Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 482726, 6 pages
doi:10.1155/2010/482726

Research Article

Upper Semicontinuity of Solution Maps for a Parametric Weak Vector Variational Inequality

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Received 7 March 2010; Accepted 9 July 2010

Academic Editor: Nikolaos Papageorgiou

This paper investigates the upper semicontinuity of the solution map for a parametric weak vector variational inequality associated to a v-hemicontinuous and weakly C-pseudomonotone operator.

1. Introduction and Preliminaries

A vector variational inequality (VVI, in short) was first introduced by Giannessi [1] in the setting of finite-dimensional Euclidean space. Later on, it was studied and generalized to infinite-dimensional spaces. Existences of the solutions for VVI have been studied extensively in various versions; see [2–6] and references therein.

Stability of the solution map for VVI or vector equilibrium problems is an important topic in optimization theory. A great deal of papers have been denoted to study the semicontinuity and continuity of the solution maps; see [7–18] and references therein. All results of the stability of the solution map for VVI in the literature are obtained based on continuity of the operator. It is well known that v-hemicontinuity is weaker than continuity. In this paper, our aim is to investigate the upper semicontinuity of the solution map for a parametric weak vector variational inequality associated to a v-hemicontinuous and weakly C-pseudomonotone operator.

Let X, Y, and W (the spaces of parameters) be Banach spaces and let $C \subset Y$ be a pointed closed and convex cone with nonempty interior $\operatorname{int} C$. Let $L(X, Y)$ be the space of all linear continuous operators from X to Y. The value of a linear operator $t \in L(X, Y)$ at $x \in X$ is denoted by $\langle t, x\rangle$. Consider the following weak vector variational inequality problem:

$$\text{find } x \in K \text{ such that } \langle T(x), y - x\rangle \notin -\operatorname{int} C, \quad \forall y \in K, \tag{WVVI}$$

where $K \subset X$ is a nonempty subset and $T : X \to L(X, Y)$ is a vector-valued mapping.

When T is perturbed by a parameter μ, which varies over a nonempty set $\Lambda \subset W$, for a given μ, we can define the parametric weak vector variational inequality problem

$$\text{find } x \in K \text{ such that } \langle T(x,\mu), y - x\rangle \notin -\operatorname{int} C, \quad \forall y \in K, \tag{PWVVI}$$

where $K \subset X$ is a nonempty subset and $T : X \times \Lambda \to L(X,Y)$ is a vector-valued mapping.

For each $\mu \in \Lambda$, we denote the solution map of (PWVVI) by $S(\mu)$, that is,

$$S(\mu) = \{x \in K \mid \langle T(x,\mu), y - x\rangle \notin -\operatorname{int} C, \ \ \forall y \in K\}. \tag{1.1}$$

Throughout the paper, we always assume that $S(\mu)$ is nonempty for all μ in a neighborhood of $\overline{\mu} \in \Lambda$. Now, we recall some basic definitions and their properties.

Definition 1.1 (see [4, 6]). Let K be a nonempty convex subset of X and let $T : K \to L(X,Y)$ be an operator. T is said to be v-hemicontinuous if and only if, for every x, $y \in K$ and $t \in [0,1]$, the mapping $t \to \langle T(ty + (1-t)x), y - x\rangle$ is continuous at 0^+.

Definition 1.2 (see [6]). Let K be a nonempty subset of X and $T : K \to L(X,Y)$ be an operator. T is weakly C-pseudomonotone on K if, for every pair of points $x \in K$, $y \in K$, one has that $\langle Tx, y - x\rangle \notin -\operatorname{int} C$ implies that $\langle T(y), y - x\rangle \notin -\operatorname{int} C$.

Proposition 1.3 (see [6, Generalized Linearization Lemma]). *Let K be a nonempty convex subset of X and let $T : K \to L(X,Y)$ be an operator. Consider the following problems:*

(I) *$x \in K$ such that $\langle Tx, y - x\rangle \notin -\operatorname{int} C$ for all $y \in K$,*

(II) *$x \in K$ such that $\langle Ty, y - x\rangle \notin -\operatorname{int} C$ for all $y \in K$.*

Then the following are obtained.

(i) *Problem (I) implies Problem (II) if T is weakly C-pseudomonotone.*

(ii) *Problem (II) implies Problem (I) if T is v-hemicontinuous.*

Let $F : \Lambda \to 2^X$ be a set-valued mapping, given that $\overline{\lambda} \in \Lambda$.

