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Research Article

New Results on the Nonoscillation of Solutions of Some Nonlinear Differential Equations of Third Order

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We give sufficient conditions so that all solutions of differential equations $(r(t)y''(t))'+q(t)k(y'(t))+p(t)y^{\alpha}(g(t)) = f(t)$, $t \geq t_0$, and $(r(t)y''(t))' + q(t)k(y'(t)) + p(t)h(y(g(t))) = f(t)$, $t \geq t_0$, are nonoscillatory. Depending on these criteria, some results which exist in the relevant literature are generalized. Furthermore, the conditions given for the functions k and h lead to studying more general differential equations.

1. Introduction

This paper is concerned with study of nonoscillation of solutions of third-order nonlinear differential equations of the form

$$\left(r(t)y''(t)\right)' + q(t)k\left(y'(t)\right) + p(t)y^{\alpha}\left(g(t)\right) = f(t), \quad t \geq t_0, \tag{1.1}$$

$$\left(r(t)y''(t)\right)' + q(t)k\left(y'(t)\right) + p(t)h\left(y\left(g(t)\right)\right) = f(t), \quad t \geq t_0, \tag{1.2}$$

where $t_0 \geq 0$ is a fixed real number, f, p, q, r, and $g \in C([0,\infty),\Re)$ such that $r(t) > 0$ and $f(t) \geq 0$ for all $t \in [0,\infty)$. $k, h \in C(R,R)$ are nondecreasing such that $h(y)y > 0$, $k(y')y' > 0$ for all $y \neq 0$, $y' \neq 0$. Throughout the paper, it is assumed, for all $g(t)$ and α appeared in (1.1) and (1.2), that $g(t) \leq t$ for all $t \geq t_0$; $\lim_{t\to\infty} g(t) = \infty$; $\alpha > 0$ is a quotient of odd integers.

It is well known from relevant literature that there have been deep and thorough studies on the nonoscillatory behaviour of solutions of second- and third-order nonlinear differential equations in recent years. See, for instance, [1–37] as some related papers or books on the subject. In the most of these studies the following differential equation and some special cases of

$$\left(r(t)y''(t)\right)' + q(t)\left(y'\right)^{\beta} + p(t)y^{\alpha} = f(t), \quad t \geq t_0, \tag{1.3}$$

have been investigated. However, much less work has been done for nonoscillation of all solutions of nonlinear functional differential equations. In this connection, Parhi [10] established some sufficient conditions for oscillation of all solutions of the second-order forced differential equation of the form

$$\left(r(t)y'(t)\right)' + p(t)y^{\alpha}\left(g(t)\right) = f(t) \tag{1.4}$$

and nonoscillation of all bounded solutions of the equations

$$\begin{gathered}\left(r(t)y'(t)\right)' + q(t)\left(y'(t)\right)^{\beta} + p(t)y^{\alpha}\left(g(t)\right) = f(t), \\ \left(r(t)y'(t)\right)' + q(t)\left(y'\left(g_1(t)\right)\right)^{\beta} + p(t)y^{\alpha}\left(g(t)\right) = f(t),\end{gathered} \tag{1.5}$$

where the real-valued functions f, p, q, r, g, and g_1 are continuous on $[0,\infty)$ with $r(t) > 0$ and $f(t) \geq 0$; $g(t) \leq t$, $g_1(t) \leq t$ for $t \geq t_0$; $\lim_{t\to\infty} g(t) = \infty$, $\lim_{t\to\infty} g_1(t) = \infty$, and both $\alpha > 0$ and $\beta > 0$ are quotients of odd integers.

Later, Nayak and Choudhury [5] considered the differential equation

$$\left(r(t)y''(t)\right)' - q(t)\left(y'(t)\right)^{\beta} - p(t)y^{\alpha}\left(g(t)\right) = f(t), \tag{1.6}$$

and they gave certain sufficient conditions on the functions involved for all bounded solutions of the above equation to be nonoscillatory.

