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Research Article

A Note on Almost Sure Central Limit Theorem in the Joint Version for the Maxima and Sums

Qing-pei Zang,[1] Zhi-xiang Wang,[1] and Ke-ang Fu[2]

[1] *School of Mathematical Science, Huaiyin Normal University, Huaian 223300, China*
[2] *School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China*

Correspondence should be addressed to Qing-pei Zang, zqphunhu@yahoo.com.cn

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Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables and denote $S_n = \sum_{k=1}^{n} X_k$, $M_n = \max_{1\leq k\leq n} X_k$. In this paper, we investigate the almost sure central limit theorem in the joint version for the maxima and sums. If for some numerical sequences $(a_n > 0)$, (b_n) we have $(M_n - b_n)/a_n \xrightarrow{\mathfrak{D}} G$ for a nondegenerate distribution G, and $f(x, y)$ is a bounded Lipschitz 1 function, then $\lim_{n\to\infty}(1/D_n)\sum_{k=1}^{n} d_k f(S_k/\sqrt{k}, (M_k - b_k)/a_k) = \iint_{-\infty}^{\infty} f(x, y)\Phi(dx)G(dy)$ almost surely, where $\Phi(x)$ stands for the standard normal distribution function, $D_n = \sum_{k=1}^{n} d_k$,and $d_k = (\exp((\log k)^{\alpha}))/k$, $0 \leq \alpha < 1/2$.

1. Introduction and Main Results

Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables and $S_n = \sum_{k=1}^{n} X_k, n \geq 1$, $M_n = \max_{1\leq k\leq n} X_k$ for $n \geq 1$. If $E(X) = 0, E(X^2) = 1$, the classical almost sure central limit theorem (ASCLT) has the simplest form as follows:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \frac{S_k}{\sqrt{k}} \leq x \right\} = \Phi(x) \tag{1.1}$$

almost surely for all $x \in R$, here and in the sequel, $I(A)$ is the indicator function of the event A, and $\Phi(x)$ stands for the standard normal distribution function. This result was first proved independently by Brosamler [1] and Schatte [2] under a stronger moment condition, since then, this type of almost sure version which mainly dealt with logarithmic average limit theorems has been extended in various directions. Fahrner and Stadtmüller [3] and

Cheng et al. [4] extended this almost sure convergence for partial sums to the case of maxima of i.i.d. random variables. Under some natural conditions, they proved that

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{M_k-b_k}{a_k}\le x\right\}=G(x) \tag{1.2}$$

almost surely for all $x\in R$, where $a_k>0$ and $b_k\in R$ satisfy

$$P\left(\frac{M_k-b_k}{a_k}\le x\right)\longrightarrow G(x) \tag{1.3}$$

for any continuity point x of G.

For Gaussian sequences, Csáki and Gonchigdanzan [5] investigated the validity of (1.2) maxima of stationary Gaussian sequences under some mild condition. Furthermore, Chen and Lin [6] extended it to nonstationary Gaussian sequences. As for some other dependent random variables, Peligrad and Shao [7] and Dudziński [8] derived some corresponding results about ASCLT. The almost sure central limit theorem in the joint version for log average in the case of independent and identically distributed random variables is obtained by Peng et al. [9]; a joint version of almost sure limit theorem for log average of maxima and partial sums in the case of stationary Gaussian random variables is derived by Dudziński [10].

All the above results are related to the almost sure logarithmic version; in this paper; inspired by the results of Berkes and Csáki [11], we further study ASCLT in the joint version for the maxima and partial sums with another weighted sequence (d_n). Now, we state our main result as follows.

Theorem 1.1. *Let $\{X,X_n;n\ge 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with non-degenerate common distribution function F, satisfying $E(X)=0$ and $E(X^2)=1$. If for some numerical sequences $(a_n>0)$, (b_n) one has*

$$\frac{M_n-b_n}{a_n}\xrightarrow{\mathfrak{D}}G \tag{1.4}$$

for a non-degenerate distribution G, and $f(x,y)$ is a bounded Lipschitz 1 function, then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k f\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right)=\iint_{-\infty}^{\infty}f(x,y)\Phi(dx)G(dy) \tag{1.5}$$

almost surely, where $\Phi(x)$ stands for the standard normal distribution function, $D_n=\sum_{k=1}^{n}d_k$, and $d_k=\exp((\log k)^{\alpha})/k$, $0\le\alpha<1/2$.

Remark 1.2. Since a set of bounded Lipschitz 1 functions is tight in a set of bounded continuous functions, Theorem 1.1 is true for all bounded continuous functions $f(x,y)$.

