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Research Article

Approximately Quadratic Mappings on Restricted Domains

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We introduce a generalized quadratic functional equation $f(rx+sy)=rf(x)+sf(y)-rsf(x-y)$, where r, s are nonzero real numbers with $r+s=1$. We show that this functional equation is quadratic if r, s are rational numbers. We also investigate its stability problem on restricted domains. These results are applied to study of an asymptotic behavior of these generalized quadratic mappings.

1. Introduction

Under what conditions does there exist a group homomorphism near an approximate group homomorphism? This question concerning the stability of group homomorphisms was posed by Ulam [1]. The case of approximately additive mappings was solved by Hyers [2] on Banach spaces. In 1950 Aoki [3] provided a generalization of the Hyers' theorem for additive mappings and in 1978 Th. M. Rassias [4] generalized the Hyers' theorem for linear mappings by allowing the Cauchy difference to be unbounded (see also [5]). The result of Rassias' theorem has been generalized by Găvruţa [6] who permitted the Cauchy difference to be bounded by a general control function. This stability concept is also applied to the case of other functional equations. For more results on the stability of functional equations, see [7–24]. We also refer the readers to the books in [25–29].

It is easy to see that the quadratic function $f(x)=x^2$ is a solution of each of the following functional equations:

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

$$f(rx+sy)+rsf(x-y)=rf(x)+sf(y), \tag{1.2}$$

where r, s are nonzero real numbers with $r+s=1$. So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function $f : X \to Y$ between real vector spaces X and Y is quadratic if and only if there exists a unique symmetric biadditive function $B : X \times X \to Y$ such that $f(x) = B(x,x)$ for all $x \in X$ (see [13, 25, 27]).

We prove that the functional equations (1.1) and (1.2) are equivalent if r, s are nonzero rational numbers. The functional equation (1.1) is a spacial case of (1.2). Indeed, for the case $r = s = 1/2$ in (1.2), we get (1.1).

In 1983 Skof [30] was the first author to solve the Hyers-Ulam problem for additive mappings on a restricted domain (see also [31–33]). In 1998 Jung [34] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains (see also [35–37]). J. M. Rassias [38] investigated the Hyers-Ulam stability of mixed type mappings on restricted domains.

2. Solutions of (1.2)

In this section we show that the functional equation (1.2) is equivalent to the quadratic equation (1.1). That is, every solution of (1.2) is a quadratic function. We recall that r, s are nonzero real numbers with $r+s=1$.

Theorem 2.1. *Let X and Y be real vector spaces and $f : X \to Y$ be an odd function satisfying* (1.2). *If r is a rational number, then $f \equiv 0$.*

Proof. Since f is odd, $f(0) = 0$. Letting $x = 0$ (resp., $y = 0$) in (1.2), we get

$$f(sy) = s(1+r)f(y), \qquad f(rx) = r^2 f(x) \tag{2.1}$$

for all $x, y \in X$. Replacing y by $-y$ in (1.2) and adding the obtained functional equation to (1.2), we get

$$f(rx+sy) + f(rx-sy) = 2rf(x) - rs[f(x+y) + f(x-y)] \tag{2.2}$$

for all $x, y \in X$. Replacing y by ry in (2.2) and using (2.1), we have

$$r[f(x+sy) + f(x-sy)] = 2f(x) - s[f(x+ry) + f(x-ry)] \tag{2.3}$$

for all $x, y \in X$. Again if we replace x by sx in (2.3) and use (2.1), we get

$$r(1+r)[f(x+y) + f(x-y)] = 2(1+r)f(x) - [f(sx+ry) + f(sx-ry)] \tag{2.4}$$

for all $x, y \in X$. Applying (1.2) and using the oddness of f, we have

$$f(sx+ry) + f(sx-ry) = 2sf(x) + rs[f(x+y) + f(x-y)] \tag{2.5}$$

for all x, y in X. So it follows from (2.4) and (2.5) that

$$f(x+y)+f(x-y)=2f(x) \tag{2.6}$$

for all x, y in X. It easily follows from (2.6) that f is additive, that is, $f(x+y)=f(x)+f(y)$ for all $x,y\in X$. So if r is a rational number, then $f(rx)=rf(x)$ for all x in X. Therefore, it follows from (2.1) that $(r^2-r)f(x)=0$ for all x in X. Since r, s are nonzero, we infer that $f\equiv 0$. □

Theorem 2.2. *Let X and Y be real vector spaces and $f:X\to Y$ be an even function satisfying* (1.2). *Then f satisfies* (1.1).