Recently, in 2007, Tunç [23] investigated nonoscillation of solutions of the third-order differential equations:

$$\begin{gathered}\left(r(t)y''(t)\right)' + q(t)y'(t) + p(t)y^{\alpha}\left(g(t)\right) = f(t), \quad t \geq t_0, \\ \left(r(t)y''(t)\right)' + q(t)\left(y'\left(g_1(t)\right)\right)^{\beta} + p(t)y^{\alpha}\left(g(t)\right) = f(t), \quad t \geq t_0.\end{gathered} \tag{1.7}$$

The motivation for the present work has come from the paper of Parhi [10], Tunç [23] and the papers mentioned above. We restrict our considerations to the real solutions of (1.1) and (1.2) which exist on the half-line $[T,\infty)$, where T (≥ 0) depends on the particular solution, and are nontrivial in any neighborhood of infinity. It is well known that a solution $y(t)$ of (1.1) or (1.2) is said to be nonoscillatory on $[T,\infty)$ if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for $t \geq t_1$; it is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$; $y(t)$ is said to be a Z-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

2. Nonoscillation Behaviors of Solutions of (1.1)

In this section, we obtain sufficient conditions for the nonoscillation of solutions of (1.1).

Theorem 2.1. *Let $q(t) \le 0$. If $\lim_{t\to\infty}(f(t)/|p(t)|) = \infty$, then all bounded solutions of (1.1) are nonoscillatory.*

Proof. Let $y(t)$ be a bounded solution of (1.1) on $[T_y, \infty)$, $T_y \ge 0$, such that $|y(t)| \le M$ for $t \ge T_y$. Since $\lim_{t\to\infty} g(t) = \infty$, there exists a $t_1 > t_0$ such that $g(t) \ge T_y$ for $t \ge t_1$. In view of the assumption $\lim_{t\to\infty}(f(t)/|p(t)|) = \infty$, it follows that there exists a $t_2 \ge t_1$ such that $f(t) > M^\alpha |p(t)|$ for $t \ge t_2$. If possible, let $y(t)$ be of nonnegative Z-type solution with consecutive double zeros at a and b ($t_2 < a < b$) such that $y(t) > 0$ for $t \in (a, b)$. So, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Multiplying (1.1) through by $y'(t)$, we get

$$\left(r(t)y'(t)y''(t)\right)' = r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)y^\alpha\left(g(t)\right)y'(t) + f(t)y'(t). \tag{2.1}$$

Integrating (2.1) from a to c, we obtain

$$\begin{aligned} 0 &= \int_a^c \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) + f(t)y'(t) - p(t)y^\alpha\left(g(t)\right)y'(t)\right]dt \\ &\ge \int_a^c \left[f(t) - p(t)y^\alpha\left(g(t)\right)\right]y'(t)dt \\ &\ge \int_a^c \left[f(t) - M^\alpha |p(t)|\right]y'(t)dt > 0, \end{aligned} \tag{2.2}$$

which is a contradiction.

Let $y(t)$ be of nonpositive Z-type solution with consecutive double zeros at a and b ($t_2 < a < b$). Then, there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$.

Integrating (2.1) from c to b yields

$$\begin{aligned} 0 &= \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) + f(t)y'(t) - p(t)y^\alpha\left(g(t)\right)y'(t)\right]dt \\ &\ge \int_c^b \left[f(t) - |p(t)|\left|y^\alpha\left(g(t)\right)\right|\right]y'(t)dt \\ &\ge \int_c^b \left[f(t) - M^\alpha |p(t)|\right]y'(t)dt > 0, \end{aligned} \tag{2.3}$$

which is a contradiction.

