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Research Article

Strong Convergence of an Implicit Iteration Process for a Finite Family of Uniformly L-Lipschitzian Mappings in Banach Spaces

Feng Gu

Department of Mathematics, Institute of Applied Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China

Correspondence should be addressed to Feng Gu, gufeng99@sohu.com

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The purpose of this paper is to prove a strong convergence theorem for a finite family of uniformly L-Lipschitzian mappings in Banach spaces. The results presented in the paper improve and extend the corresponding results announced by Chang (2001), Cho et al. (2005), Ofoedu (2006), Schu (1991) and Zeng (2003 and 2005), and many others.

1. Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, K is a nonempty closed convex subset of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2, \|f\| = \|x\| \right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.

Definition 1.1. Let $T : K \to K$ be a mapping. Therefore, the following are given.

(1) T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that, for any $x, y \in K$,

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall n \ge 1. \tag{1.2}$$

(2) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that, for any given $x, y \in K$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall n \geq 1. \tag{1.3}$$

(3) T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that, for any $x, y \in K$, there exists $j(x-y) \in J(x-y)$ as follows:

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x - y\|^2, \quad \forall n \geq 1. \tag{1.4}$$

Remark 1.2. (1) It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly L-Lipschitzian mapping, where $L = \sup_{n\geq 1} k_n$. And every asymptotically nonexpansive mapping is asymptotically pseudocontractive, but the inverse is not true, in general.

(2) The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [7], while the concept of asymptotically pseudocontractive mappings was introduced by Schu [4] who proved the following theorem.

Theorem 1.3 (see Schu [4]). *Let H be a Hilbert space, K be a nonempty bounded closed convex subset of H, and let $T : K \to K$ be a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,\infty)$ satisfying the following conditions:*

(i) $k_n \to 1$ *as* $n \to \infty$,

(ii) $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, *where* $q_n = 2k_n - 1$.

Suppose further that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ such that $\varepsilon < \alpha_n < \beta_n \leq b$, for all $n \geq 1$, where $\varepsilon > 0$ and $b \in (0, L^{-2}[(1+L^2)^{1/2} - 1])$ are some positive number. For any $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1. \tag{1.5}$$

Then $\{x_n\}$ converges strongly to a fixed point of T in K.

In [1], the first author extended Theorem 1 to a real uniformly smooth Banach space and proved the following theorem.

Theorem 1.4 (see Chang [1]). *Let E be a uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E, and $T : K \to K$ be an asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$, and let $F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 0. \tag{1.6}$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_n - x^*, j(x_n - x^*)\rangle \leq k_n \|x_n - x^*\|^2 - \phi(\|x_n - x^*\|), \quad \forall n \geq 0, \tag{1.7}$$

where $x^* \in F(T)$ is some fixed point of T in K, then $x_n \to x^*$ as $n \to \infty$.

Very recently, in [3] Ofoedu proved the following theorem.

Theorem 1.5 (see Ofoedu [3]). *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ such that $x^* \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,

(iii) $\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \quad \forall n \geq 0. \tag{1.8}$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|), \quad \forall x \in K, \tag{1.9}$$

then $\{x_n\}$ converges strongly to x^*.

Remark 1.6. It should be pointed out that although Theorem 1.5 extends Theorem 1.4 from a real uniformly smooth Banach space to an arbitrary real Banach space, it removes the boundedness condition imposed on K.

In [8], Xu and Ori introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i\}_{i\in I}$ (here $I = \{1,2,\dots,m\}$), with $\{\alpha_n\}$ as a real sequence in (0, 1), and an initial point $x_0 \in K$:

$$x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T_n x_n, \quad \forall n \geq 1, \tag{1.10}$$

where $T_n = T_{n(\bmod m)}$ (here the $\bmod m$ function takes values in I). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space.

Chidume and Shahzad [9] and Zhou and Chang [10] studied the weak and strong convergences of this implicit process to a common fixed point for a finite family of nonexpansive mappings, respectively.

Recently, Feng Gu [11] introduced a composite implicit iteration process with errors for a finite family of strictly pseudocontractive mappings $\{T_i\}_{i=1}^m$ as follows:

$$\begin{aligned} x_n &= (1-\alpha_n-\gamma_n)x_{n-1}+\alpha_n T_n y_n+\gamma_n u_n, \quad n\ge 1, \\ y_n &= (1-\beta_n-\delta_n)x_n+\beta_n T_n x_n+\delta_n v_n, \quad n\ge 1, \end{aligned} \tag{1.11}$$

where $T_n = T_{n(\bmod m)}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, are four real sequences in $[0,1]$ satisfying $\alpha_n+\gamma_n \le 1$ and $\beta_n+\delta_n\le 1$ for all $n\ge 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K, and x_0 is a given point in K. Feng Gu proved the strong convergence of this process to a common fixed point for a finite family of strictly pseudocontractive mappings $\{T_i\}_{i=1}^m$ in a real Banach space.

