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Research Article

A Generalization of the Cauchy-Schwarz Inequality with Eight Free Parameters

Mohammad Masjed-Jamei

Department of Mathematics, K.N.Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

Correspondence should be addressed to Mohammad Masjed-Jamei, mmjamei@kntu.ac.ir

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The results of the recent published paper by Masjed-Jamei et al. (2009) are extended to a larger class and some of subclasses are studied in the sequel. In other words, we generalize the well known Cauchy-Schwarz and Cauchy-Bunyakovsky inequalities having eight free parameters and then introduce some of their interesting subclasses.

1. Introduction

Let $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ be two sequences of real, numbers. It is well known that the discrete version of the Cauchy-Schwarz inequality [1, 2] can be stated as

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2, \tag{1.1}$$

while its continuous version in the space of real-valued functions $C([a,b],\mathbf{R})$, that is, the Cauchy-Bunyakovsky inequality [1, 2], can be represented as

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx. \tag{1.2}$$

To date, a large number of generalizations and refinements of inequalities (1.1) and (1.2) have been investigated in the literature (e.g., [3–6]). In this sense, recently in [7] a generalization of the Cauchy-Bunyakovsky-Schwarz inequality with four free parameters has been presented as follows.

Theorem 1.1. *If $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are two sequences of real numbers and p, q, r, s belong to a subset of* $\mathbf{R}$*, then we have*

$$\begin{aligned}
&\left(\sum_{k=1}^n a_k b_k + A_1 \sum_{k=1}^n a_k \sum_{k=1}^n b_k + B_1\left(\sum_{k=1}^n a_k\right)^2 + C_1\left(\sum_{k=1}^n b_k\right)^2\right)^2 \\
&\quad\le \left(\sum_{k=1}^n a_k^2 + A_2 \sum_{k=1}^n a_k \sum_{k=1}^n b_k + B_2\left(\sum_{k=1}^n a_k\right)^2 + C_2\left(\sum_{k=1}^n b_k\right)^2\right) \\
&\qquad\times \left(\sum_{k=1}^n b_k^2 + A_3 \sum_{k=1}^n a_k \sum_{k=1}^n b_k + B_3\left(\sum_{k=1}^n a_k\right)^2 + C_3\left(\sum_{k=1}^n b_k\right)^2\right),
\end{aligned} \tag{1.3}$$

in which

$$M_0 = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} = \frac{1}{n}\begin{pmatrix} p+s+ps+qr & r\ (1+p) & q\ (1+s) \\ 2q\ (1+p) & p(p+2) & q^2 \\ 2r\ (1+s) & r^2 & s\ (s+2) \end{pmatrix}. \tag{1.4}$$

The inequality (1.3) *generalizes the Cauchy-Schwarz inequality for $A_i = B_i = C_i = 0$ $(i = 1, 2, 3)$ and the equality holds if $a_k = b_k$ and $A_i = B_i = C_i$ for each $i = 1, 2, 3$.*

Moreover, if $f, g : [a, b] \to$ $\mathbf{R}$ *are two integrable functions on $[a, b]$ and p, q, r, s belong to a subset of* $\mathbf{R}$*, then the continuous version of inequality* (1.3) *reads as*

$$\begin{aligned}
&\left(\int_a^b f(x)g(x)dx + A_1^* \int_a^b f(x)dx \int_a^b g(x)dx + B_1^*\left(\int_a^b f(x)dx\right)^2 + C_1^*\left(\int_a^b g(x)dx\right)^2\right)^2 \\
&\quad\le \left(\int_a^b f^2(x)dx + A_2^* \int_a^b f(x)dx \int_a^b g(x)dx + B_2^*\left(\int_a^b f(x)dx\right)^2 + C_2^*\left(\int_a^b g(x)dx\right)^2\right) \\
&\qquad\times \left(\int_a^b g^2(x)dx + A_3^* \int_a^b f(x)dx \int_a^b g(x)dx + B_3^*\left(\int_a^b f(x)dx\right)^2 + C_3^*\left(\int_a^b g(x)dx\right)^2\right),
\end{aligned} \tag{1.5}$$

in which

$$M_0^* = \begin{pmatrix} A_1^* & B_1^* & C_1^* \\ A_2^* & B_2^* & C_2^* \\ A_3^* & B_3^* & C_3^* \end{pmatrix} = \frac{1}{b-a}\begin{pmatrix} p+s+ps+qr & r(1+p) & q\ (1+s) \\ 2q(1+p) & p(p+2) & q^2 \\ 2r(1+s) & r^2 & s(s+2) \end{pmatrix}. \tag{1.6}$$

