

*Research Article*

# **A New Iteration Method for Nonexpansive Mappings and Monotone Mappings in Hilbert Spaces**

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We introduce a new composite iterative scheme by the viscosity approximation method for nonexpansive mappings and monotone mappings in a Hilbert space. It is proved that the sequence generated by the iterative scheme converges strongly to a common point of set of fixed points of nonexpansive mapping and the set of solutions of variational inequality for an inverse-strongly monotone mappings, which is a solution of a certain variational inequality. Our results substantially develop and improve the corresponding results of [Chen et al. 2007 and Iiduka and Takahashi 2005]. Essentially a new approach for finding the fixed points of nonexpansive mappings and solutions of variational inequalities for monotone mappings is provided.

## **1. Introduction**

Let  $H$  be a real Hilbert space and  $C$  a nonempty closed convex subset of  $H$ . Recall that a mapping  $f : C \rightarrow C$  is a *contraction* on  $C$  if there exists a constant  $k \in (0, 1)$  such that  $\|f(x) - f(y)\| \leq k\|x - y\|$ ,  $x, y \in C$ . We use  $\Sigma_C$  to denote the collection of mappings  $f$  verifying the above inequality. That is  $\Sigma_C = \{f : C \rightarrow C \mid f \text{ is a contraction with constant } k\}$ . A mapping  $S : C \rightarrow C$  is called *nonexpansive* if  $\|Sx - Sy\| \leq \|x - y\|$ ,  $x, y \in C$ ; see [1, 2] for the results of nonexpansive mappings. We denote by  $F(S)$  the set of fixed points of  $S$ ; that is,  $F(S) = \{x \in C : x = Sx\}$ .

Let  $P_C$  be the metric projection of  $H$  onto  $C$ . A mapping  $A$  of  $C$  into  $H$  is called *monotone* if for  $x, y \in C$ ,  $\langle x - y, Ax - Ay \rangle \geq 0$ . The *variational inequality problem* is to find a  $u \in C$  such that

$$\langle v - u, Au \rangle \geq 0 \tag{1.1}$$

for all  $v \in C$ ; see [3–6]. The set of solutions of the variational inequality is denoted by  $VI(C, A)$ . A mapping  $A$  of  $C$  into  $H$  is called *inverse-strongly monotone* if there exists a positive real number  $\alpha$  such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \quad (1.2)$$

for all  $x, y \in C$ ; see [7–9]. For such a case,  $A$  is called  $\alpha$ -inverse-strongly monotone.

In 2005, Iiduka and Takahashi [10] introduced an iterative scheme for finding a common point of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strong monotone mapping as follows. For an  $\alpha$ -inverse-strongly monotone mapping  $A$  of  $C$  to  $H$  and a nonexpansive mapping  $S$  of  $C$  into itself such that  $F(S) \cap VI(C, A) \neq \emptyset$ ,  $x_1 = x \in C$ ,  $\{\alpha_n\} \subset [0, 1)$ , and  $\{\lambda_n\} \subset [0, 2\alpha]$ ,

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n) \quad (1.3)$$

for every  $n \geq 1$ . They proved that the sequence generated by (1.3) converges strongly to  $P_{F(S) \cap VI(C, A)} x$  under the conditions on  $\{\alpha_n\}$  and  $\{\lambda_n\} : \lambda_n \in [c, d]$  for some  $c, d$  with  $0 < c < d < 2\alpha$ ,

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n < \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \quad (1.4)$$

On the other hand, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [11]. In 2004, in order to extend Theorem 2.2 of Moudafi [11] to a Banach space setting, Xu [12] considered the the following explicit iterative process. For  $S : C \rightarrow C$  nonexpansive mappings,  $f \in \Sigma_C$  and  $\alpha_n \in (0, 1)$ ,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) Sx_n, \quad n \geq 1. \quad (1.5)$$

Moreover, in [12], he also studied the strong convergence of  $\{x_n\}$  generated by (1.5) as  $n \rightarrow \infty$  in either a Hilbert space or a uniformly smooth Banach space and showed that the strong  $\lim_{n \rightarrow \infty} x_n$  is a solution of a certain variational inequality.

In 2007, Chen et al. [13] considered the following iterative scheme as the viscosity approximation method of (1.3). For an  $\alpha$ -inverse-strongly-monotone mapping  $A$  of  $C$  to  $H$  and a nonexpansive mapping  $S$  of  $C$  into itself such that  $F(S) \cap VI(C, A) \neq \emptyset$ ,  $f \in \Sigma_C$ ,  $x_0 \in C$ ,  $\{\alpha_n\} \subset [0, 1)$ , and  $\{\lambda_n\} \subset [0, 2\alpha]$ ,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad n \geq 0, \quad (1.6)$$

and showed that the sequence  $\{x_n\}$  generated by (1.6) converges strongly to a point in  $F(S) \cap VI(C, A)$  under condition (1.4) on  $\{\alpha_n\}$  and  $\{\lambda_n\}$ , which is a solution of a certain variational inequality.