Inspired and motivated by the abovesaid facts, we introduced a two-step implicit iteration process with errors for a finite family of L-Lipschitzian mappings $\{T_i\}_{i=1}^m$ as follows:

$$\begin{aligned} x_n &= (1-\alpha_n-\gamma_n)x_{n-1}+\alpha_n T_n^n y_n+\gamma_n u_n, \quad n\ge 1, \\ y_n &= (1-\beta_n-\delta_n)x_n+\beta_n T_n^n x_n+\delta_n v_n, \quad n\ge 1, \end{aligned} \tag{1.12}$$

where $T_n = T_{n(\bmod m)}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$, are four real sequences in $[0,1]$ satisfying $\alpha_n+\gamma_n\le 1$ and $\beta_n+\delta_n\le 1$ for all $n\ge 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K, and x_0 is a given point in K.

Observe that if K is a nonempty closed convex subset of E and $\{T_i\}_{i=1}^m : K\to K$ be m uniformly L_i-Lipschitzian mappings. If $\alpha_n(1+\beta_n(L-1))L<1$, where $L=\max\{L_1,L_2,\dots,L_m\}$, then for given $x_{n-1}\in K$, $\gamma_n u_n$ and $\delta_n v_n\in K$, the mapping $S_n : K\to K$ defined by

$$S_n(x) = (1-\alpha_n-\gamma_n)x_{n-1}+\alpha_n T_n^n\{(1-\beta_n-\delta_n)x+\beta_n T_n^n x+\delta_n v_n\}+\gamma_n u_n, \quad \forall n\ge 1 \tag{1.13}$$

is a contractive mapping. In fact, the following are observed

$$\begin{aligned} \|S_n(x)-S_n(y)\| &= \alpha_n\|T_n^n\{(1-\beta_n-\delta_n)x+\beta_n T_n^n x+\delta_n v_n\} \\ &\qquad -T_n^n\{(1-\beta_n-\delta_n)y+\beta_n T_n^n y+\delta_n v_n\}\| \\ &\le \alpha_n L\|(1-\beta_n-\delta_n)(x-y)+\beta_n(T_n^n x-T_n^n y)\| \\ &\le \alpha_n L\{(1-\beta_n-\delta_n)\|x-y\|+\beta_n\|T_n^n x-T_n^n y\|\} \\ &\le \alpha_n L\{(1-\beta_n-\delta_n)\|x-y\|+\beta_n L\|x-y\|\} \\ &\le \alpha_n L(1-\beta_n+\beta_n L)\|x-y\| \\ &\le \alpha_n L(1+\beta_n(L-1))\|x-y\|, \ \forall x,\ y\in K. \end{aligned} \tag{1.14}$$

Since $\alpha_n L(1+\beta_n(L-1))<1$ for all $n\ge 1$, hence $S_n : K\to K$ is a contractive mapping. By Banach contractive mapping principle, there exists a unique fixed point $x_n\in K$ such that

$$\begin{aligned} x_n &= (1-\alpha_n-\gamma_n)x_{n-1}+\alpha_n T_n^n y_n+\gamma_n u_n, \quad n\ge 1, \\ y_n &= (1-\beta_n-\delta_n)x_n+\beta_n T_n^n x_n+\delta_n v_n, \quad n\ge 1. \end{aligned} \tag{1.15}$$

Therefore, if $\alpha_n L(1+\beta_n(L-1)) < 1$ for all $n \geq 1$, then the iterative sequence (1.12) can be employed for the approximation of common fixed points for a finite family of uniformly L-Lipschitzian mappings.

Especially, if $\{\alpha_n\}$ and $\{\gamma_n\}$ are two sequences in $[0,1]$ satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ is a bounded sequence in K, and x_0 is a given point in K, then the sequence $\{x_n\}$ defined by

$$x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_n^n x_n + \gamma_n u_n, \quad \forall n \geq 1 \tag{1.16}$$

is called the one-step implicit iterative sequence with errors for a finite family of operators $\{T_i\}_{i=1}^m$.

