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Research Article

Hyers-Ulam Stability of a Bi-Jensen Functional Equation on a Punctured Domain

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We obtain the Hyers-Ulam stability of a bi-Jensen functional equation: $2f((x+y)/2,z) - f(x,z) - f(y,z) = 0$ and simultaneously $2f(x,(y+z)/2) - f(x,y) - f(x,z) = 0$. And we get its stability on the punctured domain.

1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of homomorphisms: let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta \tag{1.1}$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon \tag{1.2}$$

for all $x \in G_1$? The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1949, 1950, and 1978, Bourgin [3], Aoki [4], and Rassias [5] gave a generalization of it under the conditions bounded by variables. Since then, the further generalization has been extensively investigated by a number of mathematicians, such as Găvruta, Rassias, and so forth, [6–25].

Throughout this paper, let X be a normed space and Y a Banach space. A mapping $g : X \to Y$ is called a Jensen mapping if g satisfies the functional equation $2g((x+y)/2) = g(x) + g(y)$. For a given mapping $f : X \times X \to Y$, we define

$$J_1 f(x,y,z) := 2f\left(\frac{x+y}{2}, z\right) - f(x,z) - f(y,z),$$
$$J_2 f(x,y,z) := 2f\left(x, \frac{y+z}{2}\right) - f(x,y) - f(x,z) \tag{1.3}$$

for all $x, y, z \in X$. A mapping $f : X \times X \to Y$ is called a bi-Jensen mapping if f satisfies the functional equations $J_1 f = 0$ and $J_2 f = 0$.

In 2006, Bae and Park [26] obtained the generalized Hyers-Ulam stability of a bi-Jensen mapping. The following result is a special case of Theorem 6 in [26].

Theorem A. *Let $\varepsilon > 0$ and let $f : X \times X \to Y$ be a mapping such that*

$$\|J_1 f(x,y,z)\| \le \varepsilon,$$
$$\|J_2 f(x,y,z)\| \le \varepsilon \tag{1.4}$$

for all $x, y, z \in X$. Then there exist two bi-Jensen mappings $F, F_0 : X \times X \to Y$ such that

$$\|f(x,y) - f(0,y) - F(x,y)\| \le \varepsilon,$$
$$\|f(x,y) - f(x,0) - F_0(x,y)\| \le \varepsilon \tag{1.5}$$

for all $x, y \in X$.

In Theorem A, they did not show that there exist a $k \in \mathbb{R}$ and a unique bi-Jensen mapping $F : X \times X \to Y$ such that $\|f(x,y) - F(x,y)\| \le k\varepsilon$ for all $x, y \in X$. In 2008, Jun et al. [7, 8] improved Bae and Park's results.

In Section 2, we show that there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that $\|f(x,y) - F(x,y)\| \le 4\varepsilon$ for all $x, y \in X$. In Section 3, we investigate the Hyers-Ulam stability of a bi-Jensen functional equation on the punctured domain.

2. Stability of a Bi-Jensen Functional Equation

From Lemma 1 in [8], we get the following lemma.

Lemma 2.1. *Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then*

$$f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + \left(\frac{1}{2^n} - \frac{1}{4^n}\right)(f(2^n x, 0) + f(0, 2^n y)) + \left(1 - \frac{1}{2^n}\right)^2 f(0,0) \tag{2.1}$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now we will give the Hyers-Ulam stability for a bi-Jensen mapping.

Theorem 2.2. *Let $\varepsilon > 0$ and let $f : X \times X \to Y$ be a mapping satisfying* (1.4) *for all $x, y, z \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le 4\varepsilon \tag{2.2}$$

for all $x, y \in X$ with $F(0,0) = f(0,0)$. In particular, the mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty}\left[\frac{1}{4^j} f\left(2^j x, 2^j y\right) + \left(\frac{1}{2^j} - \frac{1}{4^j}\right)\left(f\left(2^j x, 0\right) + f\left(0, 2^j y\right)\right)\right] + f(0,0) \tag{2.3}$$

for all $x, y \in X$.