In this paper, motivated by above-mentioned results, we introduce a new composite iterative scheme by the viscosity approximation method. For an  $\alpha$ -inverse-strongly monotone

mapping  $A$  of  $C$  to  $H$  and a nonexpansive mapping  $S$  of  $C$  into itself such that  $F(S) \cap VI(C, A) \neq \emptyset$ ,  $f \in \Sigma_C$ ,  $x_0 \in C$ ,  $\{\alpha_n\} \subset [0, 1)$ , and  $\{\lambda_n\} \subset [0, 2\alpha]$ ,

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n SP_C(y_n - \lambda_n A y_n), \quad n \geq 0. \end{aligned} \quad (1.7)$$

If  $\beta_n = 0$ , then the iterative scheme (1.7) reduces to the iterative scheme (1.6). Under condition (1.4) on the sequences  $\{\alpha_n\}$  and  $\{\lambda_n\}$  and appropriate condition on sequence  $\{\beta_n\}$ , we show that the sequence  $\{x_n\}$  generated by (1.7) converges strongly to a point in  $F(S) \cap VI(C, A)$ , which is a solution of a certain variational inequality. Using this result, we also obtain a strong convergence result for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping. Moreover, we investigate the problem of finding a common point of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping. The main results develop and improve the corresponding results of Chen et al. [13] and Iiduka and Takahashi [10]. We point out that the iterative scheme (1.7) is a new approach for finding the fixed points of nonexpansive mappings and solutions of variational inequalities for monotone mappings.

## 2. Preliminaries and Lemmas

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ , and  $C$  a closed convex subset of  $H$ . We write  $x_n \rightharpoonup x$  to indicate that the sequence  $\{x_n\}$  converges weakly to  $x$ .  $x_n \rightarrow x$  implies that  $\{x_n\}$  converges strongly to  $x$ . For every point  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that

$$\|x - P_C x\| \leq \|x - y\| \quad (2.1)$$

for all  $y \in C$ .  $P_C$  is called the *metric projection* of  $H$  to  $C$ . It is well known that  $P_C$  satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \quad (2.2)$$

for every  $x, y \in H$ . Moreover,  $P_C x$  is characterized by the properties

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \quad (2.3)$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad (2.4)$$

for all  $x \in H$ ,  $y \in C$ . In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \iff u = P_C(u - \lambda A u), \quad \text{for any } \lambda > 0. \quad (2.5)$$

We state some examples for inverse-strongly monotone mappings. If  $A = I - T$ , where  $T$  is a nonexpansive mapping of  $C$  into itself and  $I$  is the identity mapping of  $H$ , then  $A$  is  $1/2$ -inverse-strongly monotone and  $VI(C, A) = F(T)$ . A mapping  $A$  of  $C$  into  $H$  is called *strongly monotone* if there exists a positive real number  $\eta$  such that

$$\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^2 \quad (2.6)$$

for all  $x, y \in C$ . In such a case, we say that  $A$  is  $\eta$ -strongly monotone. If  $A$  is  $\eta$ -strongly monotone and  $\kappa$ -Lipschitz continuous, that is,  $\|Ax - Ay\| \leq \kappa \|x - y\|$  for all  $x, y \in C$ , then  $A$  is  $\eta/\kappa^2$ -inverse-strongly monotone.

If  $A$  is an  $\alpha$ -inverse-strongly monotone mapping of  $C$  into  $H$ , then it is obvious that  $A$  is  $1/\alpha$ -Lipschitz continuous. We also have that for all  $x, y \in C$  and  $\lambda > 0$ ,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda \langle x - y, Ax - Ay \rangle + \lambda^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2. \end{aligned} \quad (2.7)$$

So, if  $\lambda \leq 2\alpha$ , then  $I - \lambda A$  is a nonexpansive mapping of  $C$  into  $H$ . The following result for the existence of solutions of the variational inequality problem for inverse-strongly monotone mappings was given in Takahashi and Toyoda [14].

**Proposition 2.1.** *Let  $C$  be a bounded closed convex subset of a real Hilbert space and  $A$  an  $\alpha$ -inverse-strongly monotone mapping of  $C$  into  $H$ . Then,  $VI(C, A)$  is nonempty.*

A set-valued mapping  $T : H \rightarrow 2^H$  is called *monotone* if for all  $x, y \in H$ ,  $f \in Tx$  and  $g \in Ty$  imply  $\langle x - y, f - g \rangle \geq 0$ . A monotone mapping  $T : H \rightarrow 2^H$  is *maximal* if the graph  $G(T)$  of  $T$  is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping  $T$  is maximal if and only if for  $(x, f) \in H \times H$ ,  $\langle x - y, f - g \rangle \geq 0$  for every  $(y, g) \in G(T)$  implies  $f \in Tx$ . Let  $A$  be an inverse-strongly monotone mapping of  $C$  into  $H$  and let  $N_C v$  be the *normal cone* to  $C$  at  $v$ , that is,  $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$ , and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \quad (2.8)$$

Then  $T$  is maximal monotone and  $0 \in Tv$  if and only if  $v \in VI(C, A)$ ; see [15, 16].

We need the following lemmas for the proof of our main results.