The purpose of this paper is, by using a simple and quite different method, to study the convergence of implicit iterative sequence $\{x_n\}$ defined by (1.12) and (1.16) to a common fixed point for a finite family of L-Lipschitzian mappings instead of the assumption that T is a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in a Banach space. Our results extend and improve some recent results in [1–6]. Even in the case of $\gamma_n = \delta_n = 0$, for all $n \geq 1$ or $N = 1$ are also new.

For the main results, the following lemmas are given.

Lemma 1.7 (see Petryshyn [12]). *Let E be a real Banach space and let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y). \tag{1.17}$$

Lemma 1.8 (see Moore and Nnoli [13]). *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ be a real sequence satisfying the following conditions:*

$$0 \leq \lambda_n \leq 1, \qquad \sum_{n=0}^{\infty} \lambda_n = \infty. \tag{1.18}$$

If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \quad \forall n \geq n_0, \tag{1.19}$$

where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative number such that $\sigma_n = \circ(\lambda_n)$, then $\theta_n \to 0$ as $n \to \infty$.

Lemma 1.9. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1+\lambda_n)a_n + b_n, \quad \forall n \geq n_0, \tag{1.20}$$

where $\{\lambda_n\}$ is a sequence in (0, 1) with $\sum_{n=0}^{\infty} \lambda_n < \infty$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2. Main Results

In this section, we shall prove our main theorems in this paper.

Theorem 2.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K$, $i = 1, 2, \ldots, m$ be m uniformly L_i-Lipschitzian mappings with $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, where $F(T_i)$ is the set of fixed points of T_i in K, and let x^* be a point in F. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be four sequences in $[0, 1]$ satisfying the following conditions: $\alpha_n + \gamma_n \leq 1$, $\beta_n + \delta_n \leq 1$, for all $n \geq 1$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K, and let $\{x_n\}$ be the iterative sequence with errors defined by* (1.12)*, then the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,

(iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$,

(iv) $\sum_{n=0}^{\infty} \gamma_n < \infty$,

(v) $\sum_{n=0}^{\infty} \alpha_n \delta_n < \infty$,

(vi) $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$,

(vii) $\alpha_n (1 + \beta_n (L - 1)) L < 1$, *for all $n \geq 1$, where $L = \max\{L_1, L_2, \ldots, L_m\}$.*

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T_i^n x - x^*, j(x - x^*) \rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|), \tag{2.1}$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, $i \in I = \{1, 2, \ldots, m\}$, then $\{x_n\}$ converges strongly to x^*.

Proof. The proof is divided into two steps.

(i) First, we prove that the sequence $\{x_n\}$ defined by (1.12) is bounded.

In fact, it follows from (1.12) and Lemma 1.7 that

$$\begin{aligned}
\|x_n - x^*\|^2 &= \left\| (1 - \alpha_n - \gamma_n)(x_{n-1} - x^*) + \alpha_n (T_n^n y_n - x^*) + \gamma_n (u_n - x^*) \right\|^2 \\
&\leq (1 - \alpha_n - \gamma_n)^2 \|x_{n-1} - x^*\|^2 + 2\alpha_n \langle T_n^n y_n - x^*, j(x_n - x^*) \rangle \\
&\quad + 2\gamma_n \langle u_n - x^*, j(x_n - x^*) \rangle \\
&\leq (1 - \alpha_n)^2 \|x_{n-1} - x^*\|^2 + 2\alpha_n \langle T_n^n x_n - x^*, j(x_n - x^*) \rangle \\
&\quad + 2\alpha_n \langle T_n^n y_n - T_n^n x_n, j(x_n - x^*) \rangle + 2\gamma_n \|u_n - x^*\| \cdot \|x_n - x^*\| \\
&\leq (1 - \alpha_n)^2 \|x_{n-1} - x^*\|^2 + 2\alpha_n \left\{ k_n \|x_n - x^*\|^2 - \phi(\|x_n - x^*\|) \right\} \\
&\quad + 2\alpha_n L \left\| y_n - x_n \right\| \cdot \|x_n - x^*\| + 2\gamma_n M \|x_n - x^*\|,
\end{aligned} \tag{2.2}$$

