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*Research Article*

# The Optimal Convex Combination Bounds of Arithmetic and Harmonic Means for the Seiffert's Mean

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We find the greatest value $\alpha$ and least value $\beta$ such that the double inequality $\alpha A(a,b)+(1-\alpha)H(a,b)<P(a,b)<\beta A(a,b)+(1-\beta)H(a,b)$ holds for all $a,b>0$ with $a\neq b$. Here $A(a,b)$, $H(a,b)$, and $P(a,b)$ denote the arithmetic, harmonic, and Seiffert's means of two positive numbers $a$ and $b$, respectively.

## 1. Introduction

For $a,b>0$ with $a\neq b$ the Seiffert's mean $P(a,b)$ was introduced by Seiffert [1] as follows:

$$P(a,b)=\frac{a-b}{4\arctan\left(\sqrt{a/b}\right)-\pi}. \tag{1.1}$$

Recently, the inequalities for means have been the subject of intensive research [2–11]. In particular, many remarkable inequalities for the Seiffert's mean can be found in the literature [12–17]. The Seiffert's mean $P(a,b)$ can be rewritten as (see [14, (2.4)])

$$P(a,b)=\frac{a-b}{2\arcsin((a-b)/(a+b))}. \tag{1.2}$$

Let $A(a,b)=(a+b)/2$, $G(a,b)=\sqrt{ab}$, $H(a,b)=2ab/(a+b)$, $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, and $L(a,b)=(b-a)/(\log b-\log a)$ be the arithmetic, geometric, harmonic, identric, and logarithmic means of two positive real numbers $a$ and $b$ with $a\neq b$. Then

$$\min\{a,b\}<H(a,b)<G(a,b)<L(a,b)<I(a,b)<A(a,b)<\max\{a,b\}. \tag{1.3}$$

In [1], Seiffert proved that

$$L(a,b)<P(a,b)<I(a,b) \tag{1.4}$$

for all $a,b>0$ with $a\neq b$.

Later, Seiffert [18] established that

$$\begin{aligned} P(a,b)&>\frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)},\\ P(a,b)&>\frac{A(a,b)G(a,b)}{L(a,b)},\\ P(a,b)&>\frac{2}{\pi}A(a,b) \end{aligned} \tag{1.5}$$

for all $a,b>0$ with $a\neq b$.

In [19], Sándor proved that

$$\begin{gathered} \frac{1}{2}[A(a,b)+G(a,b)]<P(a,b)<\sqrt{A(a,b)}\sqrt{\frac{1}{2}[A(a,b)+G(a,b)]},\\ A^{2/3}(a,b)G^{1/3}(a,b)<P(a,b)<\frac{2}{3}A(a,b)+\frac{1}{3}G(a,b) \end{gathered} \tag{1.6}$$

for all $a,b>0$ with $a\neq b$.

The following bounds for the Seiffert's mean $P(a,b)$ in terms of the power mean $M_r(a,b)=((a^r+b^r)/2)^{1/r}$ $(r\neq 0)$ were presented by Jagers in [17]:

$$M_{1/2}(a,b)<P(a,b)<M_{2/3}(a,b) \tag{1.7}$$

for all $a,b>0$ with $a\neq b$.

Hästö [13] found the sharp lower power bound for the Seiffert's mean as follows:

$$M_{\log 2/\log\pi}(a,b)<P(a,b) \tag{1.8}$$

for all $a,b>0$ with $a\neq b$.

The purpose of this paper is to find the greatest value $\alpha$ and the least value $\beta$ such that the double inequality $\alpha A(a,b)+(1-\alpha)H(a,b)<P(a,b)<\beta A(a,b)+(1-\beta)H(a,b)$ holds for all $a,b>0$ with $a\neq b$.

## 2. Main Result

**Theorem 2.1.** *The double inequality* $\alpha A(a,b)+(1-\alpha)H(a,b) < P(a,b) < \beta A(a,b)+(1-\beta)H(a,b)$ *holds for all* $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha \leq 2/\pi$ *and* $\beta \geq 5/6$.

*Proof.* Firstly, we prove that

$$P(a,b) < \frac{5}{6}A(a,b) + \frac{1}{6}H(a,b), \tag{2.1}$$

$$P(a,b) > \frac{2}{\pi}A(a,b) + \left(1-\frac{2}{\pi}\right)H(a,b) \tag{2.2}$$

for all $a,b>0$ with $a\neq b$.