Proof. Let f_j be the map defined by

$$f_j(x,y) = \frac{f(2^j x, 2^j y)}{4^j} + \left(\frac{1}{2^j} - \frac{1}{4^j}\right)\left(f\left(2^j x, 0\right) + f\left(0, 2^j y\right)\right) + \left(1 - \frac{1}{2^{j-1}} + \frac{1}{4^j}\right) f(0,0) \tag{2.4}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. By (1.4), we get

$$\begin{aligned}
\|f_j(x,y) - f_{j+1}(x,y)\| &= \left\| \frac{J_1 f(2^{j+1}x, 0, 0)}{2^{j+1}} + \frac{J_1 f(2^{j+1}x, 0, 2^{j+1}y)}{2 \cdot 4^{j+1}} + \frac{J_1 f(2^{j+1}x, 0, 2^j y)}{4^{j+1}} \right. \\
&\quad - \frac{3 J_1 f(2^{j+1}x, 0, 0)}{2 \cdot 4^{j+1}} + \frac{J_2 f(0, 0, 2^{j+1}y)}{2^{j+1}} + \frac{J_2 f(2^{j+1}x, 0, 2^{j+1}y)}{2 \cdot 4^{j+1}} \\
&\quad \left. + \frac{J_2 f(2^j x, 0, 2^{j+1}y)}{4^{j+1}} - \frac{3 J_2 f(0, 0, 2^{j+1}y)}{2 \cdot 4^{j+1}} \right\| \\
&\le \left(\frac{1}{2^j} + \frac{3}{2 \cdot 4^j}\right)\varepsilon
\end{aligned} \tag{2.5}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. For given integers l, m with $0 \le l < m$, we obtain

$$\|f_l(x,y) - f_m(x,y)\| \le \sum_{j=l}^{m-1}\left(\frac{1}{2^j} + \frac{3}{2 \cdot 4^j}\right)\varepsilon \tag{2.6}$$

for all $x, y \in X$. By the above inequality, the sequence $\{f_j(x,y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{f_j(x,y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x,y) := \lim_{j\to\infty} f_j(x,y) \tag{2.7}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.6), we obtain the inequality

$$\|f(x,y) - F(x,y)\| \le 4\varepsilon \tag{2.8}$$

for all $x, y \in X$. By (1.4) and the definition of F, we get

$$\begin{aligned} J_1F(x,y,z) &= \lim_{j\to\infty} \frac{1}{4^j} J_1 f\left(2^j x, 2^j y\right) + \left(\frac{1}{2^j} - \frac{1}{4^j}\right)\left(J_1 f\left(2^j x, 0\right) + J_1 f\left(0, 2^j y\right)\right) = 0, \\ J_2F_3(x,y,z) &= \lim_{j\to\infty} \frac{1}{4^j} J_2 f\left(2^j x, 2^j y\right) + \left(\frac{1}{2^j} - \frac{1}{4^j}\right)\left(J_2 f\left(2^j x, 0\right) + J_2 f\left(0, 2^j y\right)\right) = 0 \end{aligned} \tag{2.9}$$

for all $x, y, z \in X$. So F is a bi-Jensen mapping satisfying (2.2). Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.2) with $F'(0,0) = f(0,0)$. By Lemma 2.1, we have

$$\begin{aligned} &\|F(x,y) - F'(x,y)\| \\ &\quad = \left\| \frac{(F - F')(2^n x, 2^n y)}{4^n} + \left(\frac{1}{2^n} - \frac{1}{4^n}\right)\left((F - F')(2^n x, 0) + (F - F')(0, 2^n y)\right) \right\| \\ &\quad \le \left\| \frac{(F - f)(2^n x, 2^n y)}{4^n} \right\| + \left\| \frac{(F - f)(0, 2^n y)}{2^n} \right\| + \left\| \frac{(F - f)(2^n x, 0)}{2^n} \right\| \\ &\qquad + \left\| \frac{(f - F')(2^n x, 2^n y)}{4^n} \right\| + \left\| \frac{(f - F')(0, 2^n y)}{2^n} \right\| + \left\| \frac{(f - F')(2^n x, 0)}{2^n} \right\| \\ &\quad \le \left(\frac{1}{2^{n-2}} + \frac{1}{4^{n-1}}\right)\varepsilon \end{aligned} \tag{2.10}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus the bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Example 2.3. Let $f, F, F' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the bi-Jensen mappings defined by

$$f(x,y) := 0, \qquad F(x,y) := \varepsilon, \qquad F'(x,y) := -\varepsilon \tag{2.11}$$

for all $x, y \in \mathbb{R}$. Then f, F, F' satisfy (1.4) for all $x, y, z \in \mathbb{R}$. In addition, f, F satisfy (2.2) for all $x, y \in \mathbb{R}$ and f, F' also satisfy (2.2) for all $x, y \in \mathbb{R}$. But we get $F' \ne F$. Hence the condition $F(0,0) = f(0,0)$ is necessary to show that the mapping F is unique.

