G(a,b) + (1/3)H(a,b) \geqslant M_{-1/3}(a,b)$ and $(1/3)G(a,b) + (2/3)H(a,b) \geqslant M_{-2/3}(a,b)$ for all $a,b > 0$. Here $G(a,b) = \sqrt{ab}$ and $H(a,b) = 2ab/(a+b)$ denote the geometric mean and harmonic mean of a and b, respectively.

1. Introduction

For $p \in R$, the power mean of order p of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.1}$$

Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ can be found in literature [1–12]. It is well known that $M_p(a,b)$ is continuous and increasing with respect to $p \in R$ for fixed a and b. If we denote by $A(a,b) = (a+b)/2, G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ the arithmetic mean, geometric mean and harmonic mean of a and b, respectively, then

$$\min\{a,b\} \leqslant H(a,b) = M_{-1}(a,b) \leqslant G(a,b) = M_0(a,b) \leqslant A(a,b) = M_1(a,b) \leqslant \max\{a,b\}. \tag{1.2}$$

In [13], Alzer and Janous established the following sharp double-inequality (see also [14, page 350]):

$$M_{\log 2/\log 3}(a,b) \leqslant \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leqslant M_{2/3}(a,b) \tag{1.3}$$

for all $a, b > 0$.

In [15], Mao proved

$$M_{1/3}(a,b) \leqslant \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) \leqslant M_{1/2}(a,b) \tag{1.4}$$

for all $a, b > 0$, and $M_{1/3}(a,b)$ is the best possible lower power mean bound for the sum $(1/3)A(a,b) + (2/3)G(a,b)$.

The purpose of this paper is to answer the questions: what are the greatest values p and q, and the least values r and s, such that $M_p(a,b) \leqslant (2/3)G(a,b) + (1/3)H(a,b) \leqslant M_r(a,b)$ and $M_q(a,b) \leqslant (1/3)G(a,b) + (2/3)H(a,b) \leqslant M_s(a,b)$ for all $a, b > 0$?

2. Main Results

Theorem 2.1. $(2/3)G(a,b) + (1/3)H(a,b) \geqslant M_{-1/3}(a,b)$ *for all* $a, b > 0$, *equality holds if and only if* $a = b$, *and* $M_{-1/3}(a,b)$ *is the best possible lower power mean bound for the sum* $(2/3)G(a,b) + (1/3)H(a,b)$.

Proof. If $a = b$, then we clearly see that $(2/3)G(a,b) + (1/3)H(a,b) = M_{-1/3}(a,b) = a$.

If $a \neq b$ and $a/b = t^6$, then simple computation leads to

$$\begin{aligned}
&\frac{2}{3}G(a,b) + \frac{1}{3}H(a,b) - M_{-1/3}(a,b) \\
&\quad = b\left[\frac{2t^3}{3} + \frac{2t^6}{3(1+t^6)} - \frac{8t^6}{(1+t^2)^3}\right] \\
&\quad = \frac{2bt^3}{3(1+t^2)^3(t^4-t^2+1)} \times \left[\left(t^2+1\right)^3\left(t^4-t^2+1\right) + t^3\left(t^2+1\right)^2 - 12t^3\left(t^4-t^2+1\right)\right] \\
&\quad = \frac{2bt^3}{3(1+t^2)^3(t^4-t^2+1)} \times \left[t^{10} + 2t^8 - 11t^7 + t^6 + 14t^5 + t^4 - 11t^3 + 2t^2 + 1\right] \\
&\quad = \frac{2bt^3(t-1)^4}{3(1+t^2)^3(t^4-t^2+1)} \times \left(t^6 + 4t^5 + 12t^4 + 17t^3 + 12t^2 + 4t + 1\right) \\
&\quad > 0.
\end{aligned} \tag{2.1}$$

Next, we prove that $M_{-1/3}(a,b)$ is the best possible lower power mean bound for the sum $(2/3)G(a,b) + (1/3)H(a,b)$.

For any $0 < \varepsilon < \dfrac{1}{3}$ and $0 < x < 1$, one has

$$\begin{aligned}
&\left[M_{-1/3+\varepsilon}((1+x)^2,1)\right]^{1/3-\varepsilon} - \left[\frac{2}{3}G((1+x)^2,1) + \frac{1}{3}H((1+x)^2,1)\right]^{1/3-\varepsilon} \\
&= \left[\frac{1+(1+x)^{-2/3+2\varepsilon}}{2}\right]^{-1} - \left[\frac{2}{3}(1+x) + \frac{2(1+x)^2}{3(x^2+2x+2)}\right]^{1/3-\varepsilon} \\
&= \frac{2(1+x)^{2/3-2\varepsilon}}{1+(1+x)^{2/3-2\varepsilon}} - \left(\frac{1+2x+(4/3)x^2+x^3/3}{1+x+x^2/2}\right)^{1/3-\varepsilon} \\
&= \frac{f(x)}{\left[1+(1+x)^{2/3-2\varepsilon}\right](1+x+x^2/2)^{1/3-\varepsilon}},
\end{aligned} \tag{2.2}$$

where $f(x) = 2(1+x)^{2/3-2\varepsilon}\,(1+x+(x^2/2))^{1/3-\varepsilon} - [1+(1+x)^{2/3-2\varepsilon}](1+2x+(4/3)x^2+x^3/3)^{1/3-\varepsilon}$.