The inequality (1.5) *generalizes the Cauchy-Bunyakovsky inequality for $A_i^* = B_i^* = C_i^* = 0$ $(i = 1, 2, 3)$ and the equality holds if $f(x) = g(x)$ and $A_i^* = B_i^* = C_i^*$ for each $i = 1, 2, 3$.*

The main aim of this paper is to generalize the abovementioned theorem by means of eight free parameters and then study some of its particular cases of interest.

Theorem 1.2. *Let* $\{a_k\}_{k=1}^n$ *and* $\{b_k\}_{k=1}^n$ *be two sequences of real numbers and* $\{p_i, q_i\}_{i=1}^4$ *belong to a subset of* $\mathbf{R}$*. Then the following inequality holds*

$$\begin{aligned}
&\left(A_1\sum_{k=1}^n a_k b_k + A_2\sum_{k=1}^n a_k\sum_{k=1}^n b_k + A_3\sum_{k=1}^n a_k^2 + A_4\left(\sum_{k=1}^n a_k\right)^2 + A_5\sum_{k=1}^n b_k^2 + A_6\left(\sum_{k=1}^n b_k\right)^2\right)^2 \\
&\quad\le \left(B_1\sum_{k=1}^n a_k b_k + B_2\sum_{k=1}^n a_k\sum_{k=1}^n b_k + B_3\sum_{k=1}^n a_k^2 + B_4\left(\sum_{k=1}^n a_k\right)^2 + B_5\sum_{k=1}^n b_k^2 + B_6\left(\sum_{k=1}^n b_k\right)^2\right) \\
&\quad\times \left(C_1\sum_{k=1}^n a_k b_k + C_2\sum_{k=1}^n a_k\sum_{k=1}^n b_k + C_3\sum_{k=1}^n a_k^2 + C_4\left(\sum_{k=1}^n a_k\right)^2 + C_5\sum_{k=1}^n b_k^2 + C_6\left(\sum_{k=1}^n b_k\right)^2\right),
\end{aligned} \tag{1.7}$$

in which

$$\begin{gathered}
A_1 = p_1q_2 + p_2q_1, \\
A_2 = p_4(q_1 + nq_3) + q_4(p_1 + np_3) + p_2q_3 + p_3q_2, \\
A_3 = p_1q_1, \\
A_4 = p_1q_3+p_3q_1+np_3q_3, \\
A_5 = p_2q_2, \\
A_6 = p_2q_4+p_4q_2+np_4q_4, \\
B_1 = 2p_1p_2, \\
B_2 = 2p_4(p_1 + np_3)+2p_2p_3, \\
B_3 = p_1^2, \\
B_4 = p_3(2p_1 + np_3), \\
B_5 = p_2^2, \\
B_6 = p_4(2p_2 + np_4), \\
C_1 = 2q_1q_2, \\
C_2 = 2q_4(q_1 + nq_3)+2q_2q_3, \\
C_3 = q_1^2, \\
C_4 = q_3(2q_1 + nq_3), \\
C_5 = q_2^2, \\
C_6 = q_4(2q_2 + nq_4).
\end{gathered} \tag{1.8}$$

Moreover, the equality in (1.7) *holds if* $A_i = B_i = C_i$ *for* $i = 1, 2, \ldots, 6$, *which is equivalent to* $\{p_i = q_i\}_{i=1}^{4}$.

Proof. The proof of this theorem is straightforward if one defines a positive quadratic polynomial $P_2 : \mathbf{R} \to \mathbf{R}$ as

$$\begin{aligned} P_2\left(x\,;\{p_i, q_i\}_{i=1}^{4}\right) = \sum_{k=1}^{n}\Bigg(\left(p_1 a_k + p_2 b_k + p_3 \sum_{j=1}^{n} a_j + p_4 \sum_{j=1}^{n} b_j\right) x \\ + \left(q_1 a_k + q_2 b_k + q_3 \sum_{j=1}^{n} a_j + q_4 \sum_{j=1}^{n} b_j\right)\Bigg)^2, \end{aligned} \tag{1.9}$$

