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Research Article

A Fixed Point Approach to the Stability of Pexider Quadratic Functional Equation with Involution

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We apply the fixed point method to investigate the Hyers-Ulam stability of the Pexider functional equation $f(x+y)+g(x+\sigma(y))=h(x)+k(y)$, for all $x,y\in E$, where E is a normed space and $\sigma : E \to E$ is an involution.

1. Introduction and Preliminary

A basic question in the theory of functional equations is as follows. "When is it true that a function, which approximately satisfies a functional equation must be close to an exact solution of the equation?" The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940 and affirmatively answered by Hyers in [2]. Subsequently, the result of Hyers was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. For more information, see [5–7]. Specially, Maligranda [8] and Moszner [9] provided a very interesting discussion on the definition of functional equations' stability.

Recently, the stability of functional equations has been investigated by many mathematicians. They have many applications in the Information Theory, Physics, Economic Theory and Social and Behavior Sciences. See [10–14].

A Hyers-Ulam stability theorem for the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

was proved by Skof [15] for the function $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [16] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group. Czerwik [17] proved the generalized Hyers-Ulam stability of the quadratic functional equation (1.1). Recently, Brzdęk [18], Jung [19], and Jung and Sahoo [20] investigated the Hyers-Ulam-Rassias stability of (1.1). Furthermore they proved the Hyers-Ulam-Rassias stability of the functional equation of Pexider type

$$f_1(x+y)+f_2(x-y)=f_3(x)+f_4(y). \tag{1.2}$$

The stability problem of several functional equations has been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [4, 21–37]).

Let E_1 and E_2 be real vector spaces. If an additive function $\sigma : E_1 \to E_1$ satisfies $\sigma(x+y)=\sigma(x)+\sigma(y)$ and $\sigma(\sigma(x))=x$ for all $x,y\in E_1$, then σ is called an involution of E_1, see [21, 37]. For a given involution $\sigma : E_1 \to E_1$, the functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(y), \quad \forall x,y\in E \tag{1.3}$$

is called the quadratic functional equation with involution. According to [37, Corollary 8], a function $f: E_1 \to E_2$ is a solution of (1.3) if and only if there exists an additive function $A: E_1 \to E_2$, and a biadditive symmetric function $B: E_1\times E_1 \to E_2$ such that $A(\sigma(x))=A(x)$, $B(\sigma(x),y)=-B(\sigma(x),y)$ and $f(x)=B(x,y)+A(x)$ for all $x\in E_1$.

Indeed, if we set $\sigma(x)=I$ in (1.3), where $I: E_1 \to E_1$ denotes the identity function, then (1.3) reduces to the additive functional equation

$$f(x+y)=f(x)+f(y), \quad \forall x,y\in E. \tag{1.4}$$

On the other hand, if $\sigma(x)=-I$ in (1.3), then (1.3) is transformed into the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad \forall x,y\in E. \tag{1.5}$$

Recently, Bouikhalene et al. have proved the Hyers-Ulam-Rassias stability of the quadratic functional equation with involution (1.2), see [21].

In this paper, we will apply the fixed point method to prove the Hyers-Ulam-Rassias stability of the functional equation (1.3) in the Pexider type

$$f(x+y)+g(x+\sigma(y))=2h(x)+2k(y). \tag{1.6}$$

To see the different approaches to the problem of the Pexiderized Cauchy equations' stability and further references concerning that subject we refer to [38–43].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies the following

(1) $d(x, y) = 0$, if and only if $x = y$

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(3) $d(x, z) \le d(x, y) + d(y, z)$, for all $x, y, z \in X$.

For an extensive theory of fixed point and other nonlinear methods, the reader is referred to the book of Hyers et al. [44] and Cădariu and Radu [45].

Theorem 1.1. *Let (X, d) be a generalized complete metric space. Assume that $J : X \to X$ is a strictly contractive operator with the Lipschitz constant $0 < L < 1$. If there exists a nonnegative integer k such that $d(J^{k+1}x, J^k x) < \infty$ for some $x \in X$, then the following are true:*

(a) *the sequence $\{J^n x\}$ converges to a fixed point x^* of J*

(b) *x^* is the unique fixed point of J in*

$$X^* = \left\{ y \in X : d\left(J^k x, x\right) < \infty \right\}; \tag{1.7}$$

(c) *if $y \in X^*$, then*

$$d(y, x^*) \le \frac{1}{1 - L} d(Jy, y). \tag{1.8}$$

2. Main Results

In this section, we prove the Hyers-Ulam-Rassias stability of the quadratic functional equation with involution (1.6) by applying the fixed point method.

