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Research Article

The Hermite-Hadamard Type Inequality of GA-Convex Functions and Its Application

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We established a new Hermit-Hadamard type inequality for GA-convex functions. As applications, we obtain two new Gautschi type inequalities for gamma function.

1. Introduction

Let f be a convex (concave) function on $[a,b] \subseteq \mathbb{R}$; the well-known Hermite-Hadamard's inequality [1] can be expressed as

$$f\left(\frac{a+b}{2}\right) \leq (\geq) \frac{1}{b-a}\int_a^b f(t)dt \leq (\geq) \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Recently, Hermite-Hadamard's inequality has been the subject of intensive research. In particular, many improvements, generalizations, and applications for the Hermite-Hadamard's inequality can be found in the literature [2–20].

Let $I \subseteq (0,\infty)$ be an interval; a real-valued function $f : I \to \mathbb{R}$ is said to be GA-convex (concave) on I if $f(x^{\alpha}y^{1-\alpha}) \leq (\geq)\alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in I$ and $\alpha \in [0,1]$.

In [21], Anderson et al. discussed the GA and related kinds of convexity; some applications to special functions were presented.

For $b > a > 0$, let $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, and $A(a,b) = (a+b)/2$ be the geometric, logarithmic, identric, and arithmetic means of a and b, respectively. Then

$$\min\{a,b\} < G(a,b) < L(a,b) < I(a,b) < A(a,b) < \max\{a,b\}. \tag{1.2}$$

The first purpose of this paper is to establish the following new Hermite-Hadamard type inequality for GA-convex (concave) functions.

Theorem 1.1. *If $b > a > 0$ and $f : [a,b] \to \mathbb{R}$ is a differentiable GA-convex (concave) function, then*

$$f(I(a,b)) \leq (\geq) \frac{1}{b-a}\int_a^b f(t)dt \leq (\geq) \frac{b-L(a,b)}{b-a} f(b) + \frac{L(a,b)-a}{b-a} f(a). \tag{1.3}$$

For real and positive values of x, the Euler gamma function Γ and its logarithmic derivative ψ, the so-called digamma function, are defined by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.4}$$

The ratio $\Gamma(s)/\Gamma(r)(s > r > 0)$ has attracted the attention of many mathematicians and physicists. Gautschi [22] first proved that

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp\left[(1-s)\psi(n+1)\right] \tag{1.5}$$

for $0 < s < 1$ and $n = 1, 2, 3 \ldots$.

A strengthened upper bound was given by Erber [23]:

$$\frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2}. \tag{1.6}$$

In [24], Kečkić and Vasić established the following double inequality for $b > a > 0$:

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}e^{a-b}. \tag{1.7}$$

In [25], Kershaw obtained

$$\begin{gathered}\exp\left[(1-s)\psi\left(x+s^{1/2}\right)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right], \\ \left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s}\end{gathered} \tag{1.8}$$

for $x > 0$ and $0 < s < 1$.

In [26], Zhang and Chu proved

$$\frac{\log\Gamma(b)-\log\Gamma(a)}{b-a} > \frac{b-L(a,b)}{b-a}\psi(b) + \frac{L(a,b)-a}{b-a}\psi(a) \tag{1.9}$$

for all $b > a > 0$.

In [27], Zhang and Chu presented

$$\psi(L(a,b)) < \frac{\log\Gamma(b) - \log\Gamma(a)}{b-a} < \psi(L(a,b)) + \log\frac{I(a,b)}{L(a,b)} \tag{1.10}$$

for all $b > a > 0$.

The second purpose of this paper is to establish the following two new Gautschi type inequalities by using Theorem 1.1.