in which $\{a_k\}_{k=1}^{n}$ and $\{b_k\}_{k=1}^{n}$ are real numbers and $\{p_i, q_i\}_{i=1}^{4}$ belong to a subset of $\mathbf{R}$. Because a simple calculation first reveals that

$$\begin{aligned} &P_2\left(x\,;\{p_i, q_i\}_{i=1}^{4}\right) \\ &\quad= \sum_{k=1}^{n}\left(p_1 a_k + p_2 b_k + p_3 \sum_{j=1}^{n} a_j + p_4 \sum_{j=1}^{n} b_j\right)^2 x^2 \\ &\qquad + 2x \sum_{k=1}^{n}\left(p_1 a_k + p_2 b_k + p_3 \sum_{j=1}^{n} a_j + p_4 \sum_{j=1}^{n} b_j\right)\left(q_1 a_k + q_2 b_k + q_3 \sum_{j=1}^{n} a_j + q_4 \sum_{j=1}^{n} b_j\right) \\ &\qquad + \sum_{k=1}^{n}\left(q_1 a_k + q_2 b_k + q_3 \sum_{j=1}^{n} a_j + q_4 \sum_{j=1}^{n} b_j\right)^2 \\ &\quad\geq 0, \end{aligned} \tag{1.10}$$

therefore, for any $x \in \mathbf{R}$, the discriminant Δ of P_2 must be negative, that is,

$$\begin{aligned} \frac{1}{4}\Delta &= \left(\sum_{k=1}^{n}\left(p_1 a_k + p_2 b_k + p_3 \sum_{j=1}^{n} a_j + p_4 \sum_{j=1}^{n} b_j\right)\left(q_1 a_k + q_2 b_k + q_3 \sum_{j=1}^{n} a_j + q_4 \sum_{j=1}^{n} b_j\right)\right)^2 \\ &\quad - \left(\sum_{k=1}^{n}\left(p_1 a_k + p_2 b_k + p_3 \sum_{j=1}^{n} a_j + p_4 \sum_{j=1}^{n} b_j\right)^2\right) \\ &\quad \times \left(\sum_{k=1}^{n}\left(q_1 a_k + q_2 b_k + q_3 \sum_{j=1}^{n} a_j + q_4 \sum_{j=1}^{n} b_j\right)^2\right) \\ &\leq 0. \end{aligned} \tag{1.11}$$

On the other hand, the elements of $\Delta/4$ can eventually be simplified as follows:

$$\begin{aligned}
&\sum_{k=1}^{n}\left(p_1a_k+p_2b_k+p_3\sum_{j=1}^{n}a_j+p_4\sum_{j=1}^{n}b_j\right)\left(q_1a_k+q_2b_k+q_3\sum_{j=1}^{n}a_j+q_4\sum_{j=1}^{n}b_j\right)\\
&\quad=A_1\sum_{k=1}^{n}a_kb_k+A_2\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k+B_1\sum_{k=1}^{n}a_k^2+B_2\left(\sum_{k=1}^{n}a_k\right)^2+C_1\sum_{k=1}^{n}b_k^2+C_2\left(\sum_{k=1}^{n}b_k\right)^2,\\
&\sum_{k=1}^{n}\left(p_1a_k+p_2b_k+p_3\sum_{j=1}^{n}a_j+p_4\sum_{j=1}^{n}b_j\right)^2\\
&\quad=A_3\sum_{k=1}^{n}a_kb_k+A_4\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k+B_3\sum_{k=1}^{n}a_k^2+B_4\left(\sum_{k=1}^{n}a_k\right)^2+C_3\sum_{k=1}^{n}b_k^2+C_4\left(\sum_{k=1}^{n}b_k\right)^2,\\
&\sum_{k=1}^{n}\left(q_1a_k+q_2b_k+q_3\sum_{j=1}^{n}a_j+q_4\sum_{j=1}^{n}b_j\right)^2\\
&\quad=A_5\sum_{k=1}^{n}a_kb_k+A_6\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k+B_5\sum_{k=1}^{n}a_k^2+B_6\left(\sum_{k=1}^{n}a_k\right)^2+C_5\sum_{k=1}^{n}b_k^2+C_6\left(\sum_{k=1}^{n}b_k\right)^2,
\end{aligned}\tag{1.12}$$

where $\{A_i,B_i,C_i\}_{i=1}^{6}$ are the same values as given in (1.8).