Let $x \to 0$, then the Taylor expansion leads to

$$\begin{aligned}
f(x) &= 2\left[1+\frac{2-6\varepsilon}{3}x - \frac{(1-3\varepsilon)(1+6\varepsilon)}{9}x^2 + o\left(x^2\right)\right] \\
&\quad\times\left[1+\frac{1-3\varepsilon}{3}x + \frac{(1-3\varepsilon)^2}{18}x^2 + o\left(x^2\right)\right] \\
&\quad-2\left[1+\frac{1-3\varepsilon}{3}x - \frac{(1-3\varepsilon)(1+6\varepsilon)}{18}x^2 + o\left(x^2\right)\right] \\
&\quad\times\left[1+\frac{2-6\varepsilon}{3}x - \frac{2\varepsilon(1-3\varepsilon)}{3}x^2 + o\left(x^2\right)\right] \\
&= 2\left[1+(1-3\varepsilon)x + \frac{(1-3\varepsilon)(1-9\varepsilon)}{6}x^2 + o\left(x^2\right)\right] \\
&\quad-2\left[1+(1-3\varepsilon)x + \frac{(1-3\varepsilon)(1-10\varepsilon)}{6}x^2 + o\left(x^2\right)\right] \\
&= \frac{\varepsilon(1-3\varepsilon)}{3}x^2 + o\left(x^2\right).
\end{aligned} \tag{2.3}$$

Equations (2.2) and (2.3) imply that for any $0 < \varepsilon < 1/3$ there exists $0 < \delta = \delta(\varepsilon) < 1$, such that $M_{-1/3+\varepsilon}((1+x)^2,1) > (2/3)G((1+x)^2,1) + (1/3)H((1+x)^2,1)$ for $x \in (0,\delta)$. □

Remark 2.2. For any $\varepsilon > 0$, one has

$$\lim_{t\to+\infty}\left[\frac{2}{3}G(1,t) + \frac{1}{3}H(1,t) - M_{-\varepsilon}(1,t)\right] = \lim_{t\to+\infty}\left[\frac{2}{3}\sqrt{t} + \frac{2t}{3(1+t)} - \left(\frac{2t^{\varepsilon}}{1+t^{\varepsilon}}\right)^{1/\varepsilon}\right] = +\infty. \tag{2.4}$$

Therefore, $M_0(a,b) = G(a,b)$ is the best possible upper power mean bound for the sum $(2/3)G(a,b) + (1/3)H(a,b)$.

Theorem 2.3. *$(1/3)G(a,b) + (2/3)H(a,b) \geqslant M_{-2/3}(a,b)$ for all $a,b > 0$, equality holds if and only if $a = b$, and $M_{-2/3}(a,b)$ is the best possible lower power mean bound for the sum $(1/3)G(a,b) + (2/3)H(a,b)$.*

Proof. If $a = b$, then we clearly see that $(1/3)G(a,b) + (2/3)H(a,b) = M_{-2/3}(a,b) = a$.

If $a \neq b$ and $a/b = t^6$, then elementary calculation yields

$$\begin{aligned}
&\left[\frac{1}{3}G(a,b) + \frac{2}{3}H(a,b)\right]^2 - [M_{-2/3}(a,b)]^2 \\
&= b^2\left[\left(\frac{t^3}{3} + \frac{4t^6}{3(1+t^6)}\right)^2 - \left(\frac{2t^4}{1+t^4}\right)^3\right] \\
&= \frac{b^2t^6}{9(1+t^6)^2(1+t^4)^3}\left[\left(t^4+1\right)^3\left(t^6+4t^3+1\right)^2 - 72t^6\left(t^6+1\right)^2\right] \\
&= \frac{b^2t^6}{9(1+t^6)^2(1+t^4)^3}\Big[\Big(t^{24} + 8t^{21} + 3t^{20} + 18t^{18} + 24t^{17} + 3t^{16} + 8t^{15} + 54t^{14} + 24t^{13} \\
&\qquad +2t^{12} + 24t^{11} + 54t^{10} + 8t^9 + 3t^8 + 24t^7 + 18t^6 + 3t^4 + 8t^3 + 1\Big) \\
&\qquad -\Big(72t^{18} + 144t^{12} + 72t^6\Big)\Big] \\
&= \frac{b^2t^6}{9(1+t^6)^2(1+t^4)^3}\Big(t^{24} + 8t^{21} + 3t^{20} - 54t^{18} + 24t^{17} + 3t^{16} + 8t^{15} + 54t^{14} + 24t^{13} - 142t^{12} \\
&\qquad +24t^{11} + 54t^{10} + 8t^9 + 3t^8 + 24t^7 - 54t^6 + 3t^4 + 8t^3 + 1\Big) \\
&= \frac{b^2t^6(t-1)^4}{9(1+t^6)^2(1+t^4)^3}\Big(t^{20} + 4t^{19} + 10t^{18} + 28t^{17} + 70t^{16} + 148t^{15} + 220t^{14} + 268t^{13} \\
&\qquad + 277t^{12} + 240t^{11} + 240t^{10} + 240t^9 + 277t^8 + 268t^7 + 220t^6 \\
&\qquad +148t^5 + 70t^4 + 28t^3 + 10t^2 + 4t + 1\Big) > 0.
\end{aligned} \tag{2.5}$$