**Lemma 2.2** (see [17]). *Let  $\{s_n\}$  be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n, \quad n \geq 0, \quad (2.9)$$

where  $\{\lambda_n\}$  and  $\{\beta_n\}$  satisfy the following conditions:

- (i)  $\{\lambda_n\} \subset [0, 1]$  and  $\sum_{n=0}^{\infty} \lambda_n = \infty$  or, equivalently,  $\prod_{n=0}^{\infty} (1 - \lambda_n) = 0$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \beta_n / \lambda_n \leq 0$  or  $\sum_{n=0}^{\infty} |\beta_n| < \infty$ .

Then  $\lim_{n \rightarrow \infty} s_n = 0$ .

**Lemma 2.3** (see [1], demiclosedness principle). *Let  $H$  be a real Hilbert space,  $C$  a nonempty closed convex subset of  $H$ , and  $T : C \rightarrow E$  a nonexpansive mapping. Then the mapping  $I - T$  is demiclosed on  $C$ , where  $I$  is the identity mapping; that is,  $x_n \rightarrow x$  in  $E$  and  $(I - T)x_n \rightarrow y$  imply that  $x \in C$  and  $(I - T)x = y$ .*

**Lemma 2.4.** *In a real Hilbert space  $H$ , there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad (2.10)$$

for all  $x, y \in H$ .

### 3. Main Results

In this section, we introduce a new composite iterative scheme for nonexpansive mappings and inverse-strongly monotone mappings and prove a strong convergence of this scheme.

**Theorem 3.1.** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse-strongly monotone mapping of  $C$  to  $H$  and  $S$  a nonexpansive mapping of  $C$  into itself such that  $F(S) \cap VI(C, A) \neq \emptyset$ , and  $f \in \Sigma_C$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{aligned} x_0 &\in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n SP_C(y_n - \lambda_n A y_n), \quad n \geq 0, \end{aligned} \quad (3.1)$$

where  $\{\lambda_n\} \subset [0, 2\alpha]$ ,  $\{\alpha_n\} \subset [0, 1)$ , and  $\{\beta_n\} \subset [0, 1]$ . If  $\{\alpha_n\}$ ,  $\{\lambda_n\}$  and  $\{\beta_n\}$  satisfy the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (ii)  $\beta_n \in [0, a)$  for all  $n \geq 0$  and for some  $a \in (0, 1)$ ;
- (iii)  $\lambda_n \in [c, d]$  for some  $c, d$  with  $0 < c < d < 2\alpha$ ;
- (iv)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ;  $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ ,

then  $\{x_n\}$  converges strongly to  $q \in F(S) \cap VI(C, A)$ , which is a solution of the following variational inequality:

$$\langle (I - f)(q), q - p \rangle \leq 0, \quad p \in F(S) \cap VI(C, A). \quad (3.2)$$

*Proof.* Let  $z_n = P_C(x_n - \lambda_n Ax_n)$  and  $w_n = P_C(y_n - \lambda_n Ay_n)$  for every  $n \geq 0$ . Let  $u \in F(S) \cap VI(C, A)$ . Since  $I - \lambda_n A$  is nonexpansive and  $u = P_C(u - \lambda_n Au)$  from (2.5), we have

$$\begin{aligned} \|z_n - u\| &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\| \\ &\leq \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\| \\ &\leq \|x_n - u\|. \end{aligned} \quad (3.3)$$

Similarly we have  $\|w_n - u\| \leq \|y_n - u\|$ .

We divide the proof into several steps.

*Step 1.* We show that  $\{x_n\}$  is bounded. In fact, since

$$\begin{aligned} \|y_n - u\| &= \|\alpha_n(f(x_n) - u) + (1 - \alpha_n)(Sz_n - u)\| \\ &\leq \alpha_n\|f(x_n) - u\| + (1 - \alpha_n)\|z_n - u\| \\ &\leq \alpha_n\|f(x_n) - f(u)\| + \alpha_n\|f(u) - u\| + (1 - \alpha_n)\|x_n - u\| \\ &\leq \alpha_n k\|x_n - u\| + (1 - \alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\| \\ &= (1 - (1 - k)\alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\| \\ &\leq \max\left\{\|x_n - u\|, \frac{1}{1 - k}\|f(u) - u\|\right\}, \end{aligned} \quad (3.4)$$

we have

$$\begin{aligned} \|x_{n+1} - u\| &= \|(1 - \beta_n)(y_n - u) + \beta_n(Sw_n - u)\| \\ &\leq (1 - \beta_n)\|y_n - u\| + \beta_n\|w_n - u\| \\ &\leq (1 - \beta_n)\|y_n - u\| + \beta_n\|y_n - u\| \\ &\leq \max\left\{\|x_n - u\|, \frac{1}{1 - k}\|f(u) - u\|\right\}. \end{aligned} \quad (3.5)$$

By induction, we get

$$\|x_n - u\| \leq \max\left\{\|x_0 - u\|, \frac{1}{1 - k}\|f(u) - u\|\right\}, \quad n \geq 0. \quad (3.6)$$