Without loss of generality, we assume $a>b$. Let $t=\sqrt{a/b}>1$ and $p\in\{5/6, 2/\pi\}$. Then (1.1) leads to

$$\begin{aligned} &P(a,b) - \left[pA(a,b) + (1-p)H(a,b)\right] \\ &\quad = \frac{b\left[p(t^2+1)^2 + 4(1-p)t^2\right]}{2(t^2+1)(4\arctan t - \pi)}\left[\frac{2(t^4-1)}{pt^4+(4-2p)t^2+p} - 4\arctan t + \pi\right]. \end{aligned} \tag{2.3}$$

Let

$$f(t) = \frac{2(t^4-1)}{pt^4+(4-2p)t^2+p} - 4\arctan t + \pi. \tag{2.4}$$

Then simple computations lead to

$$\lim_{t\to 1} f(t) = 0, \tag{2.5}$$

$$\lim_{t\to +\infty} f(t) = \frac{2}{p} - \pi, \tag{2.6}$$

$$f'(t) = \frac{4(t-1)^2}{(t^2+1)\left[pt^4+(4-2p)t^2+p\right]^2}g(t), \tag{2.7}$$

where

$$\begin{aligned} g(t) = {} & -p^2t^6 + \left(-2p^2-2p+4\right)t^5 + \left(p^2-12p+8\right)t^4 \\ & + \left(4p^2-20p+16\right)t^3 + \left(p^2-12p+8\right)t^2 \\ & + \left(-2p^2-2p+4\right)t - p^2. \end{aligned} \tag{2.8}$$

We divide the proof into two cases.

*Case 1.* If $p = 5/6$, then it follows from (2.8) that

$$g(t) = -\frac{1}{36}\left(25t^4 + 16t^3 + 54t^2 + 16t + 25\right)(t-1)^2 < 0 \tag{2.9}$$

for $t > 1$.

Therefore, inequality (2.1) follows from (2.3)–(2.5) and (2.7) together with (2.9).

*Case 2.* If $p = 2/\pi$, then from (2.8) we have

$$g(1) = 8(5-6p) = 8\left(5 - \frac{12}{\pi}\right) > 0, \tag{2.10}$$

$$\lim_{t\to+\infty} g(t) = -\infty, \tag{2.11}$$

$$\begin{aligned} g'(t) = {} & -6p^2t^5 + \left(-10p^2 - 10p + 20\right)t^4 + \left(4p^2 - 48p + 32\right)t^3 \\ & + \left(12p^2 - 60p + 48\right)t^2 + \left(2p^2 - 24p + 16\right)t - 2p^2 - 2p + 4, \end{aligned} \tag{2.12}$$

$$g'(1) = 24(5-6p) = 24\left(5 - \frac{12}{\pi}\right) > 0, \tag{2.13}$$

$$\lim_{t\to+\infty} g'(t) = -\infty, \tag{2.14}$$

$$\begin{aligned} g''(t) = {} & -30p^2t^4 + \left(-40p^2 - 40p + 80\right)t^3 + \left(12p^2 - 144p + 96\right)t^2 \\ & + \left(24p^2 - 120p + 96\right)t + 2p^2 - 24p + 16, \end{aligned} \tag{2.15}$$

$$g''(1) = 8\left(36 - 41p - 4p^2\right) = 8\left(36 - \frac{82}{\pi} - \frac{16}{\pi^2}\right) > 0, \tag{2.16}$$

$$\lim_{t\to+\infty} g''(t) = -\infty, \tag{2.17}$$

$$\begin{aligned} g'''(t) = {} & -120p^2t^3 + \left(-120p^2 - 120p + 240\right)t^2 \\ & + \left(24p^2 - 288p + 192\right)t + 24p^2 - 120p + 96, \end{aligned} \tag{2.18}$$

$$g'''(1) = 48\left(11 - 11p - 4p^2\right) = 48\left(11 - \frac{22}{\pi} - \frac{16}{\pi^2}\right) > 0, \tag{2.19}$$

$$\lim_{t\to+\infty} g'''(t) = -\infty, \tag{2.20}$$

$$g^{(4)}(t) = -360p^2t^2 + \left(-240p^2 - 240p + 480\right)t + 24p^2 - 288p + 192, \tag{2.21}$$

$$g^{(4)}(1) = 48\left(14 - 11p - 12p^2\right) = 48\left(14 - \frac{22}{\pi} - \frac{48}{\pi^2}\right) > 0, \tag{2.22}$$

$$\lim_{t\to+\infty} g^{(4)}(t) = -\infty, \tag{2.23}$$

$$g^{(5)}(t) = -720p^2t - 240p^2 - 240p + 480, \tag{2.24}$$

$$g^{(5)}(1) = 240\left(2 - p - 4p^2\right) = 240\left(2 - \frac{2}{\pi} - \frac{16}{\pi^2}\right) < 0. \tag{2.25}$$

From (2.24) and (2.25) we clearly see that $g^{(5)}(t) < 0$ for $t \geq 1$, hence $g^{(4)}(t)$ is strictly decreasing in $[1, +\infty)$. It follows from (2.22) and (2.23) together with the monotonicity of $g^{(4)}(t)$ that there exists $\lambda_1 > 1$ such that $g^{(4)}(t) > 0$ for $t \in [1, \lambda_1)$ and $g^{(4)}(t) < 0$ for $t \in (\lambda_1, +\infty)$, hence $g'''(t)$ is strictly increasing in $[1, \lambda_1]$ and strictly decreasing in $[\lambda_1, +\infty)$.