Theorem 2.1. *Let E_1 be a commutative semigroup (with the divisibility by 2), and let E_2 be a real Banach space. Suppose that a function $\varphi : E_1 \times E_1 \to [0, \infty)$ is given and there exists a constant L, $0 < L < 1$, such that*

$$\begin{gathered} \varphi(2x, 2y) \le 2L\varphi(x, y), \\ \varphi(x + \sigma(x), y + \sigma(y)) \le 2L\varphi(x, y), \end{gathered} \tag{2.1}$$

for all $x, y \in E_1$. Furthermore, let $f, g, h, k : E_1 \to E_2$ be even functions satisfying the inequality

$$\| f(x + y) + g(x + \sigma(y)) - 2h(x) - 2k(y) \| \le \varphi(x, y) \tag{2.2}$$

for all $x, y \in E_1$, where $\sigma : E_1 \to E_1$ is an involution of E_1 and $f(0) = g(0) = h(0) = k(0) = 0$. Then there exists a unique solution $T : E_1 \to E_2$ of (2.2) *such that*

$$\begin{gathered}
\|2f(x) - T(x)\| \le \frac{1}{4(1-L)} M'(x,x) + M(x,0), \\
\|2g(x) - T(x)\| \le \frac{1}{4(1-L)} M'(x,x) + M(x,0) + \frac{1}{2}(\varphi(x,x) + \varphi(x,-x)), \\
\|h(x) - T(x)\| \le \frac{1}{4(1-L)} M'(x,x), \\
\|k(x) - T(x)\| \le \frac{1}{4(1-L)} M'(x,x) + \frac{1}{4}(\varphi(0,x) + \varphi(x,0)),
\end{gathered} \tag{2.3}$$

for all $x \in E_1$, where

$$\begin{gathered}
M(x,y) = \varphi(x,y) + \varphi(0,y) + \varphi(y,0) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, -\left(\frac{x}{2}\right)\right), \\
M'(x,y) = M(x,y) + M(x+y,0) + M(x+\sigma(y),0), \\
T(x) = \lim_{n\to\infty} \frac{1}{2^{2n}} \left[h(2^n x) + (2^n - 1) h\left(2^{n-1}x + 2^{n-1}\sigma(x)\right)\right].
\end{gathered} \tag{2.4}$$

Proof. Letting $y = 0$ in (2.2), we obtain

$$\|f(x) + g(x) - 2h(x)\| \le \varphi(x,0). \tag{2.5}$$

Similarly, for every $y \in E_1$, we can put $x = 0$ in (2.2) to obtain

$$\|f(y) + g(\sigma(y)) - 2k(y)\| \le \varphi(0,y). \tag{2.6}$$

Since $\sigma : E_1 \to E_1$ is an involution, we replace y by $x + \sigma(x) := d$ in (2.6), then we have

$$\|f(d) + g(d) - 2k(d)\| \le \varphi(0,d). \tag{2.7}$$

Also, we can replace y by x and replace y by $-x$ in (2.2) to get

$$\|f(2x) + g(x+\sigma(x)) - 2h(x) - 2k(x)\| \le \varphi(x,x), \tag{2.8}$$

$$\|g(x-\sigma(x)) - 2h(x) - 2k(-x)\| \le \varphi(x,-x). \tag{2.9}$$

Since $\sigma : E_1 \to E_1$ is an involution, by replacing x by $x + \sigma(x) := d$ in (2.8) and using (2.1), we have

$$\|f(2d) - 2h(d) - 2k(d)\| \le \varphi(d,d). \tag{2.10}$$

Also, by replacing x by $x - \sigma(x) := d$ in (2.9), we have

$$\|g(2d) - 2h(d) - 2k(d)\| \le \varphi(d, -d). \tag{2.11}$$

In view of (2.5) and (2.7), we see that

$$\|h(d) - k(d)\| \le \frac{1}{2}(\varphi(0, d) + \varphi(d, 0)), \tag{2.12}$$

and it follows from (2.10) and (2.11) that

$$\|f(2d) - g(2d)\| \le \varphi(d, d) + \varphi(d, -d). \tag{2.13}$$

By using (2.2), (2.12) and (2.13), we have

$$\begin{aligned}
&\|f(x+y) + f(x+\sigma(y)) - 2h(x) - 2h(y)\| \\
&\quad \le \|f(x+y) + g(x+\sigma(y)) - 2h(x) - 2k(y)\| + 2\|k(y) - h(y)\| \\
&\qquad + \|f(x+\sigma(y)) - g(x+\sigma(y))\| \\
&\quad \le \varphi(x, y) + \varphi(0, y) + \varphi(y, 0) + \varphi\left(\frac{x+\sigma(y)}{2}, \frac{x+\sigma(y)}{2}\right) \\
&\qquad + \varphi\left(\frac{x+\sigma(y)}{2}, -\left(\frac{x+\sigma(y)}{2}\right)\right) \\
&\quad \le \varphi(x, y) + \varphi(0, y) + \varphi(y, 0) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, -\left(\frac{x}{2}\right)\right) = M(x, y).
\end{aligned} \tag{2.14}$$