Theorem 1.2. *If $b > a > 0$, then*

$$\begin{aligned}\psi(I(a,b)) - \frac{I(a,b)-L(a,b)}{2I(a,b)L(a,b)} - \frac{I^2(a,b)-G^2(a,b)}{12I^2(a,b)G^2(a,b)} &\le \frac{\log\Gamma(b)-\log\Gamma(a)}{b-a}\\ &\le \psi(I(a,b)) - \frac{I(a,b)-L(a,b)}{2I(a,b)L(a,b)}.\end{aligned} \tag{1.11}$$

Theorem 1.3. *If $b > a > 0$, then*

$$\begin{aligned}&\frac{b-L(a,b)}{b-a}\psi(b) + \frac{L(a,b)-a}{b-a}\psi(a) + \frac{L^2(a,b)-G^2(a,b)}{2L(a,b)G^2(a,b)} \le \frac{\log\Gamma(b)-\log\Gamma(a)}{b-a}\\ &\quad\le \frac{b-L(a,b)}{b-a}\psi(b) + \frac{L(a,b)-a}{b-a}\psi(a) + \frac{L^2(a,b)-G^2(a,b)}{2L(a,b)G^2(a,b)} + \frac{L(a,b)A(a,b)-G^2(a,b)}{6G^4(a,b)}.\end{aligned} \tag{1.12}$$

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

Lemma 2.1. *One has* $\sum_{n=1}^{\infty} 1/n^3 < 1.203$.

Proof. Simple computations lead to

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^3} &= \sum_{n=1}^{20}\frac{1}{n^3} + \sum_{n=21}^{\infty}\frac{1}{n^3} < \sum_{n=1}^{20}\frac{1}{n^3} + \sum_{n=21}^{\infty}\frac{1}{(n-1)n(n+1)}\\ &= \sum_{n=1}^{20}\frac{1}{n^3} + \sum_{n=21}^{\infty}\left[\frac{1}{2n(n-1)} - \frac{1}{2n(n+1)}\right] = \sum_{n=1}^{20}\frac{1}{n^3} + \frac{1}{2\times 20\times 21}\\ &= 1.202\cdots < 1.203.\end{aligned} \tag{2.1}$$

□

Lemma 2.2 (see [28, Lemma 2.1]). *If $x > 0$, then*

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{x^{2n+1}} + (-1)^m\theta_1\frac{B_{m+1}}{x^{2m+3}}, \tag{2.2}$$

$$\psi''(x) = -\frac{1}{x^2} - \frac{1}{x^3} + \sum_{n=1}^{m}(-1)^{n}\frac{(2n+1)B_n}{x^{2n+2}} + (-1)^{m+1}\theta_2\frac{(2m+3)B_{m+1}}{x^{2m+4}}, \tag{2.3}$$

where $0 < \theta_1, \theta_2 < 1$, $m \geq 1$, $m \in \mathbb{N}$, $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30, \ldots$.

Lemma 2.3. *Suppose that $I \subseteq (0, \infty)$ is an interval and $f : I \to \mathbb{R}$ is a real-valued function. If f is second-order differentiable on I, then f is GA-convex (concave) on I if and only if*

$$f'(x) + xf''(x) \geq (\leq) 0 \tag{2.4}$$

for all $x \in I$.

Proof. Lemma 2.3 follows easily from the basic properties of convex (concave) functions and the fact that f is GA-convex (concave) on I if and only if $g(x) = f(e^x)$ is convex (concave) on $J = \{\log x : x \in I\}$. □

Lemma 2.4 (see [29, Theorem 3]). *If $x > 0$, then*

$$0 < x^2\psi'(x+1) + x^3\psi''(x+1) < \frac{1}{2}. \tag{2.5}$$

Lemma 2.5. *$\psi(x) + 1/2x$ is GA-concave on $(0, \infty)$.*

Proof. Differentiating the well-known identity $\Gamma(x+1) = x\Gamma(x)$ we get

$$\begin{aligned} \psi'(x+1) &= -\frac{1}{x^2} + \psi'(x), \\ \psi'(x+1) &= \frac{2}{x^3} + \psi'(x). \end{aligned} \tag{2.6}$$

From inequalities (2.5) and (2.6) we have

$$x^2\psi'(x) + x^3\psi'(x) + \frac{1}{2} < 0. \tag{2.7}$$

Inequality (2.7) leads to

$$\left(\psi(x) + \frac{1}{2x}\right)' + x\left(\psi(x) + \frac{1}{2x}\right)'' = \frac{1}{x^2}\left(x^2\psi'(x) + x^3\psi''(x) + \frac{1}{2}\right) < 0. \tag{2.8}$$

□

Therefore, Lemma 2.5 follows from (2.8) and Lemma 2.3.