If possible, let $y(t)$ be oscillatory with consecutive zeros at a, b and a' ($t_2 < a < b < a'$) such that $y'(a) \le 0$, $y'(b) \ge 0$, $y'(a') \le 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exists points $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = 0$, $y'(c') = 0$, $y'(t) > 0$ for $t \in (c,b)$ and $y'(t) > 0$ for $t \in (b,c')$. Now integrating (2.1) from c to c', we get

$$\begin{aligned}
0 &= \int_c^{c'} \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) + f(t)y'(t) - p(t)y^{\alpha}\left(g(t)\right)y'(t)\right]dt \\
&\geq \int_c^{b} \left[f(t) - p(t)y^{\alpha}\left(g(t)\right)\right]y'(t)dt + \int_b^{c'} \left[f(t) - p(t)y^{\alpha}\left(g(t)\right)\right]y'(t)dt \\
&\geq \int_c^{b} \left[f(t) - |p(t)|\left|y^{\alpha}\left(g(t)\right)\right|\right]y'(t)dt + \int_b^{c'} \left[f(t) - |p(t)|\left|y^{\alpha}\left(g(t)\right)\right|\right]y'(t)dt \\
&\geq \int_c^{b} \left[f(t) - M^{\alpha}|p(t)|\right]y'(t)dt + \int_b^{c'} \left[f(t) - M^{\alpha}|p(t)|\right]y'(t)dt > 0,
\end{aligned} \tag{2.4}$$

which is a contradiction. This completes the proof of Theorem 2.1. □

Remark 2.2. For the special case $k(y'(t)) = (y'(g_1(t)))^{\beta}$, $h(y(g(t)) = y^{\alpha}(g(t))$, Theorem 2.1 has been proved by Tunç [23]. Our results include the results established in Tunç [23].

Theorem 2.3. *Let $0 \leq p(t) < f(t)$ and $q(t) \leq 0$, then all solutions $y(t)$ of* (1.1) *which satisfy the inequality*

$$1 - z^{\alpha}\left(g(t)\right) \geq 0 \tag{2.5}$$

on any interval where $y'(t) > 0$ are nonoscillatory.

Proof. Let $y(t)$ be a solution of (1.1) on $[T_y, \infty)$, $T_y > 0$. Due to $\lim_{t\to\infty} g(t) = \infty$, there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. If possible, let $y(t)$ be of nonnegative Z-type solution with consecutive double zeros at a and b ($T_y \leq a < b$) such that $y(t) > 0$ for $t \in (a,b)$. So, there exists a $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a,c)$. Integrating (2.1) from a to c, we get

$$\begin{aligned}
0 &= \int_a^{c} \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) + f(t)y'(t) - p(t)y^{\alpha}\left(g(t)\right)y'(t)\right]dt \\
&\geq \int_a^{c} \left[f(t) - p(t)y^{\alpha}\left(g(t)\right)\right]y'(t)dt \\
&\geq \int_a^{c} p(t)\left[1 - y^{\alpha}\left(g(t)\right)\right]y'(t)dt > 0,
\end{aligned} \tag{2.6}$$

which is a contradiction.

Next, let $y(t)$ be of nonpositive Z-type solution with consecutive double zeros at a and b ($T_y \leq a < b$). Then, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c,b)$.

Integrating (2.1) from c to b, we have

$$0 = \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) + f(t)y'(t) - p(t)y^{\alpha}\left(g(t)\right)y'(t)\right]dt > 0, \tag{2.7}$$

which is a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive zeros at a, b and a' ($T_y < a < b < a'$) such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. Hence, there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,b)$ and $t \in (b,c')$. Integrating (2.1) from c to c', we obtain

$$\begin{aligned}
0 &= \int_c^{c'} \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) + f(t)y'(t) - p(t)y^{\alpha}\left(g(t)\right)y'(t)\right]dt \\
&\geq \int_c^{b} \left[f(t) - p(t)y^{\alpha}\left(g(t)\right)\right]y'(t)dt + \int_b^{c'} \left[f(t) - p(t)y^{\alpha}\left(g(t)\right)\right]y'(t)dt \\
&\geq \int_b^{c'} \left[f(t) - p(t)y^{\alpha}\left(g(t)\right)\right]y'(t)dt \\
&\geq \int_c^{b} p(t)\left[1 - y^{\alpha}\left(g(t)\right)\right]y'(t)dt > 0,
\end{aligned} \tag{2.8}$$

which is a contradiction. This completes the proof of Theorem 2.3. □

Remark 2.4. For the special case $k(y') = (y')^{\beta}$, $y^{\alpha}(g(t)) = y^{\alpha}$, Theorem 2.3 has been proved by Tunç [25]. Our results include the results established in Tunç [25].