Hence, replacing the results (1.12) into (1.11) proves the first part of Theorem 1.2. The proof of second part is also clear. □

Similarly, one can consider the continuous analogue of Theorem 1.2 as follows.

Theorem 1.3. *Let $f,g:[a,b]\to\mathbf{R}$ be two integrable functions on $[a,b]$ and $\{p_i,q_i\}_{i=1}^{4}$ belong to a subset of $\mathbf{R}$. Then, the following inequality holds*

$$\begin{aligned}
&\left(A_1^*\int_a^b f(x)g(x)dx+A_2^*\int_a^b f(x)dx\int_a^b g(x)dx+A_3^*\int_a^b f^2(x)dx+A_4^*\left(\int_a^b f(x)dx\right)^2\right.\\
&\qquad\left.+A_5^*\int_a^b g^2(x)dx+A_6^*\left(\int_a^b g(x)dx\right)^2\right)^2\\
&\quad\le\left(B_1^*\int_a^b f(x)g(x)dx+B_2^*\int_a^b f(x)dx\int_a^b g(x)dx+B_3^*\int_a^b f^2(x)dx+B_4^*\left(\int_a^b f(x)dx\right)^2\right.\\
&\qquad\left.+B_5^*\int_a^b g^2(x)dx+B_6^*\left(\int_a^b g(x)dx\right)^2\right)
\end{aligned}$$

$$\times\left(C_1^*\int_a^b f(x)g(x)dx + C_2^*\int_a^b f(x)dx\int_a^b g(x)dx + C_3^*\int_a^b f^2(x)dx + C_4^*\left(\int_a^b f(x)dx\right)^2 + C_5^*\int_a^b g^2(x)dx + C_6^*\left(\int_a^b g(x)dx\right)^2\right), \tag{1.13}$$

in which

$$\begin{gathered}
A_1^* = p_1q_2 + p_2q_1,\\
A_2^* = p_4(q_1 + h^*q_3) + q_4(p_1 + h^*p_3) + p_2q_3 + p_3q_2,\\
A_3^* = p_1q_1,\\
A_4^* = p_1q_3+p_3q_1+h^*p_3q_3,\\
A_5^* = p_2q_2,\\
A_6^* = p_2q_4+p_4q_2+h^*p_4q_4,\\
B_1^* = 2p_1p_2,\\
B_2^* = 2p_4(p_1+h^*p_3)+2p_2p_3,\\
B_3^* = p_1^2,\\
B_4^* = p_3(2p_1 + h^*p_3),\\
B_5^* = p_2^2,\\
B_6^* = p_4(2p_2 + h^*p_4),\\
C_1^* = 2q_1q_2,\\
C_2^* = 2q_4(q_1+h^*q_3)+2q_2q_3,\\
C_3^* = q_1^2,\\
C_4^* = q_3(2q_1 + h^*q_3),\\
C_5^* = q_2^2,\\
C_6^* = q_4(2q_2 + h^*q_4),
\end{gathered} \tag{1.14}$$

so that $h^* = b - a$*. Moreover, the equality in* (1.13) *holds if* $A_i^* = B_i^* = C_i^*$ *for* $i = 1, 2, \ldots, 6$*, which is equivalent to* $\{p_i = q_i\}_{i=1}^4$.

Proof. The proof of this theorem is clearly similar to Theorem 1.2 if a positive quadratic polynomial $Q_2 : \mathbf{R} \to \mathbf{R}$ is defined as

$$\begin{aligned} Q_2\left(x\,; \{p_i, q_i\}_{i=1}^{4}\right) = \int_a^b \Bigg(\Bigg(& p_1 f(t) + p_2 g(t) + p_3 \int_a^b f(u)du + p_4 \int_a^b g(u)du \Bigg) x \\ & + \left(q_1 f(t) + q_2 g(t) + q_3 \int_a^b f(u)du + q_4 \int_a^b g(u)du \right) \Bigg)^2 dt \geq 0, \end{aligned} \tag{1.15}$$

and then other items are followed accordingly. □

2. Some Special Cases and a Unified Approach

As we observe in Theorems 1.2 and 1.3, there exist various subclasses of inequalities (1.7) and (1.13). In this section, we consider two of them. Naturally other cases can be studied separately.