3. Stability of a Bi-Jensen Functional Equation on the Punctured Domain

Let A be a subset of X. $X \setminus A$ and $(X \times X) \setminus (A \times A)$ are punctured domain on the spaces X and $(X \times X)$, respectively.

Throughout this paper, for a given mapping $f : X\times X \to Y$, let $f_1, A_1, A_2 : X\times X \to Y$ be the mappings defined by

$$f_1(x,y) := \frac{f(x,y) - f(-x,y) - f(x,-y) + f(-x,-y)}{4},$$
$$A_1(x,y) := \sum_{m=0}^{1}\sum_{n=0}^{1}(-1)^{m+1} J_1 f((-1)^m x, (-1)^n \cdot 3x, y), \tag{3.1}$$
$$A_2(x,y) := \sum_{m=0}^{1}\sum_{n=0}^{1}(-1)^{n+1} J_2 f(x, (-1)^m \cdot 3y, (-1)^n y)$$

for all $x, y \in X$.

Lemma 3.1. *Let A be a subset of X satisfying the following condition: for every $x \neq 0$, there exists a positive integer n_x such that $kx \notin A$ for all integer k with $|k| \geq n_x$, and such that $kx \in A$ for all integer k with $|k| < n_x$. Let $f : X \times X \to Y$ be a mapping such that*

$$J_1 f(x,y,z) = 0, \qquad J_2 f(x,y,z) = 0 \tag{3.2}$$

for all $x, y, z \in X \setminus A$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$F(x,y) = f(x,y) \tag{3.3}$$

for all $x, y \in X \setminus A$. Moreover, the equality

$$F(x,y) = f(x,y) \tag{3.4}$$

holds for all $(x,y) \in (X \times X) \setminus (A \times A)$.

Proof. Note that $J_1 f(x,y,z) = 0$, $J_2 f(x,y,z) = 0$, $A_1(x,y) = 0$, and $A_2(x,y) = 0$ for all $x, y \in X \setminus A$. Let $((f(0,y) + f(0,-y))/2) = c \in Y$ for any $y \in X \setminus A$. From (3.2), we get the equality

$$f(0,y) + f(0,-y) = f(x,0) + f(-x,0) + \frac{1}{2}(J_1 f(x,-x,y) + J_1 f(x,-x,-y) - J_2 f(x,y,-y) - J_2 f(-x,y,-y)) \tag{3.5}$$

for all $x, y \in X \setminus A$, and we know that the equality

$$\frac{f(0,y) + f(0,-y)}{2} = \frac{f(x,0) + f(-x,0)}{2} = c \tag{3.6}$$

holds for all $x, y \in X \setminus A$. From (3.2), we have

$$
\begin{aligned}
f_1(x,y) &= \frac{f_1(2x,y)}{2} + \frac{A_1(x,y)}{16} - \frac{A_1(x,-y)}{16},\\
\frac{f_1(2x,y)}{2} &= \frac{f_1(2x,2y)}{4} + \frac{A_2(2x,y)}{32} - \frac{A_2(-2x,y)}{32},\\
f(x,y) &= \frac{f(0,y)-f(0,-y)}{2} + \frac{f(0,y)+f(0,-y)}{2} + \frac{f(x,0)-f(-x,0)}{2}\\
&\quad + f_1(x,y) - \frac{1}{4}\left(2J_1f(x,-x,y) + J_2f(x,y,-y) - J_2f(-x,y,-y)\right),\\
f(x,0)-f(-x,0) &= \frac{f(2x,0)-f(-2x,0)}{2}\\
&\quad + \frac{1}{8}\left(4J_2f(x,y,-y) - 4J_1f(-x,y,-y) - 2J_1f(2x,y,-y)\right.\\
&\quad \left. +2J_1f(-2x,y,-y) + A_1(x,y) + A_1(x,-y)\right),\\
f(0,y)-f(0,-y) &= \frac{f(0,2y)-f(0,-2y)}{2}\\
&\quad + \frac{1}{8}\left(4J_1f(x,-x,y) - 4J_1f(x,-x,-y) - 2J_1f(x,-x,2y)\right.\\
&\quad \left. +2J_1f(x,-x,-2y) + A_2(x,y) + A_2(-x,y)\right)
\end{aligned}
\tag{3.7}
$$

for all $x, y \in X \setminus A$. From the above equalities, we obtain the equalities