3. Nonoscillation Behaviors of Solutions (1.2)

In this section, we give sufficient conditions so that all solutions of (1.2) are nonoscillatory.

Theorem 3.1. *Suppose that $q(t) \leq 0$ and $0 \leq p(t) < f(t)$. If $y(t)$ is a solution (1.2) such that it satisfies the inequality*

$$1 - h(z(t)) > 0 \tag{3.1}$$

on any interval where $y'(t) > 0$, then $y(t)$ is nonoscillatory.

Proof. Let $y(t)$ be a solution of (1.2) on $[T_y, \infty)$, $T_y > 0$. Due to $\lim_{t\to\infty} g(t) = \infty$, there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. If possible, let $y(t)$ be of nonnegative Z-type solution with consecutive double zeros at a and b ($T_y \leq a < b$) such that $y(t) > 0$ for $t \in (a,b)$. So, there exists a $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a,c)$. Multiplying (1.2) through by $y'(t)$, we get

$$\left(r(t)y'(t)y''(t)\right)' = r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)h\left(y\left(g(t)\right)\right)y'(t) + f(t)y'(t). \tag{3.2}$$

Integrating (3.2) from a to c, we get

$$\begin{aligned} 0 &= \int_a^c \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)h\left(y\left(g(t)\right)\right)y'(t) + f(t)y'(t)\right]dt \\ &\geq \int_a^c \left[f(t) - p(t)h\left(y\left(g(t)\right)\right)\right]y'(t)dt \\ &\geq \int_a^c f(t)\left[1 - h\left(y(t)\right)\right]y'(t)dt > 0, \end{aligned} \tag{3.3}$$

which is a contradiction.

Next, let $y(t)$ be of nonpositive Z-type solution with consecutive double zeros at a and b ($T_y \leq a < b$). Then, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c,b)$.

Integrating (3.2) from c to b, we have

$$0 = \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)h\left(y\left(g(t)\right)\right)y'(t) + f(t)y'(t)\right]dt > 0, \tag{3.4}$$

which is a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive zeros at a, b and a' ($T_y < a < b < a'$) such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. Hence, there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,b)$ and $t \in (b,c')$. Integrating (3.2) from c to c', we obtain

$$\begin{aligned} 0 &= \int_c^{c'} \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)h\left(y\left(g(t)\right)\right)y'(t) + f(t)y'(t)\right]dt \\ &\geq \int_c^b \left[f(t) - p(t)h\left(y\left(g(t)\right)\right)\right]y'(t)dt + \int_b^{c'} \left[f(t) - p(t)h\left(y\left(g(t)\right)\right)\right]y'(t)dt \\ &\geq \int_c^b \left[f(t) - p(t)h\left(y(t)\right)\right]y'(t)dt + \int_b^{c'} \left[f(t) - p(t)h\left(y(t)\right)\right]y'(t)dt \\ &\geq \int_b^{c'} \left[f(t) - p(t)h\left(y(t)\right)\right]y'(t)dt \\ &\geq \int_b^{c'} f(t)\left[1 - h\left(y(t)\right)\right]y'(t)dt > 0, \end{aligned} \tag{3.5}$$

which is a contradiction. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Suppose that* $0 \leq q \leq p \leq f$ *and* $q \neq 0$ *on any subinterval of* $[T_y, \infty)$, $T_y \geq 0$. *If* $y(t)$ *is a solution of* (1.2) *such that it satisfies the inequality*

$$1 - k\left(z'\right) - h(z) > 0 \tag{3.6}$$

on any subinteval of $[T_y, \infty)$, $T_y \geq 0$, *where* $y'(t) > 0$, *then* $y(t)$ *is nonoscillatory.*