where

$$M = \max\{\sup\|u_n - x^*\|,\ \sup\|v_n - x^*\|\}. \tag{2.3}$$

Note that

$$\begin{aligned}\|y_n - x_n\| &= \|\beta_n(T_n^n x_n - x_n) + \delta_n(v_n - x_n)\| \\ &\le \beta_n\|T_n^n x_n - x_n\| + \delta_n\|v_n - x_n\| \\ &\le (1+L)\beta_n\|x_n - x^*\| + \delta_n\{\|u_n - x^*\| + \|x_n - x^*\|\} \\ &\le \{(1+L)\beta_n + \delta_n\}\|x_n - x^*\| + \delta_n M \\ &= d_n\|x_n - x^*\| + \delta_n M,\end{aligned} \tag{2.4}$$

where $d_n = (1+L)\beta_n + \delta_n$. By the conditions (iii) and (v), the following are given:

$$\sum_{n=0}^{\infty} \alpha_n d_n < \infty. \tag{2.5}$$

Substituting (2.4) into (2.2), we have

$$\begin{aligned}\|x_n - x^*\|^2 &\le (1-\alpha_n)^2\|x_{n-1} - x^*\|^2 + 2\alpha_n k_n\|x_n - x^*\|^2 - 2\alpha_n\phi(\|x_n - x^*\|) \\ &\quad + 2\alpha_n d_n L\|x_n - x^*\|^2 + 2(\alpha_n\delta_n L + \gamma_n)M\|x_n - x^*\| \\ &\le (1-\alpha_n)^2\|x_{n-1} - x^*\|^2 + 2\alpha_n k_n\|x_n - x^*\|^2 - 2\alpha_n\phi(\|x_n - x^*\|) \\ &\quad + 2\alpha_n d_n L\|x_n - x^*\|^2 + (\alpha_n\delta_n L + \gamma_n)\left\{M^2 + \|x_n - x^*\|^2\right\},\end{aligned} \tag{2.6}$$

and hence

$$\begin{aligned}\|x_n - x^*\|^2 &\le \frac{(1-\alpha_n)^2}{A_n}\|x_{n-1} - x^*\|^2 - \frac{2\alpha_n}{A_n}\cdot\phi(\|x_{n+1} - x^*\|) + \frac{\alpha_n\delta_n L + \gamma_n}{B_n}\cdot M^2 \\ &= \left\{1 + \frac{2\alpha_n(k_n - 1) + \alpha_n^2 + 2\alpha_n d_n L + \alpha_n\delta_n L + \gamma_n}{A_n}\right\}\|x_{n-1} - x^*\|^2 \\ &\quad - \frac{2\alpha_n}{A_n}\cdot\phi(\|x_{n+1} - x^*\|) + \frac{\alpha_n\delta_n L + \gamma_n}{A_n}\cdot M^2,\end{aligned} \tag{2.7}$$

where

$$A_n = 1 - 2\alpha_n k_n - 2\alpha_n d_n L - \alpha_n\delta_n L - \gamma_n. \tag{2.8}$$

Since $\alpha_n \to 0$, $\sum_{n=0}^{\infty}\alpha_n d_n < \infty$, $\sum_{n=0}^{\infty}\alpha_n\delta_n < \infty$, and $\gamma_n \to 0$ as $n \to \infty$, there exists a positive integer n_0 such that $1/2 < A_n \le 1$ for all $n \ge n_0$. Therefore, it follows from (2.7) that

$$\begin{aligned}\|x_n - x^*\|^2 &\le \left\{1 + 2\left[2\alpha_n(k_n - 1) + \alpha_n^2 + 2\alpha_n d_n L + \alpha_n\delta_n L + \gamma_n\right]\right\}\|x_{n-1} - x^*\|^2 \\ &\quad - 2\alpha_n\phi(\|x_n - x^*\|) + 2(\alpha_n\delta_n L + \gamma_n)M^2, \quad \forall n \ge n_0,\end{aligned} \tag{2.9}$$

and so

$$\|x_n - x^*\|^2 \le \Big\{1 + 2\Big[2\alpha_n(k_n - 1) + \alpha_n^2 + 2\alpha_n d_n L + \alpha_n\delta_n L + \gamma_n\Big]\Big\}\|x_{n-1} - x^*\|^2 \\ + 2(\alpha_n\delta_n L + \gamma_n)M^2, \quad \forall n \ge n_0. \tag{2.10}$$

By the conditions (ii), (iv)~(vi), and (2.5), the following are considered:

$$\sum_{n=0}^{\infty} 2\Big[2\alpha_n(k_n - 1) + \alpha_n^2 + 2\alpha_n d_n L + \alpha_n\delta_n L + \gamma_n\Big] < \infty, \\ \sum_{n=0}^{\infty} 2(\alpha_n\delta_n L + \gamma_n)M^2 < \infty. \tag{2.11}$$

It follows from Lemma 1.9 that the limit $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Therefore, the sequence $\{\|x_n - x^*\|\}$ is bounded. Without loss of generality, we can assume that $\|x_n - x^*\|^2 \le M^*$, where M^* is a positive constant.