Therefore

$$\|f(x+y) + f(x+\sigma(y)) - 2h(x) - 2h(y)\| \le M(x, y). \tag{2.15}$$

By putting $y = 0$ in (2.15), we get

$$\|f(x) - h(x)\| \le M(x, 0). \tag{2.16}$$

Hence, (2.15) and (2.16) imply that

$$\begin{aligned}
&\|h(x+y) + h(x+\sigma(y)) - 2h(x) - 2h(y)\| \\
&\quad \le \|f(x+y) + f(x+\sigma(y)) - 2h(x) - 2h(y)\| + \|f(x+y) - h(x+y)\| \\
&\qquad + \|f(x+\sigma(y)) - h(x+\sigma(y))\| \\
&\quad \le M(x, y) + M(x+y, 0) + M(x+\sigma(y), 0) = M'(x, y).
\end{aligned} \tag{2.17}$$

Therefore

$$\|h(x+y)+h(x+\sigma(y))-2h(x)-2h(y)\| \le M'(x,y). \tag{2.18}$$

Now, we define X to be the set of all functions $f : E_1 \to E_2$ and introduce a generalized metric on X as follows:

$$d(g,h)=\inf\{C\in[0,\infty) : \|g(x)-h(x)\| \le CM'(x,x),\ \forall x\in E_1\}. \tag{2.19}$$

Let $\{f_n\}$ be a Cauchy sequence in (X,d). According to the definition of Cauchy sequences, for any given $\epsilon>0$, there exists a positive integer N_ϵ such that

$$d(f_m,f_n)\le\epsilon \tag{2.20}$$

for all $m,n\ge N_\epsilon$.

By considering the definition of the generalized metric d, we see that

$$\forall\epsilon>0,\quad \exists N_\epsilon\in N,\quad \forall m,n\ge N_\epsilon,\quad \forall x\in E_1 : \|f_m(x)-f_n(x)\|\le\epsilon M'(x,x). \tag{2.21}$$

If x is any given point in E_1, (2.21) implies that $\{f_n(x)\}$ is a Cauchy sequence in E_2. Since E_2 is complete, $\{f_n(x)\}$ converges in E_2 for each $x\in E_1$. Hence, we can define a function $f : E_1 \to E_2$ by

$$f(x)=\lim_{n\to\infty} f_n(x). \tag{2.22}$$

We define an operator $J : X \to X$ by

$$JL(x)=\frac{1}{4}[L(2x)+L(x+\sigma(x))] \tag{2.23}$$

for all $x\in E_1$.

First, we assert that J is strictly contractive on X. Given $g,h\in X$, let $C\in[0,\infty)$ be an arbitrary constant with

$$d(g,h)\le C, \tag{2.24}$$

that is,

$$\|g(x)-h(x)\|\le CM'(x,x) \tag{2.25}$$

for all $x\in E_1$.

If we replace y by x in (2.18), then we obtain

$$\|h(2x) + h(x+\sigma(x)) - 4h(x)\| \le M'(x,x) \tag{2.26}$$

for every $x \in E_1$.

It follows from (2.23) and (2.25) that

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \frac{1}{4}\|g(2x) + g(x+\sigma(x)) - h(2x) - h(x+\sigma(x))\| \\ &\le \frac{1}{4}\|g(2x) - h(2x)\| + \frac{1}{4}\|g(x+\sigma(x)) - h(x+\sigma(x))\| \\ &\le \frac{1}{4}CM'(2x,2x) + \frac{1}{4}CM'(x+\sigma(x), x+\sigma(x)) \\ &\le LCM'(x,x)\end{aligned} \tag{2.27}$$

for all $x \in E_1$, that is, $d(Jg, Jh) \le LC$. Hence, we conclude that $d(Jg, Jh) \le Ld\ (g,h)$ for any $g, h \in X$. Therefore, J is strictly contractive because L is a constant with $0 < L < 1$.

Now, we claim that $d(Jh, h) < \infty$. If we put $y = x$ in (2.26) and divide both sides by $1/4$, then we get

$$\|Jh(x) - h(x)\| = \left\|\frac{1}{4}[h(2x) + h(x+\sigma(x)) - h(x)]\right\| \le \frac{1}{4}M'(x,x) \tag{2.28}$$

for all $x \in E_1$, that is,

$$d(Jh, h) \le \frac{1}{4} < \infty. \tag{2.29}$$

Now, by Theorem 1.1 there exists a function $T : E_1 \to E_2$ which is a fixed point of J, such that $d(J^n h, T) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n h)(x) = \frac{1}{2^{2n}}\left[h(2^n x) + (2^n - 1)h\left(2^{n-1}x + 2^{n-1}\sigma(x)\right)\right] \tag{2.30}$$