Proof. Let $y(t)$ be a solution of (1.2) on $[T_y, \infty)$, $T_y > 0$. Since $\lim_{t\to\infty} g(t) = \infty$, there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. If possible, let $y(t)$ be of nonnegative Z-type solution with consecutive double zeros at a and b $(T_y \leq a < b)$ such that $y(t) > 0$ for $t \in (a,b)$. So, there exists a $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a,c)$. Integrating (3.2) from a to c, we get

$$\begin{aligned}
0 &= \int_a^c \Big[r(t)\big(y''(t)\big)^2 - q(t)k\big(y'(t)\big)y'(t) - p(t)h\big(y\big(g(t)\big)\big)y'(t) + f(t)y'(t)\Big]dt \\
&\geq \int_a^c \big[-q(t)k\big(y'(t)\big)y'(t) - p(t)h\big(y\big(g(t)\big)\big)y'(t) + f(t)y'(t)\big]dt \\
&\geq \int_a^c \big[-q(t)k\big(y'(t)\big)y'(t) - p(t)h\big(y(t)\big)y'(t) + f(t)y'(t)\big]dt \\
&\geq \int_a^c f(t)\big[1 - k\big(y'(t)\big) - p(t)h\big(y(t)\big)\big]y'(t)dt > 0,
\end{aligned} \tag{3.7}$$

which is a contradiction.

Next, let $y(t)$ be of nonpositive Z-type solution with consecutive double zeros at a and b $(T_y \leq a < b)$. Then, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c,b)$.

Integrating (3.2) from c to b, we have

$$\begin{aligned}
0 &= \int_c^b \Big[r(t)\big(y''(t)\big)^2 - q(t)k\big(y'(t)\big)y'(t) - p(t)h\big(y\big(g(t)\big)\big)y'(t) + f(t)y'(t)\Big]dt \\
&\geq \int_c^b \big[-q(t)k\big(y'(t)\big)y'(t) - p(t)h\big(y\big(g(t)\big)\big)y'(t) + f(t)y'(t)\big]dt \\
&\geq \int_c^b q(t)\big[1 - k\big(y'(t)\big) - p(t)h\big(y(t)\big)\big]y'(t)dt > 0,
\end{aligned} \tag{3.8}$$

which is a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive zeros at a, b and a' $(T_y < a < b < a')$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. Hence, there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,b)$ and $t \in (b,c')$. Integrating (3.2) from c to c', we obtain

$$\begin{aligned}
0 &= \int_c^{c'} \Big[r(t)\big(y''(t)\big)^2 - q(t)k\big(y'(t)\big)y'(t) - p(t)h\big(y\big(g(t)\big)\big)y'(t) + f(t)y'(t)\Big]dt \\
&\geq \int_c^b \big[-q(t)k\big(y'(t)\big) - p(t)h\big(y\big(g(t)\big)\big) + f(t)\big]y'(t)dt \\
&\quad + \int_b^{c'} \big[-q(t)k\big(y'(t)\big) - p(t)h\big(y\big(g(t)\big)\big) + f(t)\big]y'(t)dt
\end{aligned}$$

$$\begin{aligned}
&\geq \int_c^b \left[-q(t)k(y'(t)) - p(t)h(y(t)) + f(t)\right]y'(t)dt \\
&\quad + \int_b^{c'} \left[-q(t)k(y'(t)) - p(t)h(y(t)) + f(t)\right]y'(t)dt \\
&\geq \int_c^b q(t)\left[1 - k(y'(t)) - h(y(t))\right]y'(t)dt + \int_b^{c'} f(t)\left[1 - k(y'(t)) - h(y(t))\right]y'(t)dt > 0,
\end{aligned} \tag{3.9}$$

which is a contradiction. This completes the proof of Theorem 3.2. □

Remark 3.3. It is clear that Theorem 3.2 is not applicable to homogeneous equations:

$$(r(t)y''(t))' + q(t)k(y'(t)) + p(t)h(y(g(t))) = 0, \tag{3.10}$$

where $p(t) \geq 0$ and $q(t) \geq 0$.

Remark 3.4. For the special case $k(y') = (y')^{\gamma}$, $h(y(g(t))) = y^{\beta}$, Theorem 3.2 has been proved by N. parhi and S. parhi [19, Theorem 2.7].