Lemma 2.6. *$\psi(x) + 1/2x + 1/12x^2$ is GA-convex on $(0, \infty)$.*

Proof. Simple computation leads to

$$\left(\psi(x) + \frac{1}{2x} + \frac{1}{12x^2}\right)' + x\left(\psi(x) + \frac{1}{2x} + \frac{1}{12x^2}\right)'' = \psi'(x) + x\psi''(x) + \frac{1}{2x^2} + \frac{1}{3x^3}. \tag{2.9}$$

From (2.9) and Lemma 2.3 we know that we need only to prove that

$$\psi'(x) + x\psi''(x) + \frac{1}{2x^2} + \frac{1}{3x^3} \geq 0. \tag{2.10}$$

We divide the proof into three cases.

Case 1. $x \in [\sqrt{5}/2, \infty)$. Taking $m = 2$ in (2.2) and $m = 3$ in (2.3) we get

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5}, \tag{2.11}$$

$$\psi''(x) > -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + \frac{1}{6x^6} - \frac{1}{6x^8}. \tag{2.12}$$

Inequalities (2.11) and (2.12) together with $x \geq \sqrt{5}/2$ lead to

$$\psi'(x) + x\psi''(x) + \frac{1}{2x^2} + \frac{1}{3x^3} > \frac{2}{15x^7}\left(x^2 - \frac{5}{4}\right) \geq 0. \tag{2.13}$$

Case 2. $x \in [1, \sqrt{5}/2)$. It is well-known that

$$\log \Gamma(x) = -\gamma x + \sum_{k=1}^{\infty}\left[\frac{x}{k} - \log\left(1 + \frac{x}{k}\right)\right] - \log x, \tag{2.14}$$

where $\gamma = 0.577215\cdots$ is Euler's constant.

Differentiating (2.14) we get

$$\psi'(x) = \sum_{k=0}^{\infty}\frac{1}{(k+x)^2}, \tag{2.15}$$

$$\psi''(x) = -\sum_{k=0}^{\infty}\frac{2}{(k+x)^3}. \tag{2.16}$$

We clearly see that $x^2(k-x)/(k+x)^3$ is increasing in $[1, \sqrt{5}/2)$ for $k \geq 3$; hence (2.15) and (2.16) lead to

$$\begin{aligned}
x^3\psi'(x) + x^4\psi''(x) + \frac{x}{2} + \frac{1}{3} &= \sum_{k=0}^{\infty} \frac{x^3(k-x)}{(k+x)^3} + \frac{x}{2} + \frac{1}{3} \\
&= \frac{1}{3} - \frac{x}{2} + \frac{x^3 - x^4}{(1+x)^3} + \frac{x^3(2-x)}{(2+x)^3} + x\sum_{k=3}^{\infty} \frac{x^2(k-x)}{(k+x)^3} \\
&\geq \frac{1}{3} - \frac{x}{2} + \frac{x^3 - x^4}{(1+x)^3} + \frac{x^3(2-x)}{(2+x)^3} + x\sum_{k=3}^{\infty} \frac{k-1}{(k+1)^3} \\
&= \frac{1}{3} - \frac{x}{2} + \frac{x^3 - x^4}{(1+x)^3} + \frac{x^3(2-x)}{(2+x)^3} + x\sum_{k=3}^{\infty} \left(\frac{1}{(k+1)^2} - \frac{2}{(k+1)^3} \right).
\end{aligned} \tag{2.17}$$