2.1. Special Case 1: Theorem 1.1, Inequalities (1.3) *and* (1.5)

For example, one can conclude that inequality (1.3) is a special case of inequality (1.7) for

$$A_1 = 1, \quad A_2 = \frac{1}{n}(p + s + ps + qr), \quad A_3 = 0, \quad A_4 = \frac{1}{n}r(1+p), \quad A_5 = 0, \quad A_6 = \frac{1}{n}q(1+s), \tag{2.1}$$

where $\{A_i\}_{i=1}^{6}$ are the same values as given in (1.8). Hence, after solving the above system, one finally gets

$$(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4) = \left(1, 0, \frac{p}{n}, \frac{q}{n}, 0, 1, \frac{r}{n}, \frac{s}{n}\right). \tag{2.2}$$

This means that replacing (2.2) in (1.7) gives the same as inequality (1.3).

2.2. Special Case 2: A Generalization of Pre-Grüss and Wagner Inequalities

It is clear that for

$$(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4) = \left(1, 0, \frac{p}{n}, 0, 0, 1, 0, \frac{s}{n}\right), \tag{2.3}$$

we obtain the following inequality:

$$\left(\sum_{k=1}^{n} a_k b_k + \frac{p+s+ps}{n} \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2 + \frac{p(p+2)}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right)\left(\sum_{k=1}^{n} b_k^2 + \frac{s(s+2)}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right). \tag{2.4}$$

Now, this inequality is a generalization of Pre-Grüss inequality [2] for $p = s = -1$ and is a generalization of Wagner inequality [1, 8] for $p = s$ and $p(p+2)/n = w$.

As we said, many special cases of inequalities (1.7) and (1.13) could be considered independently. Hence, it is necessary that one classifies them by means of a unified approach. For this purpose, let us reconsider two specific matrices M and M^* defined by (1.8) and (1.14). It is clear that both matrices can directly determine all elements of inequalities (1.7) and (1.13), respectively. As a sample, substituting the arbitrary and random vector

$$(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4) = (1, 2, 3, 4, 5, 6, 7, 8), \tag{2.5}$$

in the general matrix M gives

$$M = \begin{bmatrix} 16 & 52n+60 & 5 & 21n+22 & 12 & 32n+40 \\ 4 & 24n+20 & 1 & 9n+6 & 4 & 16n+16 \\ 60 & 112n+164 & 25 & 49n+70 & 36 & 64n+96 \end{bmatrix}. \tag{2.6}$$

Therefore, the inequality corresponding to (1.7) takes the form

$$\begin{aligned} &\left(16\sum_{k=1}^{n} a_k b_k + (52n+60) \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k + 5\sum_{k=1}^{n} a_k^2 + (21n+22)\left(\sum_{k=1}^{n} a_k\right)^2 \right.\\ &\qquad \left. +12\sum_{k=1}^{n} b_k^2 + (32n+40)\left(\sum_{k=1}^{n} b_k\right)^2\right)^2 \\ &\quad \le \left(4\sum_{k=1}^{n} a_k b_k + (24n+20)\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k + \sum_{k=1}^{n} a_k^2 + (9n+6)\left(\sum_{k=1}^{n} a_k\right)^2\right.\\ &\qquad \left. + 4\sum_{k=1}^{n} b_k^2 + (16n+16)\left(\sum_{k=1}^{n} b_k\right)^2\right) \\ &\qquad \times \left(60\sum_{k=1}^{n} a_k b_k + (112n+164)\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k + 25\sum_{k=1}^{n} a_k^2 + (49n+70)\left(\sum_{k=1}^{n} a_k\right)^2\right.\\ &\qquad \left. + 36\sum_{k=1}^{n} b_k^2 + (64n+96)\left(\sum_{k=1}^{n} b_k\right)^2\right). \end{aligned} \tag{2.7}$$

Conversely, for instance the given inequality (1.5) has a characteristic matrix as

$$M^{*}(\text{Ineq.}(1.5)) = \begin{bmatrix} 1 & h^{*}(p+s+ps+qr) & 0 & h^{*}r(1+p) & 0 & h^{*}q(1+s) \\ 0 & 2h^{*}q(1+p) & 1 & h^{*}p(p+2) & 0 & h^{*}q^{2} \\ 0 & 2h^{*}r(1+s) & 0 & h^{*}r^{2} & 1 & h^{*}s(s+2) \end{bmatrix}. \tag{2.8}$$

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