Proof. Letting $x=y=0$ in (1.2), we get $f(0)=0$. Replacing x by $x+y$ in (1.2), we get

$$f(rx+y)=rf(x+y)+sf(y)-rsf(x) \tag{2.7}$$

for all $x,y\in X$. Replacing y by $-y$ in (2.7) and using the evenness of f, we get

$$f(rx-y)=rf(x-y)+sf(y)-rsf(x) \tag{2.8}$$

for all x, y in X. Adding (2.7) to (2.8), we obtain

$$f(rx+y)+f(rx-y)=r\left[f(x+y)+f(x-y)\right]+2sf(y)-2rsf(x) \tag{2.9}$$

for all $x,y\in X$. Replacing y by $x+ry$ in (2.7), we get

$$f(r(x+y)+x)=rf(2x+ry)+sf(x+ry)-rsf(x) \tag{2.10}$$

for all x, y in X. Using (2.7) in (2.10), by a simple computation, we get

$$f(2x+y)+2f(x)+f(y)=2f(x+y)+f(2x) \tag{2.11}$$

for all x, y in X. Putting $y=-x$ in (2.11), we get that $f(2x)=4f(x)$ for all $x\in X$. Therefore, it follows from (2.11) that

$$f(2x+y)+f(y)=2f(x+y)+2f(x) \tag{2.12}$$

for all x, y in X. Replacing y by $y-x$ in (2.12), we get that $f(x+y)+f(y-x)=2f(x)+2f(x)$ for all $x,y\in X$. So f satisfies (1.1). □

Theorem 2.3. *Let $f:X\to Y$ be a function between real vector spaces X and Y. If r is a rational number, then f satisfies* (1.2) *if and only if f satisfies* (1.1).

Proof. Let f_o and f_e be the odd and the even parts of f. Suppose that f satisfies (1.2). It is clear that f_o and f_e satisfy (1.2). By Theorems 2.1 and 2.2, $f_o\equiv 0$ and f_e satisfies (1.1). Since $f=f_o+f_e$, we conclude that f satisfies (1.1).

Conversely, let f satisfy (1.1). Then there exists a unique symmetric biadditive function $B : X \times X \to Y$ such that $f(x) = B(x,x)$ for all $x \in X$ (see [13]). Therefore

$$\begin{aligned} rf(x) &+ sf(y) - rsf(x-y) \\ &= rB(x,x) + sB(y,y) - rsB(x-y,x-y) \\ &= r^2B(x,x) + s^2B(y,y) + 2rsB(x,y) \quad (r,s \text{ are rational numbers}) \\ &= B(rx+sy,rx+sy) = f(rx+sy) \end{aligned} \tag{2.13}$$

for all $x,y \in X$. So f satisfies (1.2). □

Proposition 2.4. *Let $\mathcal{X}$ be a linear space with the norm $\|\cdot\|$. $\mathcal{X}$ is an inner product space if and only if there exists a real number $0 < r < 1$ such that*

$$\|rx+sy\|^2 + rs\|x-y\|^2 = r\|x\|^2 + s\|y\|^2 \tag{2.14}$$

for all $x,y \in \mathcal{X}$, where $s = 1-r$.

Proof. Let $f : \mathcal{X} \to \mathbb{R}$ be a function defined by $f(x) = \|x\|^2$. If $\mathcal{X}$ is an inner product space, then f satisfies (2.14) for all $r \in \mathbb{R}$. Conversely, let $r \in (0,1)$ and the (even) function f satisfy (2.14). So f satisfies (1.2). By Theorem 2.3, the function f satisfies (1.1), that is,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{2.15}$$

for all $x,y \in \mathcal{X}$. Therefore $\mathcal{X}$ is an inner product space (see [14]). □

Proposition 2.5. *Let $p,q,u,v \in \mathbb{R} \setminus \{0\}$ and $\mathcal{X}$ be a linear space with the norm $\|\cdot\|$. Suppose that*

$$\|rx+sy\|^p + rs\|x-y\|^q = r\|x\|^u + s\|y\|^v \tag{2.16}$$

for all x, y in $\mathcal{X}$, where $0 < r < 1$ and $s = 1-r$. Then $p = q = u = v = 2$.