It follows from inequality (2.17), Lemma 2.1, $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, and $x \in [1, \sqrt{5}/2)$ that

$$\begin{aligned}
x^3\psi'(x) + x^4\psi''(x) + \frac{x}{2} + \frac{1}{3} &> \frac{1}{3} - \frac{x}{2} + \frac{x^3 - x^4}{(1+x)^3} + \frac{x^3(2-x)}{(2+x)^3} \\
&\quad + x\left[\frac{\pi^2}{6} - 1 - \frac{1}{4} - \frac{1}{9} - 2\left(1.203 - 1 - \frac{1}{8} - \frac{1}{27}\right) \right] \\
&= \frac{1}{3} + \frac{x^3 - x^4}{(1+x)^3} + \frac{x^3(2-x)}{(2+x)^3} - x \times 0.2981\cdots \\
&> \frac{1}{3} - 0.2982x + \frac{1-x}{(1+1/x)^3} + \frac{2-x}{(1+2/x)^3} \\
&> \frac{1}{3} - 0.2982 \times \frac{\sqrt{5}}{2} + \frac{1-\sqrt{5}/2}{\left(1+2/\sqrt{5}\right)^3} + \frac{2-\sqrt{5}/2}{27} \\
&= 0.01524\cdots > 0.
\end{aligned} \tag{2.18}$$

Case 3. $x \in (0,1)$. Since $(k-x)/(k+x)^3$ is decreasing in $[0,1]$ for $k \geq 1$, hence (2.15) and (2.16) imply that

$$\begin{aligned}
x^3\psi'(x) + x^4\psi''(x) + \frac{x}{2} + \frac{1}{3} &= x^3\sum_{k=1}^{\infty} \frac{k-x}{(k+x)^3} + \frac{1}{3} - \frac{x}{2} \\
&\geq \frac{1}{3} - \frac{x}{2} + x^3\sum_{k=1}^{\infty} \frac{k-1}{(k+1)^3} \\
&= \frac{1}{3} - \frac{x}{2} + x^3\sum_{k=1}^{\infty} \left(\frac{1}{(1+k)^2} - \frac{2}{(k+1)^3} \right).
\end{aligned} \tag{2.19}$$

From (2.19), Lemma 2.1, $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, and $x \in (0,1)$ we get

$$\begin{aligned} x^3\psi'(x) + x^4\psi''(x) + \frac{x}{2} + \frac{1}{3} &> \frac{1}{3} - \frac{x}{2} + x^3\left[\frac{\pi^2}{6} - 1 - 2(1.203 - 1)\right] \\ &= \frac{1}{3} - \frac{x}{2} + x^3 \times 0.238934\cdots \\ &> \frac{1}{3} - \frac{x}{2} + 0.238x^3. \end{aligned} \tag{2.20}$$

It is not difficult to verify that

$$\min_{x\in[0,1]}\left(\frac{1}{3} - \frac{x}{2} + 0.238x^3\right) = \frac{1}{3} - \frac{1}{2}\sqrt{\frac{1}{1.428}} + 0.238\left(\sqrt{\frac{1}{1.428}}\right)^3 = 0.054390\cdots > 0. \tag{2.21}$$

Therefore, inequality (2.10) follows from (2.20) and (2.21). □

3. Proof of Theorems 1.1, 1.2, and 1.3

Proof of Theorem 1.1. Suppose that f is a GA-convex function. For any fixed $c \in (a,b)$, if $x \in [c,b]$, then $g(t) = f(e^t)$ is convex on $[\log c, \log x]$ and

$$\frac{g(\log x) - g(\log c)}{\log x - \log c} \geq g'(\log c). \tag{3.1}$$

Inequality (3.1) implies that

$$f(x) - f(c) \geq c(\log x - \log c)f'(c). \tag{3.2}$$

Let $h(x) = \int_c^x f(t)dt - (x-c)f(c) - c[x(\log x - \log c) - (x-c)]f'(c)$, then inequality (3.2) leads to that $h'(x) = f(x) - f(c) - c(\log x - \log c)f'(c) \geq 0$ for $x \in [c,b]$. Hence $h(b) \geq h(c) = 0$, namely,

$$\begin{aligned} \int_c^b f(t)dt &\geq (b-c)f(c) + c(b\log b - b\log c - b + c)f'(c) \\ &= (b-c)f(c) + c(\log b - \log c)(b - L(c,b))f'(c). \end{aligned} \tag{3.3}$$

Using a similar method we get

$$\int_a^c f(t)dt \geq (c-a)f(c) - c(\log c - \log a)(L(a,c) - a)f'(c). \tag{3.4}$$

Table 1: Comparison of $M_3(a,b)$ and $M_4(a,b)$ with $M_1(a,b)$ and $M_2(a,b)$ for some a and b.