$$f_1(x,y) = \frac{f_1(2x,y)}{2}, \tag{3.8}$$

$$f(x,0)-f(-x,0) = \frac{f(2x,0)-f(-2x,0)}{2}, \tag{3.9}$$

$$f(0,y)-f(0,-y) = \frac{f(0,2y)-f(0,-2y)}{2}, \tag{3.10}$$

$$
\begin{aligned}
f_1(x,y) &= \frac{f_1(2^nx,2^ny)}{4^n},\\
f(x,0) &= \frac{f(x,0)-f(-x,0)}{2} + \frac{f(x,0)+f(-x,0)}{2} = \frac{f(2^nx,0)-f(-2^nx,0)}{2^{n+1}} + c,\\
f(0,y) &= \frac{f(0,y)-f(0,-y)}{2} + \frac{f(0,y)+f(0,-y)}{2} = \frac{f(0,2^ny)-f(0,-2^ny)}{2^{n+1}} + c,\\
f(x,y) &= \frac{f_1(2^nx,2^ny)}{4^n} + \frac{f(0,2^ny)-f(0,-2^ny)+f(2^nx,0)-f(-2^nx,0)}{2^{n+1}} + c
\end{aligned}
\tag{3.11}
$$

for all $x, y \in X \setminus A$ and $n \in \mathbb{N}$.

Let A_x be the set defined by $A_x = \{n \in \mathbb{N} \mid nx \notin A\}$ for each $x \neq 0$. From the above equalities, we can define $F : X \times X \to Y$ by

$$F(x,y) := \begin{cases} \dfrac{f_1(2^k x, 2^k y)}{4^k} + \dfrac{f(0,2^k y) - f(0,-2^k y) + f(2^k x,0) - f(-2^k x,0)}{2^{k+1}} + c, \\ \qquad\qquad \text{for some } 2^k \in A_x \cap A_y \quad \text{if } x,\ y \neq 0, \\ \dfrac{f(2^k x,0) - f(-2^k x,0)}{2^{k+1}} + c, \quad \text{for some } 2^k \in A_x \quad \text{if } x \neq 0,\ y = 0, \\ \dfrac{f(0,2^k y) - f(0,-2^k y)}{2^{k+1}} + c \quad \text{for some } 2^k \in A_y \quad \text{if } x = 0,\ y \neq 0, \\ c \qquad\qquad \text{if } x, y = 0. \end{cases} \tag{3.12}$$

From the definition of F, we get the equalities

$$F(x,y) = f(x,y), \qquad F(0,y) = f(0,y), \qquad F(x,0) = f(x,0) \tag{3.13}$$

for all $x, y \in X \setminus A$. By (3.10), we get the equality

$$f(x,y) - F(x,y) = \frac{1}{2}\left[J_2 f\left(x, \left(2^k + 2\right)y, -2^k y\right) - J_2 F\left(x, \left(2^k + 2\right)y, -2^k y\right)\right] = 0 \tag{3.14}$$

for all $x \in X \setminus A$ and $y \neq 0$, where $2^k \in A_y$. And also we get the equality

$$f(x,y) - F(x,y) = \frac{1}{2}\left[J_1 f\left(\left(2^k + 2\right)x, -2^k x, y\right) - J_1 F\left(\left(2^k + 2\right)x, -2^k x, y\right)\right] = 0 \tag{3.15}$$

for all $x \neq 0$ and $y \in X \setminus A$, where $2^k \in A_x$. Hence the equality

$$f(x,y) = F(x,y) \tag{3.16}$$

holds for all $(x,y) \in (X \times X) \setminus (A \times A)$. From (3.8), (3.9), (3.10), and the definition of F, we easily get

$$\begin{aligned} &J_1 F(x,-x,y) = 0, \qquad J_1 F(x,-x,0) = 0, \qquad J_1 F(0,0,y) = 0, \\ &J_1 F(x,0,y) = 0, \qquad J_1 F(x,0,0) = 0, \qquad J_1 F(0,0,0) = 0 \end{aligned} \tag{3.17}$$

for all $x, y \neq 0$. And we obtain

$$J_1F(x,y,0) = \frac{J_2f(2^{k-1}(x+y),z,-z) - J_2f(-2^{k-1}(x+y),z,-z)}{2^{\mathrm{k}+1}} + \frac{-J_2f(2^kx,z,-z) + J_2f(-2^xx,z,-z)}{2^{k+2}} + \frac{-J_2f(2^ky,z,-z)}{2^{k+2}} + \frac{J_2f(-2^ky,z,-z) + J_1f(2^kx,2^ky,z) - J_1f(-2^kx,-2^ky,z)}{2^{k+2}} + \frac{J_1f(2^kx,2^ky,-z) - J_1f(-2^kx,-2^ky,-z)}{2^{k+2}} = 0 \tag{3.18}$$

