Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 579509, 6 pages
doi:10.1155/2010/579509

*Research Article*

# Multivariate Twisted $p$-Adic $q$-Integral on $\mathbb{Z}_p$ Associated with Twisted $q$-Bernoulli Polynomials and Numbers

**Seog-Hoon Rim, Eun-Jung Moon, Sun-Jung Lee, and Jeong-Hee Jin**

*Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea*

Correspondence should be addressed to Seog-Hoon Rim, shrim@knu.ac.kr

Received 19 June 2010; Accepted 2 October 2010

Academic Editor: Ulrich Abel



Recently, many authors have studied twisted $q$-Bernoulli polynomials by using the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$. In this paper, we define the twisted $p$-adic $q$-integral on $\mathbb{Z}_p$ and extend our result to the twisted $q$-Bernoulli polynomials and numbers. Finally, we derive some various identities related to the twisted $q$-Bernoulli polynomials.

## 1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < 1$.

For $n \in \mathbb{N}$, let $T_p$ be the $p$-adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n} = C_{p^\infty}, \tag{1.1}$$

where $C_{p^n} = \{\zeta \in \mathbb{C}_p \mid \zeta^{p^n} = 1 \text{ for some } n \geq 0\}$ is the cyclic group of order $p^n$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$.

For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{1.2}$$

compare with [1–3].

It is well known that the twisted $q$-Bernoulli polynomials of order $k$ are defined as

$$e^{xt}\left(\frac{t}{e^t\zeta q-1}\right)^k = \sum_{n=0}^{\infty} \beta_{n,\zeta,q}^{(k)}(x)\frac{t^n}{n!}, \quad \zeta \in T_p, \tag{1.3}$$

and $\beta_{n,\zeta,q}^k = \beta_{n,\zeta,q}^k(0)$ are called the twisted $q$-Bernoulli numbers of order $k$. When $k = 1$, the polynomials and numbers are called the twisted $q$-Bernoulli polynomials and numbers, respectively. When $k = 1$ and $q = 1$, the polynomials and numbers are called the twisted Bernoulli polynomials and numbers, respectively. When $k = 1$, $q = 1$, and $\zeta = 1$, the polynomials and numbers are called the ordinary Bernoulli polynomials and numbers, respectively.

Many authors have studied the twisted $q$-Bernoulli polynomials by using the properties of the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$ (cf. [4]). In this paper, we define the twisted $p$-adic $q$-integral on $\mathbb{Z}_p$ and extend our result to the twisted $q$-Bernoulli polynomials and numbers. Finally, we derive some various identities related to the twisted $q$-Bernoulli polynomials.

## 2. Multivariate Twisted $p$-Adic $q$-Integral on $\mathbb{Z}_p$ Associated with Twisted $q$-Bernoulli Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$. For $\zeta \in T_p$, we define the $(q,\zeta)$-numbers as

$$[k]_{q,\zeta} = \frac{1-q^k\zeta}{1-q}, \quad \text{for } k \in \mathbb{Z}_p. \tag{2.1}$$

Note that $[k]_q = [k]_{q,1} = (1-q^k)/(1-q)$.

Let us define

$$\binom{n}{k}_{q,\zeta} = \frac{[n]_{q,\zeta}!}{[k]_{q,\zeta}![n-k]_{q,\zeta}!}, \tag{2.2}$$

where $[k]_{q,\zeta}! = [k]_{q,\zeta}[k-1]_{q,\zeta}\cdots[1]_{q,\zeta}$. Note that $\binom{n}{k} = \binom{n}{k}_{1,1} = n!/k!(n-k)!$.