This implies that  $\{x_n\}$  is bounded and so  $\{y_n\}$ ,  $\{z_n\}$ ,  $\{w_n\}$ ,  $\{Ax_n\}$ , and  $\{Ay_n\}$  are bounded. Moreover, since  $\|Sz_n - u\| \leq \|x_n - u\|$  and  $\|Sw_n - u\| \leq \|y_n - u\|$ ,  $\{Sz_n\}$  and  $\{Sw_n\}$  are also bounded. By condition (i), we also obtain

$$\|y_n - Sz_n\| = \alpha_n\|f(x_n) - Sz_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \quad (3.7)$$

Step 2. We show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ . From (3.1), we have

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1 - \alpha_n)Sz_n, \\ y_{n-1} &= \alpha_{n-1} f(x_{n-1}) + (1 - \alpha_{n-1})Sz_{n-1}, \quad n \geq 1. \end{aligned} \quad (3.8)$$

Simple calculations show that

$$y_n - y_{n-1} = (1 - \alpha_n)(Sz_n - Sz_{n-1}) + (\alpha_n - \alpha_{n-1})(f(x_{n-1}) - Sz_{n-1}) + \alpha_n(f(x_n) - f(x_{n-1})). \quad (3.9)$$

Since

$$\begin{aligned} \|z_n - z_{n-1}\| &\leq \|(x_n - \lambda_n Ax_n) - (x_{n-1} - \lambda_{n-1} Ax_{n-1})\| \\ &\leq \|(x_n - \lambda_n Ax_n) - (x_{n-1} - \lambda_n Ax_{n-1})\| + |\lambda_{n-1} - \lambda_n| \|Ax_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\lambda_{n-1} - \lambda_n| \|Ax_{n-1}\| \end{aligned} \quad (3.10)$$

for every  $n \geq 1$ , we have

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq (1 - \alpha_n) \|z_n - z_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - Sz_{n-1}\| + \alpha_n k \|x_n - x_{n-1}\| \\ &\leq (1 - \alpha_n) (\|x_n - x_{n-1}\| + |\lambda_{n-1} - \lambda_n| \|Ax_{n-1}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - Sz_{n-1}\| + \alpha_n k \|x_n - x_{n-1}\| \\ &\leq (1 - (1 - k)\alpha_n) \|x_n - x_{n-1}\| + L_1 |\lambda_{n-1} - \lambda_n| + M_1 |\alpha_n - \alpha_{n-1}| \end{aligned} \quad (3.11)$$

for every  $n \geq 1$ , where  $M_1 = \sup\{\|f(x_n) - Sz_{n-1}\| : n \geq 1\}$  and  $L_1 = \sup\{\|Ax_n\| : n \geq 0\}$ .

On the other hand, from (3.1) we have

$$\begin{aligned} x_{n+1} &= (1 - \beta_n)y_n + \beta_n S\omega_n, \\ x_n &= (1 - \beta_{n-1})y_{n-1} + \beta_{n-1} S\omega_{n-1}. \end{aligned} \quad (3.12)$$

Also, simple calculations show that

$$x_{n+1} - x_n = (1 - \beta_n)(y_n - y_{n-1}) + \beta_n(S\omega_n - S\omega_{n-1}) + (\beta_n - \beta_{n-1})(S\omega_{n-1} - y_{n-1}). \quad (3.13)$$

Since

$$\begin{aligned} \|\omega_n - \omega_{n-1}\| &\leq \|(y_n - \lambda_n Ay_n) - (y_{n-1} - \lambda_{n-1} Ay_{n-1})\| \\ &\leq \|(y_n - \lambda_n Ay_n) - (y_{n-1} - \lambda_n Ay_{n-1})\| + |\lambda_{n-1} - \lambda_n| \|Ay_{n-1}\| \\ &\leq \|y_n - y_{n-1}\| + |\lambda_{n-1} - \lambda_n| \|Ay_{n-1}\| \end{aligned} \quad (3.14)$$

for every  $n \geq 1$ , it follows that

$$\begin{aligned}
 \|x_{n+1} - x_n\| &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n\|\omega_n - \omega_{n-1}\| + |\beta_n - \beta_{n-1}|\|S\omega_{n-1} - y_{n-1}\| \\
 &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n(\|y_n - y_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|Ay_{n-1}\|) \\
 &\quad + |\beta_n - \beta_{n-1}|\|S\omega_{n-1} - y_{n-1}\| \\
 &\leq \|y_n - y_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|Ay_{n-1}\| + |\beta_n - \beta_{n-1}|\|S\omega_{n-1} - y_{n-1}\|.
 \end{aligned} \tag{3.15}$$