Proof. Setting $y = 0$ in (2.16), we get

$$|r|^p\|x\|^p + rs\|x\|^q = r\|x\|^u \tag{2.17}$$

for all x in $\mathcal{X}$. If we take $x \in \mathcal{X}$ with $\|x\| = 1$ in (2.17), we get that $p = 2$. Letting $y = x$ in (2.16), we get

$$\|x\|^2 = r\|x\|^u + s\|x\|^v \tag{2.18}$$

for all x in $\mathcal{X}$. Letting $x = 0$ in (2.16), we get

$$r\|y\|^q = \|y\|^v - s\|y\|^2 \tag{2.19}$$

for all y in $\mathcal{X}$. Since $p = 2$, it follows from (2.17) and (2.19) that

$$r\|x\|^u - s\|x\|^v = (r-s)\|x\|^2 \tag{2.20}$$

for all $x \in \mathcal{X}$. Using (2.18) and (2.20), we get $\|x\|^u = \|x\|^v$ for all $x \in \mathcal{X}$. Hence $u = v$ and (2.18) implies that $u = v = 2$. Finally, $q = 2$ follows from (2.19). □

Corollary 2.6. *Let $\mathcal{X}$ be a linear space with the norm $\|\cdot\|$. $\mathcal{X}$ is an inner product space if and only if there exists a real number $0 < r < 1$ and $p, q, u, v \in \mathbb{R} \setminus \{0\}$ such that*

$$\|rx + sy\|^p + rs\|x - y\|^q = r\|x\|^u + s\|y\|^v \tag{2.21}$$

for all $x, y \in \mathcal{X}$, where $s = 1 - r$.

3. Stability of (1.2) on Restricted Domains

In this section, we investigate the Hyers-Ulam stability of the functional equation (1.2) on a restricted domain. As an application we use the result to the study of an asymptotic behavior of that equation. It should be mentioned that Skof [39] was the first author who treats the Hyers-Ulam stability of the quadratic equation. Czerwik [8] proved a Hyers-Ulam-Rassias stability theorem on the quadratic equation. As a particular case he proved the following theorem.

Theorem 3.1. *Let $\delta \geq 0$ be fixed. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta \tag{3.1}$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that $\|f(x) - Q(x)\| \leq \delta/2$ for all $x \in X$. Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

We recall that r, s are nonzero real numbers with $r + s = 1$.

Theorem 3.2. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that an even mapping $f : X \to Y$ satisfies the inequality*

$$\|f(rx + sy) + rsf(x-y) - rf(x) - sf(y)\| \leq \delta \tag{3.2}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists $K > 0$ such that f satisfies

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \frac{4(2 + |r| + |s|)}{|rs|}\delta \tag{3.3}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq K$.

Proof. Let $x, y \in X$ with $\|x\| + \|y\| \geq 2d$. Then, since $\|x+y\| + \|y\| \geq \max\{\|x\|, 2\|y\| - \|x\|\}$, we get $\|x+y\| + \|y\| \geq d$. So it follows from (3.2) that

$$\|f(rx+y) + rsf(x) - rf(x+y) - sf(y)\| \leq \delta \tag{3.4}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq 2d$. So

$$\|f(ry+x) + rsf(y) - rf(x+y) - sf(x)\| \leq \delta \tag{3.5}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq 2d$.

Let $x, y \in X$ with $\|x\| + \|y\| \geq 4d(1/|r| + |1 - 1/|r||)$. We have two cases.

Case 1. $\|y\| > 2d/|r|$. Then $\|x\| + \|x + ry\| \geq |r|\|y\| \geq 2d$.

Case 2. $\|y\| \leq 2d/|r|$. Then we have $\|x\| \geq 2d(1/|r| + 2|1 - 1/|r||)$. So

$$\|x\| + \left\|x + ry\right\| \geq 2\|x\| - |r|\left\|y\right\| \geq 2d\left(\frac{2}{|r|} + 4\left|1 - \frac{1}{|r|}\right| - 1\right) \geq 2d. \tag{3.6}$$

Therefore we get that $\|x\| + \|x + ry\| \geq 2d$ from Cases 1 and 2. Hence by (3.4) we have

$$\|f(r(x+y)+x) + rsf(x) - rf(2x+ry) - sf(x+ry)\| \leq \delta \tag{3.7}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq 4d(1/|r| + |1 - 1/|r||)$. Set $M := 4d(1/|r| + |1 - 1/|r||)$. Then

$$\left\|x+y\right\| + \|x\| \geq \frac{M}{2} \geq 2d, \qquad \|2x\| + \left\|y\right\| \geq M \geq 4d \tag{3.8}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq M$. From (3.4) and (3.5), we get the following inequalities:

$$\begin{gathered}
\|f(r(x+y)+x) + rsf(x+y) - rf(2x+y) - sf(x)\| \leq \delta, \\
\left\|rf(ry+2x) + r^2 sf(y) - r^2 f(2x+y) - rsf(2x)\right\| \leq \delta|r|, \\
\left\|sf(ry+x) + rs^2 f(y) - rsf(x+y) - s^2 f(x)\right\| \leq \delta|s|.
\end{gathered} \tag{3.9}$$