Definition 1.4 (see [19, 20]). (i) F is called lower semicontinuous (l.s.c) at $\overline{\lambda}$ if, for any open set V satisfying $V \cap F(\overline{\lambda}) \neq \emptyset$, there exists $\delta > 0$ such that for every $\lambda \in B(\overline{\lambda}, \delta)$, $V \cap F(\lambda) \neq \emptyset$.

(ii) F is called upper semicontinuous (u.s.c) at $\overline{\lambda}$ if, for any open set V satisfying $F(\overline{\lambda}) \subset V$, there exists $\delta > 0$ such that, for every $\lambda \in B(\overline{\lambda}, \delta)$, $F(\lambda) \subset V$.

We say F is l.s.c (resp., u.s.c) on Λ, if it is l.s.c (resp., u.s.c) at each $\lambda \in \Lambda$. F is said to be continuous on Λ if it is both l.s.c and u.s.c on Λ.

Proposition 1.5 (see [19, 21]). (i) *F is l.s.c at $\overline{\lambda}$ if and only if, for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \overline{\lambda}$ and any $\overline{x} \in F(\overline{\lambda})$, there exists $x_n \in F(\lambda_n)$ such that $x_n \to \overline{x}$.*

(ii) *If F has compact values (i.e., $F(\lambda)$ is a compact set for each $\lambda \in \Lambda$), then F is u.s.c at $\overline{\lambda}$ if and only if, for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \overline{\lambda}$ and for any $x_n \in F(\lambda_n)$, there exist $\overline{x} \in F(\overline{\lambda})$ and a subsequence $\{x_{n_k}\}$ of x_n such that $x_{n_k} \to \overline{x}$.*

2. Main Results

In this section, we mainly discuss the upper semicontinuity of the solution map for (PWVVI).

Lemma 2.1. *Let K be a nonempty compact convex subset of X. Suppose that, for any $\mu \in \Lambda$, $T(\cdot,\mu)$ is v-hemicontinuous and weakly C-pseudomonotone on K. Then, $S(\cdot)$ has compact values on Λ, that is, $S(\mu)$ is a compact set for each $\mu \in \Lambda$.*

Proof. For any $\mu \in \Lambda$, take any sequence $x_n \in S(\mu)$ with $x_n \to x$; we have

$$\langle T(x_n,\mu), y - x_n\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.1}$$

By Proposition 1.3 and the weakly C-pseudomonotonicity of $T(\cdot,\mu)$, we get

$$\langle T(y,\mu), y - x_n\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.2}$$

From $T(y,\mu) \in L(X,Y)$, we have $\langle T(y,\mu), y - x_n\rangle \to \langle T(y,\mu), y - x\rangle$ as $n \to \infty$. It follows from the closedness of $Y \setminus -\operatorname{int} C$ and (2.2) that

$$\langle T(y,\mu), y - x\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.3}$$

Moreover, by Proposition 1.3 and the v-hemicontinuity of $T(\cdot,\mu)$, we have

$$\langle T(x,\mu), y - x\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.4}$$

That is $x \in S(\mu)$. Thus, $S(\mu)$ is a closed set. Furthermore, it follows from $S(\mu) \subset K$ and the compactness of K that $S(\mu)$ is a compact set. The proof is complete. □

Theorem 2.2. *Let K be a nonempty compact convex subset of X. Suppose that the following conditions are satisfied.*

(i) *For any $\mu \in \Lambda$, $T(\cdot,\mu)$ is v-hemicontinuous on K,*

(ii) *For any $\mu \in \Lambda$, $T(\cdot,\mu)$ is weakly C-pseudomonotone on K,*

(iii) *For any $x \in X$, $T(x,\cdot)$ is continuous on Λ.*

Then, $S(\cdot)$ is u.s.c on Λ.

Proof. For any $\mu_0 \in \Lambda$, any sequences $\{\mu_n\} \subset \Lambda$ with $\mu_n \to \mu_0$, and $x_n \in S(\mu_n)$, we have $x_n \in K$ and

$$\langle T(x_n,\mu_n), y - x_n\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.5}$$

Since K is a compact set, there are an $x_0 \in K$ and a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x_0$. Particularly, from (2.5), we get

$$\langle T(x_{n_k},\mu_{n_k}), y - x_{n_k}\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.6}$$

By Proposition 1.3 and (iii), we can obtain that

$$\langle T(y,\mu_{n_k}), y - x_{n_k}\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.7}$$