Substituting (3.11) into (3.15), we derive

$$\begin{aligned}
 \|x_{n+1} - x_n\| &\leq (1 - (1 - k)\alpha_n)\|x_n - x_{n-1}\| + L_1|\lambda_{n-1} - \lambda_n| + M_1|\alpha_n - \alpha_{n-1}| \\
 &\quad + |\lambda_{n-1} - \lambda_n|\|Ay_{n-1}\| + |\beta_n - \beta_{n-1}|\|S\omega_{n-1} - y_{n-1}\| \\
 &\leq (1 - (1 - k)\alpha_n)\|x_n - x_{n-1}\| + L_2|\lambda_{n-1} - \lambda_n| + M_1|\alpha_n - \alpha_{n-1}| + M_2|\beta_n - \beta_{n-1}|,
 \end{aligned} \tag{3.16}$$

where  $L_2 = \sup\{L_1 + \|Ay_n\| : n \geq 1\}$  and  $M_2 = \sup\{\|S\omega_n - y_n\| : n \geq 0\}$ . From conditions (i) and (iv), it is easy to see that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} (1 - k)\alpha_n &= 0, & \sum_{n=0}^{\infty} (1 - k)\alpha_n &= \infty, \\
 \sum_{n=0}^{\infty} (M_1|\alpha_{n+1} - \alpha_n| + M_2|\beta_{n+1} - \beta_n| + L_2|\lambda_{n+1} - \lambda_n|) &< \infty.
 \end{aligned} \tag{3.17}$$

Applying Lemma 2.2 to (3.16), we have

$$\|x_{n+1} - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.18}$$

By (3.11), we also have that  $\|y_{n+1} - y_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

*Step 3.* We show that  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|x_n - Sz_n\| = 0$ . Indeed, it follows that

$$\begin{aligned}
 \|x_{n+1} - y_n\| &= \beta_n\|S\omega_n - y_n\| \\
 &\leq \beta_n(\|S\omega_n - Sz_n\| + \|Sz_n - y_n\|) \\
 &\leq a(\|\omega_n - z_n\| + \|Sz_n - y_n\|) \\
 &\leq a(\|y_n - x_n\| + \|Sz_n - y_n\|) \\
 &\leq a(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|Sz_n - y_n\|),
 \end{aligned} \tag{3.19}$$

which implies that

$$\|x_{n+1} - y_n\| \leq \frac{a}{1 - a} (\|x_{n+1} - x_n\| + \|Sz_n - y_n\|). \tag{3.20}$$

Obviously, by (3.7) and Step 2, we have  $\|x_{n+1} - y_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . This implies that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.21)$$

By (3.7) and (3.21), we also have

$$\|x_n - Sz_n\| \leq \|x_n - y_n\| + \|y_n - Sz_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.22)$$

*Step 4.* We show that  $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$ . To this end, let  $u \in F(S) \cap VI(C, A)$ . Then, by convexity of  $\|\cdot\|^2$ , we have

$$\begin{aligned} \|y_n - u\|^2 &= \|\alpha_n f(x_n) + (1 - \alpha_n)Sz_n - u\|^2 \\ &\leq \alpha_n \|f(x_n) - u\|^2 + (1 - \alpha_n) \|Sz_n - u\|^2 \\ &\leq \alpha_n \|f(x_n) - u\|^2 + (1 - \alpha_n) \|z_n - u\|^2 \\ &\leq \alpha_n \|f(x_n) - u\|^2 + (1 - \alpha_n) \left[ \|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha) \|Ax_n - Au\|^2 \right] \\ &\leq \alpha_n \|f(x_n) - u\|^2 + \|x_n - u\|^2 + (1 - \alpha_n)c(d - 2\alpha) \|Ax_n - Au\|^2. \end{aligned} \quad (3.23)$$

So we obtain

$$\begin{aligned} &-(1 - \alpha_n)c(d - 2\alpha) \|Ax_n - Au\|^2 \\ &\leq \alpha_n \|f(x_n) - u\|^2 + (\|x_n - u\| + \|y_n - u\|)(\|x_n - u\| - \|y_n - u\|) \\ &\leq \alpha_n \|f(x_n) - u\|^2 + (\|x_n - u\| + \|y_n - u\|) \|x_n - y_n\|. \end{aligned} \quad (3.24)$$

Since  $\alpha_n \rightarrow 0$  and  $\|x_n - y_n\| \rightarrow 0$  by condition (i) and (3.21), we have  $\|Ax_n - Au\| \rightarrow 0$  ( $n \rightarrow \infty$ ). Moreover, from (2.2) we obtain

$$\begin{aligned} \|z_n - u\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\|^2 \\ &\leq \langle x_n - \lambda_n Ax_n - (u - \lambda_n Au), z_n - u \rangle \\ &= \frac{1}{2} \left\{ \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\|^2 + \|z_n - u\|^2 \right. \\ &\quad \left. - \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au) - (z_n - u)\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|x_n - u\|^2 + \|z_n - u\|^2 - \|x_n - z_n\|^2 \right. \\ &\quad \left. + 2\lambda_n \langle x_n - z_n, Ax_n - Au \rangle - \lambda_n^2 \|Ax_n - Au\|^2 \right\}, \end{aligned} \quad (3.25)$$

and so

$$\|z_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \langle x_n - z_n, Ax_n - Au \rangle - \lambda_n^2 \|Ax_n - Au\|^2. \quad (3.26)$$

And hence

$$\begin{aligned} \|y_n - u\|^2 &\leq \alpha_n \|f(x_n) - u\|^2 + (1 - \alpha_n) \|z_n - u\|^2 \\ &\leq \alpha_n \|f(x_n) - u\|^2 + \|x_n - u\|^2 - (1 - \alpha_n) \|x_n - z_n\|^2 \\ &\quad + 2(1 - \alpha_n) \lambda_n \langle x_n - z_n, Ax_n - Au \rangle - (1 - \alpha_n) \lambda_n^2 \|Ax_n - Au\|^2. \end{aligned} \quad (3.27)$$