for each $n \in \mathbf{N}$. Since $d(J^n h, T) \to 0$ as $n \to \infty$, there exists a sequence $\{C_n\}$ such that $C_n \to 0$ as $n \to \infty$ and $d(J^n h, T) \le C_n$ for every $n \in \mathbf{N}$. Hence, by the definition of d, we have

$$\|J^n h(x) - T(x)\| \le C_n M'(x,x) \tag{2.31}$$

for all $x \in E_1$. Thus, for each fixed $x \in E_1$, we have

$$\lim_{n\to\infty}\|J^n h(x) - T(x)\| = 0. \tag{2.32}$$

Therefore

$$T(x) = \lim_{n\to\infty} \frac{1}{2^{2n}} \left[h(2^n x) + (2^n - 1) h\left(2^{n-1}x + 2^{n-1}\sigma(x)\right) \right] \tag{2.33}$$

for all $x \in E_1$. It follows from (2.1), (2.2), and (2.33) that

$$\begin{aligned}
&\|T(x+y) + T(x+\sigma(y)) - 2T(x) - 2T(y)\| \\
&\quad = \lim_{n\to\infty} \frac{1}{2^{2n}} \Big\| h(2^n x + 2^n y) + (2^n - 1) h\left(2^{n-1}(x+y) + 2^{n-1}(\sigma(x)+\sigma(y))\right) \\
&\qquad\qquad + h(2^n x + 2^n \sigma(y)) + (2^n - 1) h\left(2^{n-1}(x+\sigma(y)) + 2^{n-1}(\sigma(x)+y)\right) \\
&\qquad\qquad - 2h(2^n x) + 2(2^n - 1) h\left(2^{n-1}x + 2^{n-1}\sigma(x)\right) \\
&\qquad\qquad -2h(2^n y) + 2(2^n - 1) h\left(2^{n-1}y + 2^{n-1}\sigma(y)\right) \Big\| \\
&\quad \le \lim_{n\to\infty} \frac{1}{2^{2n}} \| h(2^n x + 2^n y) + h(2^n x + 2^n \sigma(y)) - 2h(2^n x) - 2h(2^n y) \| \\
&\qquad + \lim_{n\to\infty} \frac{(2^n - 1)}{2^{2n}} \Big\| h\left(2^{n-1}(x+\sigma(x)) + 2^{n-1}(y+\sigma(y))\right) \\
&\qquad\qquad + h\left(2^{n-1}(x+\sigma(x)) + 2^{n-1}(y+\sigma(y))\right) \\
&\qquad\qquad -2h\left(2^{n-1}x + 2^{n-1}\sigma(x)\right) - 2h\left(2^{n-1}x + 2^{n-1}\sigma(x)\right) \Big\| \\
&\quad \le \lim_{n\to\infty} \frac{1}{2^{2n}} M'(2^n x, 2^n y) + \lim_{n\to\infty} \frac{(2^n - 1)}{2^{2n}} M'(2^n(x+\sigma(x)), 2^n(y+\sigma(y))) = 0
\end{aligned} \tag{2.34}$$

for all $x, y \in E_1$, which implies that T is a solution of (1.6).

By Theorem 1.1 (c) and (2.29), we obtain

$$d(h, T) \le \frac{1}{1-L} d(h, Jh) < \frac{1}{4(1-L)}, \tag{2.35}$$

that is, (2.3) is true for all $x \in E_1$. Assume that $T_1 : E_1 \to E_2$ is another solution of (2.2) satisfying (2.3). We know that T_1 is a fixed point of J. In view of (2.3) and the definition of d, we can conclude that (2.35) is true with T_1 in place of T. Due to Theorem 1.1 (b), we get $T = T_1$. This proves the uniqueness of T.

By (2.16), (2.28), we obtain

$$\begin{aligned}\|f(x)-T(x)\| &\le \|f(x)-h(x)\|+\|h(x)-T(x)\|\\ &\le M(x,0)+\frac{1}{4(1-L)}M'(x,x).\end{aligned}\tag{2.36}$$

Also by (2.12), (2.28), we obtain

$$\begin{aligned}\|k(x)-T(x)\| &\le \|k(x)-h(x)\|+\|h(x)-T(x)\|\\ &\le \frac{1}{4}\big(\varphi(0,x)+\varphi(x,0)\big)+\frac{1}{4(1-L)}M'(x,x),\end{aligned}\tag{2.37}$$

and by (2.13), (2.28), we obtain

$$\begin{aligned}\|g(x)-T(x)\| &\le \|f(x)-g(x)\|+\|f(x)-T(x)\|\\ &\le \frac{1}{2}\big(\varphi(x,x)+\varphi(x,-x)\big)+M(x,0)+\frac{1}{4(1-L)}M'(x,x).\end{aligned}\tag{2.38}$$

□

In the following, we will investigate some special cases of Theorem 2.1.