Remark 1.3. Under the conditions of Theorem 1.1, it can be seen that the result for indicator functions by routine approximation arguments is similar, for example, to those in Lacey and Philipp [12], that is,

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k I\left(\frac{S_k}{\sqrt{k}}\le x,\frac{M_k-b_k}{a_k}\le y\right)=\Phi(x)G(y) \tag{1.6}$$

almost surely.

Example 1.4. The ASCLT has already received applications in many fields, including condensed matter physics, statistical mechanics, ergodic theory and dynamical systems, and control and information and quanitle estimation. As an example, we assume that $\{X, X_n; n \ge 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with standard normal distribution function $\Phi(x)$, and in (1.4), we choose

$$a_n=(2\log n)^{-1/2}, \tag{1.7}$$

$$b_n=(2\log n)^{1/2}-\frac{1}{2}(2\log n)^{-1/2}(\log\log n+\log 4\pi) \tag{1.8}$$

which imply that (see Leadbetter et al. [13, Theorem 4.3.3])

$$\frac{M_n-b_n}{a_n}\xrightarrow{\mathfrak{D}}\wedge, \tag{1.9}$$

where $\wedge$ is one of the extreme value distributions, that is,

$$\wedge(y)=\exp\{-\exp(-y)\}. \tag{1.10}$$

Then, we can derive a corresponding result in Theorem 1.1.

2. Proof of Our Main Result

In this section, denote $S_n = \sum_{k=1}^{n} X_k, S_{k,n} = \sum_{i=k+1}^{n} X_i, M_n = \max_{1\le i\le n} X_i$, and $M_{k,n} = \max_{k+1\le i\le n} X_i$, for $n \ge 1$, unless it is specially mentioned. Here $a \ll b$ and $a \sim b$ stand for $a = O(b)$ and $a/b \to 1$, respectively. $\Phi(x)$ is the standard normal distribution function.

Proof of Theorem 1.1. Firstly, by Theorem 1.1. in [14] and our assumptions, we have

$$\lim_{n\to\infty}P\left(\frac{S_n}{\sqrt{n}}\le x,\frac{M_n-b_n}{a_n}\le y\right)=\Phi(x)G(y) \tag{2.1}$$

for $x, y \in R$. Then, in view of the dominated convergence theorem, we have

$$Ef\left(\frac{S_n}{\sqrt{n}},\frac{M_n-b_n}{a_n}\right)\longrightarrow\iint_{-\infty}^{\infty}f(x,y)\Phi(dx)G(dy). \tag{2.2}$$

Hence, to complete the proof, it is sufficient to show that

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\left(f\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right)-Ef\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right)\right)=0 \tag{2.3}$$

almost surely. Let

$$\xi_k=f\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right)-Ef\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right). \tag{2.4}$$

For $l>k$, it follows that

$$\begin{aligned}\left|E(\xi_k\xi_l)\right| &= \left|\operatorname{Cov}\left(f\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right),f\left(\frac{S_l}{\sqrt{l}},\frac{M_l-b_l}{a_l}\right)\right)\right|\\ &\le \left|\operatorname{Cov}\left(f\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right),f\left(\frac{S_l}{\sqrt{l}},\frac{M_l-b_l}{a_l}\right)-f\left(\frac{S_l}{\sqrt{l}},\frac{M_{k,l}-b_l}{a_l}\right)\right)\right|\\ &\quad+\left|\operatorname{Cov}\left(f\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right),f\left(\frac{S_l}{\sqrt{l}},\frac{M_{k,l}-b_l}{a_l}\right)-f\left(\frac{S_{k,l}}{\sqrt{l}},\frac{M_{k,l}-b_l}{a_l}\right)\right)\right|\\ &\quad+\left|\operatorname{Cov}\left(f\left(\frac{S_k}{\sqrt{k}},\frac{M_k-b_k}{a_k}\right),f\left(\frac{S_{k,l}}{\sqrt{l}},\frac{M_{k,l}-b_l}{a_l}\right)\right)\right|\\ &:= L_1+L_2+L_3.\end{aligned} \tag{2.5}$$

For L_3, by the independence of $\{X_n; n\ge 1\}$, we have

$$L_3=0. \tag{2.6}$$

Now, we are in a position to estimate L_1. We use the fact that f is bounded and Lipschitzian, then it follows that

$$\begin{aligned}L_1 &\ll E\left|f\left(\frac{S_l}{\sqrt{l}},\frac{M_l-b_l}{a_l}\right)-f\left(\frac{S_l}{\sqrt{l}},\frac{M_{k,l}-b_l}{a_l}\right)\right|\\ &\ll E\left(\min\left(\frac{M_l-M_{k,l}}{a_l},2\right)\right)\\ &= E\left(\min\left(\frac{M_l-M_{k,l}}{a_l},2\right)\right)I\left(M_l\ne M_{k,l}\right)\\ &\ll P(M_l\ne M_{k,l})\\ &= P(M_k>M_{k,l})\\ &= k\int_{-\infty}^{\infty}(F(x))^{l+k-1}dF(x)\\ &\le \frac{k}{l}.\end{aligned} \tag{2.7}$$