Now we construct the twisted $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$I_{q,\zeta}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{q,\zeta}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{q,\zeta}\left(x+p^N\mathbb{Z}_p\right) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x\zeta^x, \tag{2.3}$$

where $\mu_{q,\zeta}(x+p^N\mathbb{Z}_p) = q^x\zeta^x/[p^N]_q$. From the definition of the twisted $p$-adic $q$-integral on $\mathbb{Z}_p$, we can consider the twisted $q$-Bernoulli polynomials and numbers of order $k$ as follows:

$$\begin{aligned}\beta_{n,q,\zeta}^{(k)}(x) &= \int_{\mathbb{Z}_p^k} [x_1+x_2+\cdots+x_k+x]_q^n d\mu_{q,\zeta}(x_1)d\mu_{q,\zeta}(x_2)\cdots d\mu_{q,\zeta}(x_k)\\ &= \lim_{N\to\infty}\frac{1}{[p^N]_q^k}\sum_{x_1,\ldots,x_k=0}^{p^N-1}[x_1+x_2+\cdots+x_k+x]_q^n q^{x_1+x_2+\cdots+x_k}\zeta^{x_1+x_2+\cdots+x_k}\\ &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\lim_{N\to\infty}\frac{1}{[p^N]_q^k}\sum_{x_1,\ldots,x_k=0}^{p^N-1} q^{(l+1)x_1+\cdots+(l+1)x_k}\zeta^{x_1+\cdots+x_k}\\ &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{(l+1)^k}{[l+1]_{q,\zeta}^k}.\end{aligned} \tag{2.4}$$

In the special case $x = 0$ in (2.4), $\beta_{n,q,\zeta}^{(k)}(0) = \beta_{n,q,\zeta}^{(k)}$ are called the twisted $q$-Bernoulli numbers of order $k$.

If we take $k = 1$ and $\zeta = 1$ in (2.4), we can easily see that

$$\beta_{n,q}(x) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l+1}{[l+1]_q}. \tag{2.5}$$

compare with [4].

**Theorem 2.1.** *For $k \in \mathbb{Z}_+$ and $\zeta \in T_p$, we have*

$$\beta_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{(l+1)^k}{[l+1]_{q,\zeta}^k}. \tag{2.6}$$

Moreover, if we take $x = 0$ in Theorem 2.1, then we have the following identity for the twisted $q$-Bernoull numbers

$$\beta_{n,q,\zeta}^{(k)} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{(l+1)^k}{[l+1]_{q,\zeta}^k}. \tag{2.7}$$

From the definition of multivariate twisted $p$-adic $q$-integral, we also see that

$$\begin{aligned}
\beta_{n,q,\zeta}^{(k)}(x) &= \int_{\mathbb{Z}_p^k} [x_1 + x_2 + \cdots + x_k + x]_q^n d\mu_{q,\zeta}(x_1) d\mu_{q,\zeta}(x_2) \cdots d\mu_{q,\zeta}(x_k) \\
&= \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \int_{\mathbb{Z}_p^k} [x_1 + x_2 + \cdots + x_k]_q^l d\mu_{q,\zeta}(x_1) d\mu_{q,\zeta}(x_2) \cdots d\mu_{q,\zeta}(x_k) \\
&= \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \beta_{l,q,\zeta}^{(k)}.
\end{aligned} \tag{2.8}$$

**Corollary 2.2.** *For $k \in \mathbb{Z}_+$ and $\zeta \in T_p$, one obtains*

$$\beta_{n,q,\zeta}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \beta_{l,q,\zeta}^{(k)}. \tag{2.9}$$

Note that

$$q^{n(x_1+\cdots+x_k)} = \sum_{l=0}^{n} \binom{n}{l} (q-1)^l [x_1 + \cdots + x_k]_q^l. \tag{2.10}$$

We have

$$\int_{\mathbb{Z}_p^k} q^{n(x_1+\cdots+x_k)} d\mu_{q,\zeta}(x_1) d\mu_{q,\zeta}(x_2) \cdots d\mu_{q,\zeta}(x_k) = \sum_{l=0}^{n} \binom{n}{l} (q-1)^l \beta_{l,q,\zeta}^{(k)}. \tag{2.11}$$

It is easy to see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p^k} q^{n(x_1+\cdots+x_k)} d\mu_{q,\zeta}(x_1) d\mu_{q,\zeta}(x_2) \cdots d\mu_{q,\zeta}(x_k) \\
&\quad = \lim_{N\to\infty} \frac{1}{[p^N]_q^k} \sum_{x_1,\ldots,x_k=0}^{p^N-1} q^{n(x_1+\cdots+x_k)} q^{x_1+\cdots+x_k} \zeta^{x_1+\cdots+x_k} = \frac{(n+1)^k}{[n+1]_{q,\zeta}^k}.
\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we obtain the following theorem.