Remark 2.2. Let E_1 be a commutative semigroup (with the divisibility by 2), and let E_2 be a a real Banach space. Suppose that a function $\varphi : E_1 \times E_1 \to [0,\infty)$ is given and there exists a constant L, $0 < L < 1$, such that

$$\begin{aligned}\varphi(x,y) &\le \frac{L}{8}\varphi(2x,2y),\\ \varphi(x+\sigma(x),y+\sigma(y)) &\le \frac{L}{4}\varphi(2x,2y),\end{aligned}\tag{2.39}$$

for all $x,y \in E_1$. Furthermore, let $f,g,h,k : E_1 \to E_2$ be even functions satisfying the inequality

$$\|f(x+y)+g(x+\sigma(y))-2h(x)-2k(y)\| \le \varphi(x,y)\tag{2.40}$$

for all $x, y \in E_1$, where $\sigma : E_1 \longrightarrow E_1$ is an involution of E_1 and $f(0) = g(0) = h(0) = k(0) = 0$. Then there exists a unique solution $T : E_1 \longrightarrow E_2$ of (2.40) such that

$$\begin{aligned}
\|2f(x) - T(x)\| &\leq \frac{L}{4(1-L)} M'(x,x) + M(x,0), \\
\|2g(x) - T(x)\| &\leq \frac{L}{4(1-L)} M'(x,x) + M(x,0) + \frac{1}{2}(\varphi(x,x) + \varphi(x,-x)), \\
\|h(x) - T(x)\| &\leq \frac{L}{4(1-L)} M'(x,x), \\
\|k(x) - T(x)\| &\leq \frac{L}{4(1-L)} M'(x,x) + \frac{1}{4}(\varphi(0,x) + \varphi(x,0))
\end{aligned} \tag{2.41}$$

for all $x \in E_1$, where

$$\begin{aligned}
M(x,y) &= \varphi(x,y) + \varphi(0,y) + \varphi(y,0) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, -\left(\frac{x}{2}\right)\right), \\
M'(x,y) &= M(x,y) + M(x+y,0) + M(x+\sigma(y),0), \\
T(x) &= \lim_{n\to\infty}\left(2^{2n}\left[h\left(\frac{x}{2^n}\right) + \left(\frac{1}{2^n} - 1\right)h\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right)\right]\right).
\end{aligned} \tag{2.42}$$

Remark 2.3. Let E_1 and E_2 be real Banach spaces. Let the hypotheses of Theorem 2.1 hold. If we put

$$\phi(x,y) = \delta, \quad \delta > 0 \tag{2.43}$$

for all $x, y \in E_1$, then, there exists a unique solution $T : E_1 \longrightarrow E_2$ such that

$$\begin{aligned}
\|2f(x) - T(x)\| &\leq \frac{25}{2}\delta, \\
\|2g(x) - T(x)\| &\leq \frac{27}{2}\delta, \\
\|h(x) - T(x)\| &\leq \frac{15}{2}\delta, \\
\|K(x) - T(x)\| &\leq 8\delta,
\end{aligned} \tag{2.44}$$

for all $x \in E_1$, where

$$T(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}\left[h(2^n x) + (2^n - 1)h\left(2^{n-1}x + 2^{n-1}\sigma(x)\right)\right]. \tag{2.45}$$

Also, if we put $\phi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$ for $0 \le p < 1$ and $\epsilon > 0$, then there exists a unique solution $T : E_1 \to E_2$ such that

$$\begin{aligned}
\|2f(x) - T(x)\| &\le \frac{1}{2(2-2^p)}\epsilon\left(10 + \frac{3+(-1)^p}{2^{p-1}} + 2(-2)^p\right)\epsilon\|x\|^p \\
&\quad + \left(1 + \frac{3+(-1)^p}{2^p}\right)\epsilon\|x\|^p, \\
\|2g(x) - T(x)\| &\le \frac{1}{2(2-2^p)}\epsilon\left(10 + \frac{3+(-1)^p}{2^{p-1}} + 2(-2)^p\right)\epsilon\|x\|^p \\
&\quad + \left(\frac{5}{2} + \frac{3+(-1)^p}{2^p} + \frac{(-1)^p}{2}\right)\epsilon\|x\|^p, \\
\|h(x) - T(x)\| &\le \frac{1}{2(2-2^p)}\epsilon\left(10 + \frac{3+(-1)^p}{2^{p-1}} + 2(-2)^p\right)\epsilon\|x\|^p, \\
\|k(x) - T(x)\| &\le \frac{1}{2(2-2^p)}\epsilon\left(10 + \frac{3+(-1)^p}{2^{p-1}} + 2(-2)^p\right)\epsilon\|x\|^p + \frac{1}{2}\epsilon\|x\|^p.
\end{aligned} \tag{2.46}$$