(a,b)	$M_1(a,b)$	$M_2(a,b)$	$M_3(a,b)$	$M_4(a,b)$
(1,20)	1.95847476···	1.76003014···	2.06182987···	2.03819859···
(2,30)	2.48099790···	2.27655813···	2.53158738···	2.51880271···
(3,10)	1.71034106···	1.66603361···	1.72366288···	1.72124442···
(5,20)	2.38826702···	2.32373887···	2.39918236···	2.39615827···
(10,20)	2.63689471···	2.61972436···	2.63920555···	2.63830472···
(15,40)	3.22695356···	3.19333175···	3.23147416···	3.22857750···
(1,50)	2.81088747···	2.47487539···	2.93055857···	2.89622376···
(50,80)	4.09342163···	4.08651410···	4.09511617···	4.09356690···
(100,200)	4.84411811···	4.83351060···	4.85077727···	4.84425912···
(1,1000)	5.23783238···	4.83563978···	5.59613902···	5.30668508···

Table 2: Comparison of $N_2(a,b)$ and $N_3(a,b)$ with $N_1(a,b)$ for some a and b.

(a,b)	$N_1(a,b)$	$N_2(a,b)$	$N_3(a,b)$
(1,20)	2.06618225···	2.06487349···	2.05761307···
(2,30)	2.53521205···	2.53251160···	2.52368386···
(3,10)	1.72432365···	1.72424671···	1.72268730···
(5,20)	2.40015332···	2.39940479···	2.39674582···
(10,20)	2.63950581···	2.63923737···	2.63837307···
(15,40)	3.23247604···	3.23149444···	3.22862423···
(1,50)	2.93896993···	2.93201528···	2.91418376···
(50,80)	4.09566787···	4.09511692···	4.09356845···
(100,200)	4.85329204···	4.85077759···	4.84425980···
(1,1000)	5.77619986···	5.59622174···	5.31858214···

Let $c = I(a,b)$, then

$$\left(\log b - \log c\right)(b - L(c,b)) = \left(\log c - \log a\right)(L(a,c) - a) = I(a,b) - \frac{ab}{L(a,b)}. \tag{3.5}$$

From inequalities (3.3) and (3.4) together with (3.5) we clearly see that

$$\int_a^b f(t)dt \geq (b-a)f(I(a,b)). \tag{3.6}$$

Next for any $x \in [a,b]$, let $y = (\log x - \log a)/(\log b - \log a)$, then $0 \leq y \leq 1$ and $x = a^{1-y}b^y$. From the definition of GA-convex function and the transformation to variable of integration we get

$$
\begin{aligned}
\int_a^b f(x)dx &= \int_0^1 f\left(a^{1-y}b^y\right)d\left(a^{1-y}b^y\right) \le a\int_0^1 [(1-y)f(a)+yf(b)]d\left(\frac{b}{a}\right)^y \\
&= a\int_0^1 [f(a)+(f(b)-f(a))y]d\left(\frac{b}{a}\right)^y \\
&= (b-a)f(a)+a(f(b)-f(a))\int_0^1 yd\left(\frac{b}{a}\right)^y \\
&= (b-a)f(a)+a(f(b)-f(a))\left(\frac{b}{a}-\frac{b/a-1}{\log b-\log a}\right) \\
&= bf(b)-af(a)-(f(b)-f(a))L(a,b) \\
&= (b-L(a,b))f(b)+(L(a,b)-a)f(a).
\end{aligned}
\tag{3.7}
$$

□

Therefore, Theorem 1.1 follows from inequalities (3.6) and (3.7).