for all $x, y \neq 0$ with $x + y \neq 0$, where $2^k \in A_x \cap A_y \cap A_{x+y}$ and $z \notin A$. From this, we have

$$J_1F(x,y,z) = \frac{J_1f(2^kx,2^ky,2^kz) - J_1f(-2^kx,-2^ky,2^kz)}{4^{k+1}} + \frac{J_1f(-2^kx,-2^ky,-2^kz) - J_1f(2^kx,2^ky,-2^kz)}{4^{k+1}} + J_1F(x,y,0) = 0 \tag{3.19}$$

for all $x, y, z \neq 0$ with $x + y \neq 0$, where $2^k \in A_x \cap A_y \cap A_z$. From the above equalities, we get

$$J_1F(x,y,z) = 0 \tag{3.20}$$

for all $x, y, z \in X$. By the similar method, we have

$$J_2F(x,y,z) = 0 \tag{3.21}$$

for all $x, y, z \in X$. Hence F is a bi-Jensen mapping. Let F' be another bi-Jensen mapping satisfying

$$F'(x,y) = f(x,y) = F(x,y) \tag{3.22}$$

for all $(x, y) \in (X \times X) \setminus (A \times A)$. Using the above equality, we show that the equalities

$$\begin{aligned} F'(x,y) - F(x,y) &= \frac{1}{2}\left(J_1F'((k+2)x,-kx,y) - J_1F((k+2)x,-kx,y)\right) = 0, \\ F'(0,y) - F(0,y) &= \frac{1}{2}\left(J_1F'(kx,-kx,y) - J_1F(kx,-kx,y)\right) = 0 \end{aligned} \tag{3.23}$$

hold for all $x \neq 0$ and $y \in X$ as we desired, where $k \in A_x$. □

Corollary 3.2. *Let $f : X \times X \to Y$ be a mapping such that*

$$J_1 f(x,y,z) = 0, \qquad J_2 f(x,y,z) = 0 \tag{3.24}$$

for all $x,y,z \in X \setminus \{0\}$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$F(x,y) = f(x,y) \tag{3.25}$$

for all $(x,y) \neq (0,0)$.

Example 3.3. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the mapping defined by

$$f(x,y) := \begin{cases} \{(x+3)(y+4) & \text{for} \quad (x,y) \neq (0,0), \\ 1 & \text{for} \quad (x,y) = (0,0), \end{cases} \tag{3.26}$$

and let F be the mapping defined by $F(x,y) := (x+3)(y+4)$ for all $x,y \in \mathbb{R}$. Then the mappings f, F satisfy the conditions of Corollary 3.2 with $f(0,0) \neq F(0,0)$.

Now, we prove the Hyers-Ulam stability of a bi-Jensen functional equation on the punctured domain $X \setminus A$.

Theorem 3.4. *Let $\varepsilon > 0$ and $x_0 \in X \setminus A$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|J_1 f(x,y,z)\| \leq \varepsilon, \qquad \|J_2 f(x,y,z)\| \leq \varepsilon \tag{3.27}$$

for all $x,y,z \in X \setminus A$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \leq \frac{17}{2}\varepsilon \tag{3.28}$$

holds for all $(x,y) \in (X \times X) \setminus (A \times A)$ with $F(0,0) = (f(x_0,0) + f(-x_0,0))/2$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty}\left(\frac{f_1(2^j x, 2^j y)}{4^j} + \frac{f(0,2^j y) + f(2^j x, 0)}{2^{j+1}}\right) + \frac{f(x_0,0) + f(-x_0,0)}{2} \tag{3.29}$$

for all $x,y \in X$.