Via Cauchy-Schwarz inequality, we have

$$\begin{aligned} L_2 &\ll E\left| f\left(\frac{S_l}{\sqrt{l}}, \frac{M_{k,l}-b_l}{a_l}\right) - f\left(\frac{S_{k,l}}{\sqrt{l}}, \frac{M_{k,l}-b_l}{a_l}\right)\right| \\ &\ll E\left|\frac{S_k}{\sqrt{l}}\right| \\ &\le \frac{1}{\sqrt{l}}\left(ES_k^2\right)^{1/2} \\ &\le \left(\frac{k}{l}\right)^{1/2}. \end{aligned} \tag{2.8}$$

Thus, using (2.6), (2.7), and (2.8), it follows that

$$|E(\xi_k\xi_l)| \ll \left(\frac{k}{l}\right)^{1/2} \tag{2.9}$$

for $l > k$. Then, we have

$$\begin{aligned} E\left(\sum_{k=1}^{n} d_k\xi_k\right)^2 &\le \sum_{k=1}^{n}\sum_{l=1}^{n} d_k d_l |E(\xi_k\xi_l)| \\ &\ll \sum_{k=1}^{n}\sum_{l=1}^{n} d_k d_l \left(\frac{k}{l}\right)^{1/2} \\ &= \sum_{\substack{1\le k\le l\le n;\\ l/k\le(\log n)^{4\alpha}}} d_k d_l \left(\frac{k}{l}\right)^{1/2} + \sum_{\substack{1\le k\le l\le n;\\ l/k>(\log n)^{4\alpha}}} d_k d_l \left(\frac{k}{l}\right)^{1/2} \\ &:= T_1 + T_2. \end{aligned} \tag{2.10}$$

It is obvious that

$$T_2 \le \sum_{1\le k\le l\le n} d_k d_l (\log n)^{-2\alpha} \le D_n^2 (\log n)^{-2\alpha}. \tag{2.11}$$

In view of the definition of numerical sequence (d_l) and by L'Hospital rule and fixed $0 < \alpha < 1/2$, we have

$$D_n \sim \frac{1}{\alpha}(\log n)^{1-\alpha} \exp\left((\log n)^{\alpha}\right). \tag{2.12}$$

Then

$$
\begin{aligned}
T_1 &\leq \sum_{1\leq k\leq n} d_k \sum_{1\leq k\leq l\leq \min(n,k(\log n)^{4\alpha})} \frac{\exp\left((\log l)^{\alpha}\right)}{l} \\
&\leq \exp\left((\log n)^{\alpha}\right) \sum_{1\leq k\leq n} d_k \sum_{1\leq k\leq l\leq \min(n,k(\log n)^{4\alpha})} \frac{1}{l} \\
&\ll D_n \exp\left((\log n)^{\alpha}\right) \log\log n \\
&\ll D_n^2 (\log\log n)(\log n)^{\alpha-1} \\
&\ll D_n^2 (\log n)^{-(1+\varepsilon)\alpha}
\end{aligned}
\tag{2.13}
$$

for $\varepsilon > 0$ such that $\varepsilon < \min(1, (1/\alpha) - 2)$. From (2.11), (2.13), and the Markov inequality, we derive

$$
P\left(\left|\sum_{k=1}^{n} d_k \xi_k\right| > \varepsilon D_n\right) \ll (\log n)^{-(1+\varepsilon)\alpha} \tag{2.14}
$$

for the above ε and $0 < \alpha < 1/2$. We can choose subsequence $n_k = \exp(k^{(1-\beta)/\alpha})$, where $\beta > 0$ such that $(1+\varepsilon)(1-\beta) > 1$. Then, by Borel-Cantelli lemma, we derive

$$
\lim_{k\to\infty} \frac{1}{D_{n_k}} \sum_{j=1}^{n_k} d_j \xi_j = 0 \tag{2.15}
$$

almost surely. For $n_k \leq n < n_{k+1}$ we have

$$
\left|\frac{1}{D_n}\sum_{k=1}^{n} d_k \xi_k\right| \leq \left|\frac{1}{D_{n_k}}\sum_{j=1}^{n_k} d_j \xi_j\right| + 2\left(\frac{D_{n_{k+1}}}{D_{n_k}} - 1\right) \tag{2.16}
$$

almost surely. Since $D_{n_{k+1}}/D_{n_k} \to 1$, the convergence of the subsequence implies that the whole sequence converges almost surely. Hence the proof of (1.5) is completed for $0 < \alpha < 1/2$. Via the same arguments, we can obtain (1.5) for $\alpha = 0$. □

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