Then we have

$$\begin{aligned} (1 - \alpha_n) \|x_n - z_n\|^2 &\leq \alpha_n \|f(x_n) - u\|^2 + (\|x_n - u\| + \|y_n - u\|) (\|x_n - u\| - \|y_n - u\|) \\ &\quad + 2(1 - \alpha_n) \lambda_n \langle x_n - z_n, Ax_n - Au \rangle - (1 - \alpha_n) \lambda_n^2 \|Ax_n - Au\|^2 \\ &\leq \alpha_n \|f(x_n) - u\|^2 + (\|x_n - u\| + \|y_n - u\|) \|x_n - y_n\| \\ &\quad + 2(1 - \alpha_n) \lambda_n \langle x_n - z_n, Ax_n - Au \rangle - (1 - \alpha_n) \lambda_n^2 \|Ax_n - Au\|^2. \end{aligned} \quad (3.28)$$

Since  $\alpha_n \rightarrow 0$ ,  $\|x_n - y_n\| \rightarrow 0$  and  $\|Ax_n - Au\| \rightarrow 0$ , we get  $\|x_n - z_n\| \rightarrow 0$ . Also by (3.21), we have

$$\|y_n - z_n\| \leq \|y_n - x_n\| + \|x_n - z_n\| \rightarrow 0 \quad (n \rightarrow \infty). \quad (3.29)$$

*Step 5.* We show that  $\limsup_{n \rightarrow \infty} \langle f(q) - q, y_n - q \rangle \leq 0$  for  $q \in F(S) \cap \text{VI}(C, A)$ , where  $q$  is a solution of the variational inequality

$$\langle (I - f)(q), q - p \rangle \leq 0, \quad p \in F(S) \cap \text{VI}(C, A). \quad (3.30)$$

To this end, choose a subsequence  $\{z_{n_i}\}$  of  $\{z_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, Sz_n - q \rangle = \lim_{i \rightarrow \infty} \langle f(q) - q, Sz_{n_i} - q \rangle. \quad (3.31)$$

Since  $\{z_{n_i}\}$  is bounded, there exists a subsequence  $\{z_{n_{i_j}}\}$  of  $\{z_{n_i}\}$  which converges weakly to  $z$ . We may assume without loss of generality that  $z_{n_{i_j}} \rightharpoonup z$ . Since  $\|Sz_{n_{i_j}} - z_{n_{i_j}}\| \leq \|Sz_{n_{i_j}} - x_{n_{i_j}}\| + \|x_{n_{i_j}} - z_{n_{i_j}}\| \rightarrow 0$  by Steps 4 and 5, we have  $Sz_{n_{i_j}} \rightarrow z$ . Then we can obtain  $z \in F(S) \cap \text{VI}(C, A)$ . Indeed, let us first show that  $z \in \text{VI}(C, A)$ . Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \quad (3.32)$$

Then  $T$  is maximal monotone. Let  $(v, w) \in G(T)$ . Since  $w - Av \in N_C v$  and  $z_n \in C$ , we have

$$\langle v - z_n, w - Av \rangle \geq 0. \quad (3.33)$$

On the other hand, from  $z_n = P_C(x_n - \lambda_n Ax_n)$ , we have  $\langle v - z_n, z_n - (x_n - \lambda_n Ax_n) \rangle \geq 0$  and hence

$$\left\langle v - z_n, \frac{z_n - x_n}{\lambda_n} + Ax_n \right\rangle \geq 0. \quad (3.34)$$

Therefore we have

$$\begin{aligned} \langle v - z_{n_i}, w \rangle &\geq \langle v - z_{n_i}, Av \rangle \\ &\geq \langle v - z_{n_i}, Av \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i} \right\rangle \\ &= \left\langle v - z_{n_i}, Av - Ax_{n_i} - \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\ &= \langle v - z_{n_i}, Av - Az_{n_i} \rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle \\ &\quad - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle. \end{aligned} \quad (3.35)$$

Hence we have  $\langle v - z, w \rangle \geq 0$  as  $i \rightarrow \infty$ . Since  $T$  is maximal monotone, we have  $z \in T^{-1}0$  and hence  $z \in \text{VI}(C, A)$ .