Since $T(y,\cdot)$ is continuous and

$$\begin{aligned}
&\left\| \langle T(y,\mu_{n_k}), y - x_{n_k}\rangle - \langle T(y,\mu_0), y - x_0\rangle \right\| \\
&\quad \le \left\| \langle T(y,\mu_{n_k}), y - x_{n_k}\rangle - \langle T(y,\mu_0), y - x_{n_k}\rangle \right\| \\
&\qquad + \left\| \langle T(y,\mu_0), y - x_{n_k}\rangle - \langle T(y,\mu_0), y - x_0\rangle \right\| \\
&\quad \le \left\| T(y,\mu_{n_k}) - T(y,\mu_0) \right\| \left\| y - x_{n_k} \right\| + \left\| T(y,\mu_0) \right\| \left\| x_{n_k} - x_0 \right\|,
\end{aligned} \tag{2.8}$$

we get $\langle T(y,\mu_{n_k}), y - x_{n_k}\rangle \to \langle T(y,\mu_0), y - x_0\rangle$, as $n_k \to \infty$. It follows from the closedness of $Y \setminus -\operatorname{int} C$ and (2.7) that

$$\langle T(y,\mu_0), y - x_0\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K. \tag{2.9}$$

Moreover, by Proposition 1.3 and (ii), we have

$$\langle T(x_0,\mu_0), y - x_0\rangle \in Y \setminus -\operatorname{int} C, \quad \forall y \in K, \tag{2.10}$$

that is $x_0 \in S(\mu_0)$.

Thus, for any sequence $\mu_n \subset \Lambda$ with $\mu_n \to \mu_0$ and for any $x_n \in S(\mu_n)$, there exist $x_0 \in S(\mu_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. By Proposition 1.5 and Lemma 2.1, we have $S(\cdot)$ is u.s.c at μ_0. From the arbitrariness of μ_0, we can get $S(\cdot)$ is u.s.c on Λ. The proof is complete. □

Remark 2.3. In [7–10], the upper semicontinuity of the solution map for (PVVI) has been discussed based on the continuity of the operator. Note that v-hemicontinuity is weaker than continuity. Moreover, together with the assumption of weakly C-pseudomonotonicity, v-hemicontinuity may not derive the continuity of the operator. Thus, it is necessary to investigate the upper semicontinuity of the solution map for (PVVI) associated to a v-hemicontinuous and weakly C-pseudomonotone operator. Now we give an example to illustrate our result.

Example 2.4. Let $X = R^2$, $Y = R$, $K = [0,1] \times [0,1]$, $\Lambda = [0,1]$ and

$$T(x,\mu) = \begin{cases} \dfrac{\mu x_1^2 x_2}{x_1^2 + x_2^2}, & \text{if } x \neq (0,0)^\top, \\ 1, & \text{if } x = (0,0)^\top. \end{cases} \tag{2.11}$$

Then,

$$\langle T(x,\mu), y-x\rangle = \begin{cases} \dfrac{\mu x_1^2 x_2}{x_1^2+x_2^2}(y_1-x_1, y_2-x_2)^{\top}, & \text{if } x \neq (0,0)^{\top}, \\ 0, & \text{if } x = (0,0)^{\top}. \end{cases} \tag{2.12}$$

It is clear that conditions (ii) and (iii) of Theorem 2.2 are satisfied. For any ray $x_2 = kx_1 (0 \leq k < \infty)$, $T(\cdot,\mu)$ is continuous. Thus, $T(\cdot,\mu)$ is v-hemicontinuous on K and condition (i) of Theorem 2.2 is satisfied. By Theorem 2.2, we conclude that $S(\cdot)$ is u.s.c on Λ. In fact,

$$S(\mu) = \begin{cases} \{0\}, & \text{if } \mu \neq 0, \\ [0,1], & \text{if } \mu = 0. \end{cases} \tag{2.13}$$

Then, by the definition of upper semicontinuity, it follows readily that the solutions map $S(\mu)$ is u.s.c on Λ.

However, for $x = (x_1, x_2)^{\top} \to x_0 = (0,0)^{\top}$ with $x_2 = x_1^2$, we have $T(x,\mu) = 1/2$, but $T(x_0,\mu) = 0$. Thus, for any $\mu \in \Lambda$, $T(\cdot,\mu)$ is not continuous at $(0,0)^{\top}$. Therefore, the theorems concerning the upper semicontinuity in the literatures are not applicable.

Acknowledgments

The authors would like to thank two anonymous referees for their valuable comments and suggestions, which helped to improve the paper. This research was partially supported by the National Natural Science Foundation of China (Grant no. 10871216) and Chongqing University Postgraduates Science and Innovation Fund (Project no. 201005B1A0010338).

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