Similarly, let $\epsilon, p, q \ge 0$ be real numbers such that $p+q<1$. If we put $\phi(x,y) = \epsilon(\|x\|^p)(\|y\|^q)$ (see [22, 28]), then there exists a unique solution $T : E_1 \to E_2$ such that

$$\begin{aligned}
\|2f(x) - T(x)\| &\le \frac{1}{4-2^{p+q+1}}\left(5 + \frac{1}{2^{p+q-1}}\right)\epsilon\left(\|x\|^{p+q}\right) + \frac{\epsilon}{2^{p+q}}\left(\|x\|^{p+q}\right), \\
\|2g(x) - T(x)\| &\le \frac{1}{4-2^{p+q+1}}\left(5 + \frac{1}{2^{p+q-1}}\right)\epsilon\left(\|x\|^{p+q}\right) + \left(\frac{1}{2^{p+q}} + 1\right)\epsilon\left(\|x\|^{p+q}\right), \\
\|h(x) - T(x)\| &\le \frac{1}{4-2^{p+q+1}}\left(5 + \frac{1}{2^{p+q-1}}\right)\epsilon\left(\|x\|^{p+q}\right), \\
\|k(x) - T(x)\| &\le \frac{1}{4-2^{p+q+1}}\left(5 + \frac{1}{2^{p+q-1}}\right)\epsilon\left(\|x\|^{p+q}\right).
\end{aligned} \tag{2.47}$$

Let $\epsilon, p, q \ge 0$ be real numbers such that $p+q<1$. Another control function is $\epsilon(\|x\|^p\|y\|^q + \|x\|^{p+q} + \|y\|^{p+q})$ (see [35]). Then, there exists a unique solution $T : E_1 \to E_2$ such that

$$\begin{aligned}
\|2f(x) - T(x)\| &\le \frac{1}{4-2^{p+q+1}}\left(15 + \frac{9}{2^{p+q-1}}\right)\|x\|^{p+q} + \left(1 + \frac{3}{2^{p+q-1}}\right)\epsilon\|x\|^{p+q}, \\
\|2g(x) - T(x)\| &\le \frac{1}{4-2^{p+q+1}}\left(15 + \frac{9}{2^{p+q-1}}\right)\|x\|^{p+q} + \left(4 + \frac{3}{2^{p+q-1}}\right)\epsilon\|x\|^{p+q}, \\
\|h(x) - T(x)\| &\le \frac{1}{4-2^{p+q+1}}\left(15 + \frac{9}{2^{p+q-1}}\right)\|x\|^{p+q}, \\
\|k(x) - T(x)\| &\le \frac{1}{4-2^{p+q+1}}\left(15 + \frac{9}{2^{p+q-1}}\right)\|x\|^{p+q} + \frac{1}{2}\epsilon\|x\|^{p+q}
\end{aligned} \tag{2.48}$$

for all $x \in E_1$.

Theorem 2.4. *Let E_1 be a commutative semigroup (with the divisibility by 2) and let E_2 be a real Banach space. Suppose that a function $\varphi : E_1 \times E_1 \to [0,\infty)$ is given and there exists a constant L, $0 < L < 1$, such that*

$$\begin{gathered}\varphi(2x,2y) \le L\varphi(x,y),\\ \varphi(x+\sigma(x), y+\sigma(y)) \le L\varphi(x,y),\end{gathered} \tag{2.49}$$

for all $x,y \in E_1$. Furthermore, let $f,g,h,k : E_1 \to E_2$ be odd functions satisfying the inequality

$$\|f(x+y) + g(x+\sigma(y)) - h(x) - k(y)\| \le \varphi(x,y), \tag{2.50}$$

for all $x,y \in E_1$, where $\sigma : E_1 \to E_1$ is an involution of E_1. Then, there exists a unique solution $T : E_1 \to E_2$ of (2.50) such that

$$\begin{gathered}\|f(x) - T(x)\| \le \frac{1}{4(1-L)}M'(x,x) + \frac{3}{2}(\varphi(x,0)+\varphi(0,x)),\\ \|g(x) - T(x)\| \le \frac{1}{2(1-L)}M'(x,x) + 2\varphi(x,0) + \varphi(0,x),\\ \|h(x) - T(x)\| \le \frac{1}{2(1-L)}M'(x,x),\\ \|k(x) - T(x)\| \le \frac{1}{2(1-L)}M'(x,x) + \frac{1}{2}(\varphi(x,0)+\varphi(0,x)),\end{gathered} \tag{2.51}$$

for all $x \in E_1$, where

$$\begin{aligned}M(x,y) &= \varphi(x,y) + \varphi(x+y,0) + \varphi(0,x+y) + \frac{3}{2}(\varphi(x,0)+\varphi(0,x))\\ &\quad + \varphi(x,-x) + \frac{1}{2}(\varphi(y,0)+\varphi(0,y)) + \varphi(y,-y),\\ M'(x,y) &= M(x,y) + M(x,\sigma(y)),\\ T(x) &= \lim_{n\to\infty}\frac{1}{2^n}\Big[h(2^n x) + (2^n-1)h\Big(2^{n-1}x + 2^{n-1}\sigma(x)\Big)\Big].\end{aligned} \tag{2.52}$$