From (2.19) and (2.20) together with the monotonicity of $g'''(t)$ we know that there exists $\lambda_2 > 1$ such that $g'''(t) > 0$ for $t \in [1, \lambda_2)$ and $g'''(t) < 0$ for $t \in (\lambda_2, +\infty)$, hence, $g''(t)$ is strictly increasing in $[1, \lambda_2]$ and strictly decreasing in $[\lambda_2, \infty)$.

From (2.16) and (2.17) together with the monotonicity of $g''(t)$ we clearly see that there exists $\lambda_3 > 1$ such that $g'(t)$ is strictly increasing in $[1, \lambda_3]$ and strictly decreasing in $[\lambda_3, \infty)$. It follows from (2.13) and (2.14) together with the monotonicity of $g'(t)$ that there exists $\lambda_4 > 1$ such that $g(t)$ is strictly increasing in $[1, \lambda_4]$ and strictly decreasing in $[\lambda_4, \infty)$. Then (2.7), (2.10) and (2.11) imply that there exists $\lambda_5 > 1$ such that $f(t)$ is strictly increasing in $(1, \lambda_5]$ and strictly decreasing in $[\lambda_5, \infty)$.

Note that (2.6) becomes

$$\lim_{t\to+\infty} f(t) = 0 \tag{2.26}$$

for $p = 2/\pi$.

It follows from (2.5) and (2.26) together with the monotonicity of $f(t)$ that

$$f(t) > 0 \tag{2.27}$$

for $t > 1$.

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with (2.27).

Secondly, we prove that $(5/6)A(a,b) + (1/6)H(a,b)$ is the best possible upper convex combination bound of arithmetic and harmonic means for the Seiffert's mean $P(a,b)$.

For any $t > 1$ and $\beta \in \mathbb{R}$, from (1.1) we have

$$\begin{aligned} P\left(1,t^2\right) - \left[\beta A\left(1,t^2\right) + (1-\beta)H\left(1,t^2\right)\right] &= \frac{t^2-1}{4\arctan t - \pi} - \frac{\beta}{2}\left(1+t^2\right) - 2(1-\beta)\frac{t^2}{1+t^2} \\ &= \frac{h(t)}{(4\arctan t - \pi)(1+t^2)}, \end{aligned} \tag{2.28}$$

where

$$h(t) = \left(t^4 - 1\right) - \frac{\beta}{2}\left(t^2+1\right)^2(4\arctan t - \pi) - 2(1-\beta)t^2(4\arctan t - \pi). \tag{2.29}$$

It follows from (2.29) that

$$h(1) = h'(1) = h''(1) = 0, \tag{2.30}$$

$$h'''(1) = 4(5 - 6\beta). \tag{2.31}$$

If $\beta < 5/6$, then (2.31) leads to

$$h'''(1) > 0. \tag{2.32}$$

From (2.32) and the continuity of $h'''(t)$ we clearly see that there exists $\delta = \delta(\beta) > 0$ such that

$$h'''(t) > 0 \tag{2.33}$$

for $t \in [1, 1 + \delta)$. Then (2.30) and (2.33) imply that

$$h(t) > 0 \tag{2.34}$$

for $t \in (1, 1 + \delta)$.

Therefore, $P(1, t^2) > \beta A(1, t^2) + (1 - \beta)H(1, t^2)$ for $t \in (1, 1 + \delta)$ follows from (2.28) and (2.34).

Finally, we prove that $(2/\pi)A(a, b) + (1 - 2/\pi)H(a, b)$ is the best possible lower convex combination bound of arithmetic and harmonic means for the Seiffert's mean $P(a, b)$.

For $\alpha > 2/\pi$, then from (1.1) one has

$$\lim_{x \to +\infty} \frac{\alpha A(1, x) + (1 - \alpha)H(1, x)}{P(1, x)} = \frac{\pi}{2}\alpha > 1. \tag{2.35}$$

Inequality (2.35) implies that for any $\alpha > 2/\pi$ there exists $X = X(\alpha) > 1$ such that $\alpha A(1, x) + (1 - \alpha)H(1, x) > P(1, x)$ for $x \in (X, +\infty)$. □

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