Next, we prove that $M_{-2/3}(a,b)$ is the best possible lower power mean bound for the sum $(1/3)G(a,b) + (2/3)H(a,b)$.

For any $0 < \varepsilon < 2/3$ and $0 < x < 1$, one has

$$\begin{aligned}
&\left[M_{-2/3+\varepsilon}(1,(1+x)^2)\right]^{2/3-\varepsilon} - \left[\frac{1}{3}G(1,(1+x)^2) + \frac{2}{3}H(1,(1+x)^2)\right]^{2/3-\varepsilon} \\
&\quad = \frac{2(1+x)^{(4-6\varepsilon)/3}}{1+(1+x)^{(4-6\varepsilon)/3}} - \frac{(1+2x+(7/6)x^2+(1/6)x^3)^{(2-3\varepsilon)/3}}{(1+x+(1/2)x^2)^{(2-3\varepsilon)/3}} \\
&\quad = \frac{f(x)}{\left[1+(1+x)^{(4-6\varepsilon)/3}\right](1+x+(1/2)x^2)^{(2-3\varepsilon)/3}},
\end{aligned} \tag{2.6}$$

where $f(x) = 2(1+x)^{(4-6\varepsilon)/3}(1+x+x^2/2)^{(2-3\varepsilon)/3} - (1+2x+(7/6)x^2+(1/6)x^3)^{(2-3\varepsilon)/3}[1+(1+x)^{(4-6\varepsilon)/3}]$.

Let $x \to 0$, then the Taylor expansion leads to

$$\begin{aligned}
f(x) &= 2\left[1+\frac{4-6\varepsilon}{3}x + \frac{(2-3\varepsilon)(1-6\varepsilon)}{9}x^2 + o\left(x^2\right)\right] \\
&\quad \times \left[1+\frac{2-3\varepsilon}{3}x + \frac{(2-3\varepsilon)^2}{18}x^2 + o\left(x^2\right)\right] \\
&\quad - 2\left[1+\frac{4-6\varepsilon}{3}x + \frac{(2-3\varepsilon)(1-4\varepsilon)}{6}x^2 + o\left(x^2\right)\right] \\
&\quad \times \left[1+\frac{2-3\varepsilon}{3}x + \frac{(2-3\varepsilon)(1-6\varepsilon)}{18}x^2 + o\left(x^2\right)\right] \\
&= 2\left[1+(2-3\varepsilon)x + \frac{(2-3\varepsilon)(4-9\varepsilon)}{6}x^2 + o\left(x^2\right)\right] \\
&\quad - 2\left[1+(2-3\varepsilon)x + \frac{(2-3\varepsilon)(4-10\varepsilon)}{6}x^2 + o\left(x^2\right)\right] \\
&= \frac{\varepsilon(2-3\varepsilon)}{3}x^2 + o\left(x^2\right).
\end{aligned} \tag{2.7}$$

Equations (2.6) and (2.7) imply that for any $0 < \varepsilon < 2/3$ there exists $0 < \delta = \delta(\varepsilon) < 1$, such that

$$M_{-2/3+\varepsilon}\left(1,(1+x)^2\right) > (1/3)G\left(1,(1+x)^2\right) + (2/3)H\left(1,(1+x)^2\right) \tag{2.8}$$

for $x \in (0,\delta)$. □

Remark 2.4. For any $\varepsilon > 0$, one has

$$\lim_{t\to+\infty}\left[\frac{1}{3}G(1,t) + \frac{2}{3}H(1,t) - M_{-\varepsilon}(1,t)\right] = \lim_{t\to+\infty}\left[\frac{1}{3}\sqrt{t} + \frac{4t}{3(1+t)} - \left(\frac{2t^{\varepsilon}}{1+t^{\varepsilon}}\right)^{1/\varepsilon}\right] = +\infty. \tag{2.9}$$

Therefore, $M_0(a,b) = G(a,b)$ is the best possible upper power mean bound for the sum $(1/3)G(a,b) + (2/3)H(a,b)$.

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