Proof. By (3.27), we get

$$\begin{aligned}
&\left\| \frac{f_1(2^j x, 2^j y)}{4^j} - \frac{f_1(2^{j+1} x, 2^{j+1} y)}{4^{j+1}} \right\| \\
&\quad = \frac{1}{4^{j+2}} \left\| A_1\left(2^j x, 2^j y\right) - A_1\left(2^j x, -2^j y\right) + \frac{1}{2} A_2\left(2^{j+1} x, 2^j y\right) - \frac{1}{2} A_2\left(-2^{j+1} x, 2^j y\right) \right\| \le \frac{3\varepsilon}{4^{j+1}}, \\
&\left\| \frac{f(0, 2^j y) - f(0, -2^j y)}{2^{j+1}} - \frac{f(0, 2^{j+1} y) - f(0, -2^{j+1} y)}{2^{j+2}} \right\| \\
&\quad = \frac{1}{2^{j+4}} \Big\| +4 J_1 f\left(x, -x, 2^j y\right) - 4 J_1 f\left(x, -x, -2^j y\right) - 2 J_1 f\left(x, -x, 2^{j+1} y\right) \\
&\qquad + 2 J_1 f\left(x, -x, -2^{j+1} y\right) + A_2\left(x, 2^j y\right) + A_2\left(-x, 2^j y\right) \Big\| \le \frac{5\varepsilon}{2^{j+2}}, \\
&\left\| \frac{f(0, y) + f(0, -y)}{2} - \frac{f(x, 0) + f(-x, 0)}{2} \right\| \\
&\quad = \frac{1}{4} \left\| J_1 f(x, -x, y) + J_1 f(x, -x, -y) - J_2 f(x, y, -y) - J_2 f(-x, y, -y) \right\| \le \varepsilon
\end{aligned} \tag{3.30}$$

for all $x, y \in X \setminus A$ and $j \in \mathbb{N}$. For given integers l, m $(0 \le l < m)$, we have

$$\left\| \frac{f_1(2^l x, 2^l y)}{4^l} - \frac{f_1(2^m x, 2^m y)}{4^m} \right\| \le \sum_{j=l}^{m-1} \frac{3\varepsilon}{4^{j+1}}, \tag{3.31}$$

$$\left\| \frac{f(0, 2^l y) - f(0, -2^l y)}{2^{l+1}} - \frac{f(0, 2^m y) - f(0, -2^m y)}{2^{m+1}} \right\| \le \sum_{j=l}^{m-1} \frac{5\varepsilon}{2^{j+2}}, \tag{3.32}$$

$$\left\| \frac{f(2^l x, 0) - f(-2^l x, 0)}{2^{l+1}} - \frac{f(2^m x, 0) - f(-2^m x, 0)}{2^{m+1}} \right\| \le \sum_{j=l}^{m-1} \frac{5\varepsilon}{2^{j+2}}, \tag{3.33}$$

$$\left\| \frac{f(x, 0) + f(-x, 0)}{2} - \frac{f(0, 2^m y) + f(0, -2^m y)}{2} \right\| \le \varepsilon, \tag{3.34}$$

$$\left\| \frac{f(0, y) + f(0, -y)}{2} - \frac{f(2^m x, 0) + f(-2^m x, 0)}{2} \right\| \le \varepsilon \tag{3.35}$$

for all $x, y \in X \setminus A$. The sequences $\{(f_1(2^j x, 2^j y))/4^j\}$, $\{(f(0, 2^j y) - f(0, -2^j y))/2^{j+1}\}$, and $\{(f(2^j x, 0) - f(-2^j x, 0))/2^{j+1}\}$ are Cauchy sequences for all $x, y \in X \setminus A$. Since Y is complete, the above sequences converge for all $x, y \in X \setminus A$. From (3.34) and (3.35), we have

$$\lim_{j \to \infty} \frac{f(0, 2^j y) + f(0, -2^j y)}{2^{j+1}} = \lim_{j \to \infty} \frac{f(2^j x, 0) + f(-2^j x, 0)}{2^{j+1}} = 0 \tag{3.36}$$

for all $x, y \in X$. Using the inequalities (3.31)–(3.35) and the above equality, we can define the mappings $F_1, F_2, F_3 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} \frac{f_1(2^j x, 2^j y)}{4^j},$$
$$F_2(x,y) := \lim_{j\to\infty} \frac{f(0,2^j y)}{2^j} = \lim_{j\to\infty} \frac{f(0,2^j \mathrm{y}) - f(0,-2^j y)}{2^{j+1}},$$
$$F_3(x,y) := \lim_{j\to\infty} \frac{f(2^j x,0)}{2^j} = \lim_{j\to\infty} \frac{f(2^j x,0) - f(-2^j x,0)}{2^{j+1}} \tag{3.37}$$