Proof of Theorem 1.2. From Lemmas 2.5 and 2.6 together with Theorem 1.1 we clearly see that

$$
\psi(I(a,b))+\frac{1}{2I(a,b)} \ge \frac{1}{b-a}\int_a^b \left(\psi(x)+\frac{1}{2x}\right)dx = \frac{\log\Gamma(b)-\log\Gamma(a)}{b-a}+\frac{1}{2L(a,b)}, \tag{3.8}
$$

$$
\begin{aligned}
&\psi(I(a,b))+\frac{1}{2I(a,b)}+\frac{1}{12I^2(a,b)} \le \frac{1}{b-a}\int_a^b\left(\psi(x)+\frac{1}{2x}+\frac{1}{12x^2}\right)dx \\
&\quad= \frac{\log\Gamma(b)-\log\Gamma(a)}{b-a}+\frac{1}{2L(a,b)}+\frac{1}{12ab} = \frac{\log\Gamma(b)-\log\Gamma(a)}{b-a}+\frac{1}{2L(a,b)}+\frac{1}{12G^2(a,b)}.
\end{aligned}
\tag{3.9}
$$

□

Therefore, Theorem 1.2 follows from (3.8) and (3.9).

Proof of Theorem 1.3. From Lemmas 2.5 and 2.6 together with Theorem 1.1 we get

$$
\frac{1}{b-a}\int_a^b\left(\psi(x)+\frac{1}{2x}\right)dx \ge \frac{b-L(a,b)}{b-a}\left(\psi(b)+\frac{1}{2b}\right)+\frac{L(a,b)-a}{b-a}\left(\psi(a)+\frac{1}{2a}\right), \tag{3.10}
$$

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b\left(\psi(x)+\frac{1}{2x}+\frac{1}{12x^2}\right)dx &\le \frac{b-L(a,b)}{b-a}\left(\psi(b)+\frac{1}{2b}+\frac{1}{12b^2}\right) \\
&\quad+\frac{L(a,b)-a}{b-a}\left(\psi(a)+\frac{1}{2a}+\frac{1}{12a^2}\right).
\end{aligned}
\tag{3.11}
$$

Inequalities (3.10) and (3.11) lead to

$$\frac{\log\Gamma(b)-\log\Gamma(a)}{b-a} \geq \frac{b-L(a,b)}{b-a}\psi(b)+\frac{L(a,b)-a}{b-a}\psi(a)+\frac{L^2(a,b)-G^2(a,b)}{2L(a,b)G^2(a,b)}, \tag{3.12}$$

$$\frac{\log\Gamma(b)-\log\Gamma(a)}{b-a} \leq \frac{b-L(a,b)}{b-a}\psi(b)+\frac{L(a,b)-a}{b-a}\psi(a)+\frac{L^2(a,b)-G^2(a,b)}{2L(a,b)G^2(a,b)} + \frac{L(a,b)A(a,b)-G^2(a,b)}{6G^4(a,b)}. \tag{3.13}$$

□

Therefore, Theorem 1.3 follows from (3.12) and (3.13).

Remark 3.1. Making use of a computer and the mathematica software we can show that the bounds in Theorems 1.2 and 1.3 are stronger than that in inequalities (1.9) and (1.10) for some a and b. In fact, if we let $M_1(a,b) = ((b-L(a,b))/(b-a))\psi(b)+((L(a,b)-a)/(b-a))\psi(a)$, $M_2(a,b)=\psi(L(a,b))$, $M_3(a,b)=\psi(I(a,b))-((I(a,b)-L(a,b))/(2I(a,b)L(a,b)))-((I^2(a,b)-G^2(a,b))/(12I^2(a,b)G^2(a,b)))$, $M_4(a,b) = ((b-L(a,b))/(b-a))\psi(b)+((L(a,b)-a)/(b-a))\psi(a)+((L^2(a,b)-G^2(a,b))/(2L(a,b)G^2(a,b)))$, $N_1(a,b)=\psi(L(a,b))+\log I(a,b)/L(a,b)$, $N_2(a,b)=\psi(I(a,b))-((I(a,b)-L(a,b))/(2I(a,b)L(a,b)))$ and $N_3(a,b)=((b-L(a,b))/(b-a))\psi(b)+((L(a,b)-a)/(b-a))\psi(a)+(L^2(a,b)-G^2(a,b))/(2L(a,b)G^2(a,b))+(L(a,b)A(a,b)-G^2(a,b))/(6G^4(a,b))$, then we have Tables 1 and 2 via elementary computation.

Remark 3.2. We clear see that the lower bound in Theorem 1.3 is stronger than that in inequality (1.9) for all $a,b>0$.

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