Theorem 3.5. *Let $p(t) \geq 0$, $q(t) \leq 0$, and $h(y) \leq y$ for all $y > 0$. If $p(t)$ and $f(t)$ are once continuously differentiable functions such that $p'(t) \geq 0$, $f'(t) \leq 0$, and $2f(t) - p(t) \geq 0$, then all solutions $y(t)$ of* (1.2) *for which $|y(t)| \leq 1$ ultimately are nonoscillatory.*

Proof. Let $y(t)$ be a solution of (1.2) on $[T_y, \infty)$, $T_y > 0$, such that $|y(t)| \leq 1$ for $t \geq T_1 > T_y$. Since $\lim_{t\to\infty} g(t) = \infty$, there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. If possible, let $y(t)$ be of nonnegative Z-type solution with consecutive double zeros at a and b ($T_1 \leq a < b$) such that $y(t) > 0$ for $t \in (a, b)$. So, there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Integrating (3.2) from a to c, we get

$$0 = \int_a^c \Big[r(t)(y''(t))^2 - q(t)k(y'(t))y'(t) - p(t)h(y(g(t)))y'(t) + f(t)y'(t)\Big]dt. \tag{3.11}$$

But

$$\begin{gathered}
\int_a^c f(t)y'(t)dt = f(t)y(t)\big|_a^c - \int_a^c f'(t)y(t)dt \geq f(c)y(c), \\
\int_a^c p(t)h(y(g(t)))y'(t)dt \leq \frac{1}{2}p(c)y^2(c).
\end{gathered} \tag{3.12}$$

Therefore

$$\begin{aligned}
&\int_a^c \left[-p(t)h(y(g(t)))y'(t) + f(t)y'(t)\right]dt \\
&\quad \geq f(c)y(c) - \frac{1}{2}p(c)y^2(c) \geq \frac{p(c)}{2}y(c) - \frac{1}{2}p(c)y^2(c) = \frac{1}{2}p(c)\Big[y(c) - y^2(c)\Big] > 0,
\end{aligned} \tag{3.13}$$

since $|y(t)| \leq 1$ for $t \geq T_1$. So (3.11) yields

$$0 = \int_a^c \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)h\left(y\left(g(t)\right)\right)y'(t) + f(t)y'(t)\right]dt > 0, \tag{3.14}$$

which is a contradiction.

Next, let $y(t)$ be of nonpositive Z-type solution with consecutive double zeros at a and b ($T_1 \leq a < b$). Then, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c,b)$.

Integrating (3.2) from c to b, we have

$$0 = \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)h\left(y\left(g(t)\right)\right)y'(t) + f(t)y'(t)\right]dt > 0, \tag{3.15}$$

which is a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive zeros at a, b and a' ($T_y < a < b < a'$) such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. So there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = 0$, $y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,c')$. We consider two cases, namely, $y''(b) \leq 0$ and $y''(b) > 0$. Suppose that $y''(b) \leq 0$. Integrating (3.2) from c to b, we get

$$\begin{aligned} 0 &\geq r(b)y'(b)y''(b) \\ &= \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)h\left(y\left(g(t)\right)\right)y'(t) + f(t)y'(t)\right]dt \\ &> 0, \end{aligned} \tag{3.16}$$

which is a contradiction. Let $y''(b) > 0$. Integrating (3.2) from b to c', we get

$$-r(b)y'(b)y''(b) = \int_b^{c'} \left[r(t)\left(y''(t)\right)^2 - q(t)k\left(y'(t)\right)y'(t) - p(t)h\left(y\left(g(t)\right)\right)y'(t) + f(t)y'(t)\right]dt. \tag{3.17}$$

We proceed as in nonnegative Z-type to conclude that $0 \geq -r(b)y'(b)y''(b) > 0$. This is a contradiction. So $y(t)$ is nonoscillatory. This completes the proof of Theorem 3.5. □

Remark 3.6. If $f \equiv 0$ in Theorem 3.5, then $p \equiv 0$ and hence the theorem is not applicable to homogeneous equation:

$$\left(r(t)y''(t)\right)' + q(t)k\left(y'(t)\right) + p(t)h\left(y\left(g(t)\right)\right) = 0. \tag{3.18}$$

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