(ii) Now, we consider (2.9) and prove that $x_n \to x^*$.

Taking $\theta_n = \|x_{n-1} - x^*\|$, $\lambda_n = 2\alpha_n$, and

$$\sigma_n = 2\Big[2\alpha_n(k_n - 1) + \alpha_n^2 + 2\alpha_n d_n L + \alpha_n\delta_n L + \gamma_n\Big]M^* + 2(\alpha_n\delta_n L + \gamma_n)M^2, \tag{2.12}$$

then (2.9) can be written as

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n\phi(\theta_{n+1}) + \sigma_n, \quad \forall n \ge n_0. \tag{2.13}$$

By the conditions (i)~(vi), we know that all the conditions in Lemma 1.8 are satisfied. Therefore, it follows that

$$\lim_{n\to\infty}\|x_n - x^*\| = 0, \tag{2.14}$$

that is, $x_n \to x^*$ as $n \to \infty$. This completes the proof of Theorem 2.1. □

Remark 2.2. (1) Theorem 2.1 extends and improves the corresponding results in Chang [1], Cho et al. [2], Ofoedu [3], Schu [4], and Zeng [5, 6].

(2) The method given by the proof of Theorem 2.1 is quite different from the method given in Ofoedu [3].

(3) Theorem 2.1 extends and improves Theorem 3.2 of Ofoedu [3]; it abolishes the assumption that T is an asymptotically pseudocontractive mapping.

The following theorem can be obtained from Theorem 2.1 immediately.

Theorem 2.3. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, let $T_i : K \to K$, $i = 1, 2, \ldots, m$ be m uniformly L_i-Lipschitzian mappings with $F = \bigcap_{i=1}^m F(T_i) \ne \emptyset$, where $F(T_i)$ is the set of fixed points of T_i in K, and let x^* be a point in F. Let $\{k_n\} \subset [1, \infty)$*

be a sequence with $k_n \to 1$. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two sequences in $[0,1]$ satisfying the following conditions: $\alpha_n + \gamma_n \le 1$, for all $n \ge 1$. Let $\{u_n\}$ be a bounded sequence in K, and let $\{x_n\}$ be the iterative sequence with errors defined by (1.16), then the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$,

(iv) $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$,

(v) $\alpha_n L < 1$, *for all* $\forall\ n \ge 1$, *where* $L = \max\{L_1, L_2, \ldots, L_m\}$.

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T_i^n x - x^*, j(x - x^*)\rangle \le k_n \|x - x^*\|^2 - \phi(\|x - x^*\|), \tag{2.15}$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K, i \in I = \{1, 2, \ldots, m\}$, then $\{x_n\}$ converges strongly to x^*.

Proof. Taking $\beta_n = \delta_n = 0$ in Theorem 2.1, then the conclusion of Theorem 2.3 can be obtained from Theorem 2.1 immediately. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, let $T : K \to K$ be a uniformly L-Lipschitzian mappings with $F = F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in K, and let x^* be a point in F. Let $\{k_n\} \subset [1,\infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two sequences in $[0,1]$ satisfying the following condition: $\alpha_n + \gamma_n \le 1$, $\forall\ n \ge 1$, and let $\{u_n\}$ be a bounded sequence in K satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$,

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$,

(iv) $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$.

(v) $\alpha_n L < 1$, $\forall\ n \ge 1$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_n = (1 - \alpha_n - \gamma_n) x_{n-1} + \alpha_n T^n x_n + \gamma_n u_n. \tag{2.16}$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \le k_n \|x - x^*\|^2 - \phi(\|x - x^*\|), \tag{2.17}$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, then $\{x_n\}$ converges strongly to x^*.

Proof. Taking $m = 1$ in Theorem 2.3, then the conclusion of Theorem 2.4 can be obtained from Theorem 2.3 immediately. This completes the proof of Theorem 2.4. □

Remark 2.5. In Theorem 2.4 without the assumption that T is an asymptotically pseudocontractive mapping, Theorem 2.4 extends and improves Theorem 3.2 of Ofoedu [3].

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