for all $x, y \in X$. By (3.27) and the definition of F_1, we obtain

$$J_1F_1(x,y,z) = \lim_{j\to\infty}\left[\frac{J_1 f(2^j x, 2^j y, 2^j z) - J_1 f(-2^j x, -2^j y, 2^j z)}{4^{j+1}} - \frac{J_1 f(2^j \mathrm{x}, 2^j y, -2^j z) - J_1 f(-2^j x, -2^j y, -2^j z)}{4^{j+1}}\right] = 0,$$
$$J_2F_1(x,y,z) = \lim_{j\to\infty}\left[\frac{J_2 f(2^j x, 2^j y, 2^j z) - J_2 f(-2^j x, 2^j y, 2^j z)}{4^{j+1}} - \frac{J_2 f(2^j x, -2^j y, -2^j z) - J_2 f(-2^j x, -2^j y, -2^j z)}{4^{j+1}}\right] = 0 \tag{3.38}$$

for all $x, y, z \neq 0$. Since $J_2F_2(x, y, -y) = 0$ and

$$J_2F_2(x,y,z) = \lim_{j\to\infty}\left(\frac{J_1 f(w,-w,2^{j-1}(y+z))}{2^j} - \frac{J_1 f(w,-w,2^j y)}{2^{j+1}} - \frac{J_1 f(w,-w,2^j z)}{2^{j+1}} + \frac{J_2 f(w,2^j y,2^j z)}{2^{j+1}} + \frac{J_2 f(-w,2^j y,2^j z)}{2^{j+1}}\right) = 0 \tag{3.39}$$

for all $x, y, z \neq 0$ with $y + z \neq 0$, where $w \notin A$, we have

$$J_1F_2(x,y,z) = 0, \qquad J_2F_2(x,y,z) = 0 \tag{3.40}$$

for all $x, y, z \neq 0$. Similarly, the equalities

$$J_1F_3(x,y,z) = 0, \qquad J_2F_3(x,y,z) = 0 \tag{3.41}$$

hold for all $x, y, z \neq 0$. By Lemma 3.1, There exist bi-Jensen mappings $F_1', F_2', F_3' : X \times X \to Y$ such that

$$F_1'(x,y) = F_1(x,y), \qquad F_2'(x,y) = F_2(x,y), \qquad F_3'(x,y) = F_3(x,y) \tag{3.42}$$

for all $(x, y) \neq (0,0)$. Since the equalities

$$\begin{aligned} F_1'(0,0) &= \frac{F_1'(x,0) + F_1'(-x,0)}{2} = \frac{F_1(x,0) + F_1(-x,0)}{2} = F_1(0,0), \\ F_2'(0,0) &= \frac{F_2'(x,0) + F_2'(-x,0)}{2} = \frac{F_2(x,0) + F_2(-x,0)}{2} = F_2(0,0), \\ F_3'(0,0) &= \frac{F_3'(x,0) + F_3'(-x,0)}{2} = \frac{F_3(x,0) + F_3(-x,0)}{2} = F_3(0,0) \end{aligned} \tag{3.43}$$

hold, F_1, F_2, F_3 are bi-Jensen mappings. Putting $l = 0$ and taking $m \to \infty$ in (3.31), (3.32), and (3.33), one can obtain the inequalities

$$\begin{gathered} \|f_1(x,y) - F_1(x,y)\| \leq \varepsilon, \qquad \left\| \frac{1}{2}(f(0,y) - f(0,-y)) - F_2(x,y) \right\| \leq \frac{5\varepsilon}{2}, \\ \left\| \frac{1}{2}(f(x,0) - f(-x,0)) - F_3(x,y) \right\| \leq \frac{5\varepsilon}{2} \end{gathered} \tag{3.44}$$

for all $x, y \in X \setminus A$. By (3.30) and the above equalities, we get

$$\begin{aligned} \|f(x,y) - F(x,y)\| &\leq \left\| f(x,y) - f_1(x,y) - f(0,y) - \frac{f(x,0) - f(-x,0)}{2} \right\| \\ &\quad + \left\| \frac{f(0,y) + f(0,-y)}{2} - \frac{f(x_0,0) + f(-x_0,0)}{2} \right\| + \|f_1(x,y) - F_1(x,y)\| \\ &\quad + \left\| \frac{f(0,y) - f(0,-y)}{2} - F_2(x,y) \right\| + \left\| \frac{f(x,0) - f(-x,0)}{2} - F_3(x,y) \right\| \\ &\leq \left\| -\frac{1}{2} J_1 f(x,-x,y) - \frac{1}{4} J_2 f(x,y,-y) + \frac{1}{4} J_2 f(-x,y,-y) \right\| + 7\varepsilon \\ &\leq 8\varepsilon \end{aligned} \tag{3.45}$$