Proof. As in the Theorem 2.1, if we put $y = 0$, $x = 0$ (and replace y by x), $y = x$, and $y = -x$ in (2.50) separately, then we obtain

$$\|f(x) + g(x) - 2h(x)\| \le \varphi(x,0), \tag{2.53}$$

$$\|f(y) + g(\sigma(y)) - 2k(y)\| \le \varphi(0,y), \tag{2.54}$$

$$\|f(2x) + g(x+\sigma(x)) - 2h(x) - 2k(x)\| \le \varphi(x,x), \tag{2.55}$$

$$\|g(x-\sigma(x)) - 2h(x) + 2k(x)\| \le \varphi(x,-x), \tag{2.56}$$

for all $x \in E_1$, respectively.

We replace y by $x - \sigma(x) := d$ in (2.54). Since $\sigma : E_1 \to E_1$ is an involution, then

$$\|f(d) - g(d) - 2k(d)\| \leq \varphi(0, d). \tag{2.57}$$

Also, replace y by $x + \sigma(x) := d$ in (2.54), then

$$\|f(d) + g(d) - 2k(d)\| \leq \varphi(0, d). \tag{2.58}$$

Also, we replace x by $x - \sigma(x) := d$ in (2.55), then

$$\|f(2d) - 2h(d) - 2k(d)\| \leq \varphi(d, d). \tag{2.59}$$

We replace x by $x - \sigma(x) := d$ in (2.56), then

$$\|g(2d) - 2h(d) + 2k(d)\| \leq \varphi(d, -d). \tag{2.60}$$

Due to (2.53) and (2.57), we have

$$\|2f(d) - 2h(d) - 2k(d)\| \leq \varphi(d, 0) + \varphi(0, d), \tag{2.61}$$

$$\|2g(d) - 2h(d) + 2k(d)\| \leq \varphi(d, 0) + \varphi(0, d). \tag{2.62}$$

Combining (2.54) with (2.61) yields

$$\|h(2d) + k(2d) - 2h(d) - 2k(d)\| \leq \frac{1}{2}\big(\varphi(d, 0) + \varphi(0, d)\big) + \varphi(d, d). \tag{2.63}$$

Due to (2.60) and (2.62), we have

$$\|h(2d) - k(2d) - 2h(d) + 2k(d)\| \leq \frac{1}{2}\big(\varphi(d, 0) + \varphi(0, d)\big) + \varphi(d, -d). \tag{2.64}$$

Now, it follows from (2.63) and (2.64) that

$$\begin{aligned} \|h(2d) - 2h(d)\| &\leq \frac{1}{2}\big(\varphi(d, 0) + \varphi(0, d)\big) + \varphi(d, -d), \\ \|k(2d) - 2k(d)\| &\leq \frac{1}{2}\big(\varphi(d, 0) + \varphi(0, d)\big) + \varphi(d, -d). \end{aligned} \tag{2.65}$$

By (2.50), (2.61), (2.62), and (2.65), we have

$$\begin{aligned}
&\|h(x+y)+k(x+y)+h(x+\sigma(y))-k(x+\sigma(y))-h(2x)-k(2y)\| \\
&\quad\le \|f(x+y)+g(x+\sigma(y))-2h(x)-2k(y)\| \\
&\qquad+\|h(x+y)+k(x+y)-f(x+y)\| \\
&\qquad+\|h(x+\sigma(y))-k(x+\sigma(y))-g(x+\sigma(y))\| \\
&\qquad+\|h(2x)-2h(x)\|+\|k(2y)-2k(y)\| \\
&\quad\le \varphi(x,y)+\varphi(x+y,0)+\varphi(0,x+y)+\frac{3}{2}(\varphi(x,0)+\varphi(0,x)) \\
&\qquad+\varphi(x,-x)+\frac{1}{2}(\varphi(y,0)+\varphi(0,y))+\varphi(y,-y)=M(x,y)
\end{aligned} \tag{2.66}$$

for all $x,y\in E_1$. If we replace y in (2.66) by $\sigma(y)$, we get

$$\|h(x+\sigma(y))+k(x+\sigma(y))+h(x+y)-k(x+y)-h(2x)-k(2\sigma(y))\|\le M(x,\sigma(y)). \tag{2.67}$$

By (2.66) and (2.67), we get

$$\|2h(x+y)+2h(x+\sigma(y))-h(2x)-h(2\sigma(y))\|\le M(x,y)+M(x,\sigma(y)). \tag{2.68}$$