**Theorem 2.3.** *For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has*

$$\sum_{l=0}^{n} \binom{n}{l} (q-1)^l \beta_{l,q,\zeta}^{(k)} = \frac{(n+1)^k}{[n+1]_{q,\zeta}^k}. \tag{2.13}$$

Now we consider the modified extension of the twisted $q$-Bernoulli polynomials of order $k$ as follows:

$$B_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{ix} \int_{\mathbb{Z}_p^k} q^{\sum_{l=1}^{k}(k-l+i)x_i} d\mu_{q,\zeta}(x_1) \cdots d\mu_{q,\zeta}(x_k). \tag{2.14}$$

In the special case $x = 0$, we write $B_{n,q,\zeta}^{(k)} = B_{n,q,\zeta}^{(k)}(0)$, which are called the modified extension of the twisted $q$-Bernoulli numbers of order $k$.

From (2.14), we derive that

$$\begin{aligned} B_{n,q,\zeta}^{(k)}(x) &= \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(i+k)\cdots(i+1)}{[i+k]_{q,\zeta}\cdots[i+1]_{q,\zeta}} q^{ix} \\ &= \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\binom{i+k}{k} k!}{\binom{i+k}{k}_{q,\zeta} [k]_{q,\zeta}!} q^{ix}. \end{aligned} \tag{2.15}$$

Therefore, we obtain the following theorem.

**Theorem 2.4.** *For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has*

$$B_{n,q,\zeta}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\binom{i+k}{k} k!}{\binom{i+k}{k}_{q,\zeta} [k]_{q,\zeta}!} q^{ix}. \tag{2.16}$$

Now, we define $B_{n,q,\zeta}^{(-k)}(x)$ as follows:

$$B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \frac{(-1)^i \binom{n}{i} q^{ix}}{\int_{\mathbb{Z}_p^k} q^{\sum_{l=1}^{k}(k-l+i)x_i} d\mu_{q,\zeta}(x_1) \cdots d\mu_{q,\zeta}(x_k)}. \tag{2.17}$$

By (2.17), we can see that

$$B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\binom{i+k}{k}_{q,\zeta} [k]_{q,\zeta}!}{\binom{i+k}{k} k!} q^{ix}. \tag{2.18}$$

Therefore, we obtain the following theorem.

**Theorem 2.5.** *For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ and $\zeta \in T_p$, one has*

$$B_{n,q,\zeta}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{i+k}{k}_{q,\zeta} \frac{\binom{n+k}{n-i}[k]_{q,\zeta}!}{\binom{n+k}{k}k!} q^{ix}. \quad (2.19)$$

In (2.19), we can see the relations between the binomial coefficients and the modified extension of the twisted $q$-Bernoulli polynomials of order $k$.

## Acknowledgments

The authors would like to thank the anonymous referee for his/her excellent detail comments and suggestions. This Research was supported by Kyungpook National University Research Fund, 2010.

## References

[1] L.-C. Jang, "Multiple twisted $q$-Euler numbers and polynomials associated with $p$-adic $q$-integrals," *Advances in Difference Equations*, Article ID 738603, 11 pages, 2008.
[2] T. Kim, "$q$-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
[3] T. Kim, "$q$-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.
[4] T. Kim, "Sums of products of $q$-Bernoulli numbers," *Archiv der Mathematik*, vol. 76, no. 3, pp. 190–195, 2001.