for all $x, y \in X \setminus A$, where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + \frac{f(x_0,0) + f(-x_0,0)}{2} \tag{3.46}$$

and $z \notin A$. By (3.45), we get the inequalities

$$\begin{aligned}\|f(x,y)-F(x,y)\| &= \frac{1}{2}\|J_1 f((k+2)x,-kx,y)+f((k+2)x,y)-F((k+2)x,y)\\ &\qquad +f(-kx,y)-F(-kx,y)\| \le \frac{17}{2}\varepsilon,\\ \|f(0,y)-F(0,y)\| &= \frac{1}{2}\|J_1 f(kx,-kx,y)+f(kx,y)-F(kx,y)+f(-kx,y)-F(-kx,y)\|\\ &\le \frac{17}{2}\varepsilon\end{aligned} \tag{3.47}$$

for all $x \neq 0$ and $y \notin A$, where $k \in A_x$, and the inequalities

$$\begin{aligned}\|f(x,y)-F(x,y)\| &\le \frac{17}{2}\varepsilon,\\ \|f(x,0)-F(x,0)\| &\le \frac{17}{2}\varepsilon\end{aligned} \tag{3.48}$$

for all $y \neq 0$ and $x \notin A$. Hence F is a bi-Jensen mapping satisfying (3.28).

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (3.28) with $F'(0,0) = F(0,0)$. By Lemma 2.1, we have

$$\begin{aligned}&\|F(x,y)-F'(x,y)\|\\ &\quad\le \left\|\frac{1}{4^n}(F-f)(2^n x,2^n y)+\left(\frac{1}{2^n}-\frac{1}{4^n}\right)((F-f)(2^n x,0)+(F-f)(0,2^n y))\right\|\\ &\qquad+\left\|\frac{1}{4^n}(f-F')(2^n x,2^n y)+\left(\frac{1}{2^n}-\frac{1}{4^n}\right)((f-F')(2^n x,0)+(f-F')(0,2^n y))\right\| \le \frac{17\varepsilon}{2^{n-1}}\end{aligned} \tag{3.49}$$

for all $x, y \in X \setminus A$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X \setminus A$. By Lemma 3.1, $F = F'$ as we desired. □

Example 3.5. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the mapping defined by

$$f(x,y) := \begin{cases}\dfrac{\varepsilon}{2} & \text{if } (x,y) = (0,0),\\ 0 & \text{if } (x,y) \neq (0,0).\end{cases} \tag{3.50}$$

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the mapping defined by $F(x,y) = 1$ for all $x, y \in X$. Then f satisfies the conditions in Theorem 3.4, and F is a bi-Jensen mapping satisfying (3.28) but $F(0,0) \neq f(0,0)$.

Corollary 3.6. *Let $f : X \times X \to Y$ be a mapping satisfying* (3.13) *and* (3.27) *for all $x, y, z \in X \setminus \{0\}$. Then there exists a bi-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le 8\varepsilon \tag{3.51}$$

for all $(x, y) \ne (0, 0)$.

Proof. Let F, F_2, F_3 be as in the proof of Theorem 3.4. By (3.30), we obtain

$$\begin{aligned}
\|f(0,y) - F(0,y)\| &\le \left\| \frac{f(0,y) + f(0,-y)}{2} - \frac{f(x_0,0) + f(-x_0,0)}{2} \right\| \\
&\quad + \left\| \frac{f(0,y) - f(0,-y)}{2} - F_2(x,y) \right\| \le \frac{7\varepsilon}{2}, \\
\|f(x,0) - F(x,0)\| &\le \left\| \frac{f(x,0) + f(-x,0)}{2} - \frac{f(0,y) + f(0,-y)}{2} \right\| \\
&\quad + \left\| \frac{f(0,y) + f(0,-y)}{2} - \frac{f(x_0,0) + f(-x_0,0)}{2} \right\| \\
&\quad + \left\| \frac{f(x,0) - f(-x,0)}{2} - F_3(x,y) \right\| \le \frac{9\varepsilon}{2}
\end{aligned} \tag{3.52}$$

for $x, y \ne 0$. From the above inequalities and (3.45), we get the inequality

$$\|f(x,y) - F(x,y)\| \le 8\varepsilon \tag{3.53}$$

for all $(x, y) \ne (0, 0)$. □

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