On the another hand, by Steps 3 and 4,  $\|z_n - Sz_n\| \leq \|z_n - x_n\| + \|x_n - Sz_n\| \rightarrow 0$ . So, by Lemma 2.3, we obtain  $z \in F(S)$  and hence  $z \in F(S) \cap \text{VI}(C, A)$ . Then by (3.30) we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f(q) - q, Sz_{n_i} - q \rangle &= \lim_{i \rightarrow \infty} \langle f(q) - q, Sz_{n_i} - q \rangle = \langle f(q) - q, z - q \rangle \\ &= \langle (I - f)(q), q - z \rangle \leq 0. \end{aligned} \quad (3.36)$$

Thus, from (3.7) we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f(q) - q, y_n - q \rangle &\leq \limsup_{n \rightarrow \infty} \langle f(q) - q, y_n - Sz_n \rangle + \limsup_{n \rightarrow \infty} \langle f(q) - q, Sz_n - q \rangle \\ &\leq \limsup_{n \rightarrow \infty} \|f(q) - q\| \|y_n - Sz_n\| + \limsup_{n \rightarrow \infty} \langle f(q) - q, Sz_n - q \rangle \\ &\leq 0. \end{aligned} \quad (3.37)$$

Step 6. We show that  $\lim_{n \rightarrow \infty} \|x_n - q\| = 0$  for  $q \in F(S) \cap VI(C, A)$ , where  $q$  is a solution of the variational inequality

$$\langle (I - f)(q), q - p \rangle \leq 0, \quad p \in F(S) \cap VI(C, A). \quad (3.38)$$

Indeed, from Lemma 2.4, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|y_n - q\|^2 = \|\alpha_n(f(x_n) - q) + (1 - \alpha_n)(Sz_n - q)\|^2 \\ &\leq (1 - \alpha_n)\|Sz_n - q\|^2 + 2\alpha_n\langle f(x_n) - q, y_n - q \rangle \\ &\leq (1 - \alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\langle f(x_n) - f(q), y_n - q \rangle + 2\alpha_n\langle f(q) - q, y_n - q \rangle \\ &\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n k \|x_n - q\| \|y_n - q\| + 2\alpha_n\langle f(q) - q, y_n - q \rangle \\ &\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n k \|x_n - q\| (\|y_n - x_n\| + \|x_n - q\|) \\ &\quad + 2\alpha_n\langle f(q) - q, y_n - q \rangle \\ &= (1 - 2(1 - k)\alpha_n)\|x_n - q\|^2 + 2\alpha_n k \|y_n - x_n\| \|x_n - q\| + 2\alpha_n\langle f(q) - q, y_n - q \rangle \\ &\leq (1 - \bar{\alpha}_n)\|x_n - q\|^2 + \bar{\alpha}_n \bar{\beta}_n, \end{aligned} \quad (3.39)$$

where

$$\bar{\alpha}_n = 2(1 - k)\alpha_n, \quad \bar{\beta}_n = \frac{kB}{1 - k} \|y_n - x_n\| + \frac{1}{1 - k} \langle f(q) - q, y_n - q \rangle, \quad (3.40)$$

and  $B = \sup\{\|x_n - q\| : n \geq 0\}$ . It is easily seen that  $\bar{\alpha}_n \rightarrow 0$ ,  $\sum_{n=1}^{\infty} \bar{\alpha}_n = \infty$ , and  $\limsup_{n \rightarrow \infty} \bar{\beta}_n \leq 0$ . Thus by Lemma 2.2, we obtain  $x_n \rightarrow q$ . This completes the proof.  $\square$

*Remark 3.2.* (1) Theorem 3.1 improves the corresponding results in Chen et al. [13] and Iiduka and Takahashi [10]. In particular, if  $\beta_n = 0$  and  $f(x_n) = x$  is constant in (3.1), then Theorem 3.1 reduces to Theorem 3.1 of Iiduka and Takahashi [10].

(2) We obtain a new composite iterative scheme for a nonexpansive mapping if  $A = 0$  in Theorem 3.1 as follows (see also Jung [18]):

$$\begin{aligned} x_0 &\in C, \\ y_n &= \alpha_n f(x_n) + (1 - \alpha_n) Sx_n, \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n S y_n, \quad n \geq 0. \end{aligned} \quad (3.41)$$

As a direct consequence of Theorem 3.1, we have the following result.

**Corollary 3.3.** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse-strongly monotone mapping of  $C$  to  $H$  such that  $VI(C, A) \neq \emptyset$ , and  $f \in \Sigma_C$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{aligned}x_0 &\in C, \\y_n &= \alpha_n f(x_n) + (1 - \alpha_n)P_C(x_n - \lambda_n A x_n), \\x_{n+1} &= (1 - \beta_n)y_n + \beta_n P_C(y_n - \lambda_n A y_n), \quad n \geq 0,\end{aligned}\tag{3.42}$$

where  $\{\lambda_n\} \subset [0, 2\alpha]$ ,  $\{\alpha_n\} \subset [0, 1)$ , and  $\{\beta_n\} \subset [0, 1]$ . If  $\{\alpha_n\}$ ,  $\{\lambda_n\}$ , and  $\{\beta_n\}$  satisfy the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (ii)  $\beta_n \in [0, a)$  for all  $n \geq 0$  and for some  $a \in (0, 1)$ ,
- (iii)  $\lambda_n \in [c, d]$  for some  $c, d$  with  $0 < c < d < 2\alpha$ ,
- (iv)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ;  $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ ,

then  $\{x_n\}$  converges strongly to  $q \in VI(C, A)$ , which is a solution of the following variational inequality:

$$\langle (I - f)(q), q - p \rangle \leq 0, \quad p \in VI(C, A).\tag{3.43}$$

## 4. Applications

In this section, as in [10, 13], we obtain two theorems in a Hilbert space by using Theorem 3.1.