Using (3.7) and the above inequalities, we get

$$\|f(2x+y) + 2f(x) + f(y) - 2f(x+y) - f(2x)\| \leq \frac{2 + |r| + |s|}{|rs|}\delta \tag{3.10}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq M$. If $x, y \in X$ with $\|x\| + \|y\| \geq 2M$, then $\|x\| + \|y - x\| \geq M$. So it follows from (3.10) that

$$\|f(x+y) + 2f(x) + f(y-x) - 2f(y) - f(2x)\| \leq \frac{2 + |r| + |s|}{|rs|}\delta. \tag{3.11}$$

Letting $y = 0$ in (3.11), we get

$$\|4f(x) - f(2x) - 2f(0)\| \leq \frac{2 + |r| + |s|}{|rs|}\delta \tag{3.12}$$

for all $x, y \in X$ with $\|x\| \geq 2M$. Letting $x = 0$ (and $y \in X$ with $\|y\| \geq 2M$) in (3.11), we get $\|f(0)\| \leq ((2 + |r| + |s|)/|rs|)\delta$. Therefore it follows from (3.11) and (3.12) that

$$\begin{aligned}
&\|f(x+y) + f(y-x) - 2f(x) - 2f(y)\| \\
&\quad \leq \|f(x+y) + 2f(x) + f(y-x) - 2f(y) - f(2x)\| \\
&\quad\quad + \|4f(x) - f(2x) - 2f(0)\| + 2\|f(0)\| \\
&\quad \leq \frac{4(2 + |r| + |s|)}{|rs|}\delta
\end{aligned} \tag{3.13}$$

for all $x, y \in X$ with $\|x\| \geq 2M$. Since f is even, the inequality (3.13) holds for all $x, y \in X$ with $\|y\| \geq 2M$. Therefore

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \frac{4(2 + |r| + |s|)}{|rs|}\delta \tag{3.14}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq 4M$. This completes the proof by letting $K := 4M$. □

Theorem 3.3. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that an even mapping $f : X \to Y$ satisfies the inequality* (3.2) *for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then f satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \frac{19(2 + |r| + |s|)}{|rs|}\delta \tag{3.15}$$

for all $x, y \in X$.

Proof. By Theorem 3.2 there exists $K > 0$ such that f satisfies (3.3) for all $x, y \in X$ with $\|x\| + \|y\| \geq K$ and $\|f(0)\| \leq ((2 + |r| + |s|)/|rs|)\delta$ (see the proof of Theorem 3.2). Using Theorem 2 of [38], we get that

$$\begin{aligned}
\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &\leq \frac{18(2 + |r| + |s|)}{|rs|}\delta + \|f(0)\| \\
&\leq \frac{19(2 + |r| + |s|)}{|rs|}\delta
\end{aligned} \tag{3.16}$$

all $x, y \in X$. □

Theorem 3.4. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that an even mapping $f : X \to Y$ satisfies the inequality* (3.2) *for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that $Q(x) = \lim_{n\to\infty} 4^{-n} f(2^n x)$ and*

$$\|f(x) - Q(x)\| \leq \frac{19(2 + |r| + |s|)}{2|rs|}\delta \tag{3.17}$$

for all $x \in X$.

Proof. The result follows from Theorems 3.1 and 3.3. □

Skof [39] has proved an asymptotic property of the additive mappings and Jung [34] has proved an asymptotic property of the quadratic mappings (see also [36]). We prove such a property also for the quadratic mappings.

Corollary 3.5. *An even mapping $f : X \to Y$ satisfies* (1.2) *if and only if the asymptotic condition*

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \longrightarrow 0, \quad \text{as } \|x\| + \|y\| \longrightarrow \infty \tag{3.18}$$

holds true.

Proof. By the asymptotic condition (3.18), there exists a sequence $\{\delta_n\}$ monotonically decreasing to 0 such that

$$\|f(rx + sy) + rsf(x - y) - rf(x) - sf(y)\| \leq \delta_n \tag{3.19}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq n$. Hence, it follows from (3.19) and Theorem 3.4 that there exists a unique quadratic mapping $Q_n : X \to Y$ such that

$$\|f(x) - Q_n(x)\| \leq \frac{19(2 + |r| + |s|)}{2|rs|}\delta_n \tag{3.20}$$

for all $x \in X$. Since $\{\delta_n\}$ is a monotonically decreasing sequence, the quadratic mapping Q_m satisfies (3.20) for all $m \geq n$. The uniqueness of Q_n implies $Q_m = Q_n$ for all $m \geq n$. Hence, by letting $n \to \infty$ in (3.20), we conclude that f is quadratic. □

Corollary 3.6. *Let r be rational. An even mapping $f : X \to Y$ is quadratic if and only if the asymptotic condition* (3.18) *holds true.*

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