A mapping  $T : C \rightarrow C$  is called *strictly pseudocontractive* if there exists  $\alpha$  with  $0 \leq \alpha < 1$  such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \alpha \|(I - T)x - (I - T)y\|^2\tag{4.1}$$

for every  $x, y \in C$ . If  $\alpha = 0$ , then  $T$  is nonexpansive. Put  $A = I - T$ , where  $T : C \rightarrow C$  is a strictly pseudocontractive mapping with  $\alpha$ . Then  $A$  is  $(1 - \alpha)/2$ -inverse-strongly monotone; see [7]. Actually, we have, for all  $x, y \in C$ ,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + \alpha \|Ax - Ay\|^2.\tag{4.2}$$

On the other hand, since  $H$  is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \alpha \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay \rangle.\tag{4.3}$$

Hence we have

$$\langle x - y, Ax - Ay \rangle \geq \frac{1 - \alpha}{2} \|Ax - Ay\|^2.\tag{4.4}$$

Using Theorem 3.1, we first get a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

**Theorem 4.1.** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $T$  be an  $\alpha$ -strictly pseudocontractive mapping of  $C$  into itself and  $S$  a nonexpansive mapping of  $C$  into itself such that  $F(S) \cap F(T) \neq \emptyset$ , and  $f \in \Sigma_C$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{aligned}x_0 &\in C, \\y_n &= \alpha_n f(x_n) + (1 - \alpha_n)S((1 - \lambda_n)x_n + \lambda_n T x_n), \\x_{n+1} &= (1 - \beta_n)y_n + \beta_n S((1 - \lambda_n)y_n + \lambda_n T y_n), \quad n \geq 0,\end{aligned}\tag{4.5}$$

where  $\{\lambda_n\} \subset [0, 1 - \alpha]$ ,  $\{\alpha_n\} \subset [0, 1)$ , and  $\{\beta_n\} \subset [0, 1]$ . If  $\{\alpha_n\}$ ,  $\{\lambda_n\}$ , and  $\{\beta_n\}$  satisfy the conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (ii)  $\beta_n \in [0, a)$  for all  $n \geq 0$  and for some  $a \in (0, 1)$ ,
- (iii)  $\lambda_n \in [c, d]$  for some  $c, d$  with  $0 < c < d < 1 - \alpha$ ,
- (iv)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ;  $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ ,

then  $\{x_n\}$  converges strongly to  $q \in F(S) \cap F(T)$ , which is a solution of the following variational inequality:

$$\langle (I - f)(q), q - p \rangle \leq 0, \quad p \in F(S) \cap F(T).\tag{4.6}$$

*Proof.* Put  $A = I - T$ . Then  $A$  is  $(1 - \alpha)/2$ -inverse-strongly monotone. We have  $F(T) = VI(C, A)$  and  $P_C(x_n - \lambda_n A x_n) = (1 - \lambda_n)x_n + \lambda_n T x_n$ . Thus, the desired result follows from Theorem 3.1.  $\square$

Using Theorem 3.1, we also have the following result.

**Theorem 4.2.** *Let  $H$  be a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse-strongly monotone mapping of  $H$  into itself and  $S$  a nonexpansive mapping of  $H$  into itself such that  $F(S) \cap A^{-1}0 \neq \emptyset$ , and  $f \in \Sigma_C$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{aligned}x_0 &\in H, \\y_n &= \alpha_n f(x_n) + (1 - \alpha_n)S(x_n - \lambda_n A x_n), \\x_{n+1} &= (1 - \beta_n)y_n + \beta_n S(y_n - \lambda_n A y_n), \quad n \geq 0,\end{aligned}\tag{4.7}$$

where  $\{\lambda_n\} \subset [0, 2\alpha]$ ,  $\{\alpha_n\} \subset [0, 1)$ , and  $\{\beta_n\} \subset [0, 1]$ . If  $\{\alpha_n\}$ ,  $\{\lambda_n\}$ , and  $\{\beta_n\}$  satisfy the conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (ii)  $\beta_n \in [0, a)$  for all  $n \geq 0$  and for some  $a \in (0, 1)$ ,
- (iii)  $\lambda_n \in [c, d]$  for some  $c, d$  with  $0 < c < d < 2\alpha$ ,
- (iv)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ;  $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ ,

then  $\{x_n\}$  converges strongly to  $q \in F(S) \cap A^{-1}0$ , which is a solution of the following variational inequality:

$$\langle (I - f)(q), q - p \rangle \leq 0, \quad p \in F(S) \cap A^{-1}0. \quad (4.8)$$

*Proof.* We have  $A^{-1}0 = VI(H, A)$ . So, putting  $P_H = I$ , by Theorem 3.1, we obtain the desired result.  $\square$

*Remark 4.3.* If  $\beta_n = 0$  in Theorems 4.1 and 4.2, then Theorems 4.1 and 4.2 reduce to Chen et al. [13, Theorems 4.1 and 4.2]. Theorems 4.1 and 4.2 also extend in Iiduka and Takahashi [10, Theorems 4.1 and 4.2] to the viscosity methods in composite iterative schemes.

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