If we replace y in (2.66) by $\sigma(y)$ and combine (2.53) with (2.58), we get

$$\|2h(x+y)+2h(x+\sigma(y))-h(2x)-h(2y)\|\le M(x,y)+M(x,\sigma(y)). \tag{2.69}$$

By (2.65) and (2.69), we get

$$\begin{aligned}
\|2h(x+y)+2h(x+\sigma(y))-2h(x)-2h(y)\| &\le M(x,y)+M(x,\sigma(y)) \\
&= M'(x,y).
\end{aligned} \tag{2.70}$$

Therefore

$$\|h(x+y)+h(x+\sigma(y))-h(x)-h(y)\|\le M'(x,y). \tag{2.71}$$

We define X to be the set of all functions $f: E_1 \to E_2$ and introduce a generalized metric on X as follows:

$$d(g,h)=\inf\{C\in[0,\infty):\|g(x)-h(x)\|\le CM'(x,x),\ \forall x\in E_1\}. \tag{2.72}$$

Also, we define an operator $J : X \to X$ by

$$JL(x) = \frac{1}{2}[L(2x) + L(x + \sigma(x))] \tag{2.73}$$

for all $x \in E_1$.

Then, there exists a function $T : E_1 \to E_2$ which is a fixed point of J, such that $d(J^n h, T) \to 0$ as $n \to \infty$. By induction, we can show that

$$(J^n h)(x) = \frac{1}{2^n}\Big[h(2^n x) + (2^n - 1)h\Big(2^{n-1}x + 2^{n-1}\sigma(x)\Big)\Big] \tag{2.74}$$

for each $n \in \mathbf{N}$. Since $d(J^n h, T) \to 0$ as $n \to \infty$, there exists a sequence $\{C_n\}$ such that $C_n \to 0$ as $n \to \infty$ and $d(J^n h, T) \le C_n$ for every $n \in \mathbf{N}$. Hence,

$$\|J^n h(x) - T(x)\| \le C_n M'(x, x) \tag{2.75}$$

for all $x \in E_1$. Thus, for each fixed $x \in E_1$, we have

$$\lim_{n\to\infty} \|J^n h(x) - T(x)\| = 0. \tag{2.76}$$

Then

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n}\Big[h(2^n x) + (2^n - 1)h\Big(2^{n-1}x + 2^{n-1}\sigma(x)\Big)\Big] \tag{2.77}$$

for all $x \in E_1$. Hence, T is a solution of (2.51). □

Remark 2.5. Let E_1 and E_2 be real Banach spaces. Let the hypotheses of Theorem 2.4 hold. If we replace the control function $\phi(x, y)$ by $\delta > 0$, then, there exists a unique solution $T : E_1 \to E_2$ such that

$$\begin{aligned} \|f(x) - T(x)\| &\le 12\delta, \\ \|g(x) - T(x)\| &\le 21\delta, \\ \|h(x) - T(x)\| &\le 18\delta, \\ \|k(x) - T(x)\| &\le 19\delta, \end{aligned} \tag{2.78}$$

for all $x \in E_1$, where

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n}\Big[h(2^n x) + (2^n - 1)h\Big(2^{n-1}x + 2^{n-1}\sigma(x)\Big)\Big]. \tag{2.79}$$

If we put $\phi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$ in which $\epsilon > 0$ and p is a nonnegative number less than 1, then, there exists a unique solution $T : E_1 \to E_2$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{2^{p-1}}{(2^p - 1)}\left(12 + 2(-1)^p\right)\epsilon\|x\|^p + 3\left(\epsilon\|x\|^p\right), \\
\|g(x) - T(x)\| &\leq \frac{2^{p}}{(2^p - 1)}\left(12 + 2(-1)^p\right)\epsilon\|x\|^p + 3\left(\epsilon\|x\|^p\right), \\
\|h(x) - T(x)\| &\leq \frac{2^{p}}{(2^p - 1)}\left(12 + 2(-1)^p\right)\epsilon\|x\|^p, \\
\|k(x) - T(x)\| &\leq \frac{2^{p}}{(2^p - 1)}\left(12 + 2(-1)^p\right)\epsilon\|x\|^p + \left(\epsilon\|x\|^p\right).
\end{aligned} \tag{2.80}$$

Let $\epsilon > 0$ and $p, q > 2$. If we put $\phi(x,y) = \epsilon(\|x\|^p)(\|y\|^q)$, then, there exists a unique solution $T : E_1 \to E_2$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{3}{2\left(2^{(p+q+2)} - 1\right)}\epsilon\|x\|^{p+q}, \\
\|g(x) - T(x)\| &\leq \frac{3}{\left(2^{(p+q+2)} - 1\right)}\epsilon\|x\|^{p+q}, \\
\|h(x) - T(x)\| &\leq \frac{3}{\left(2^{(p+q+2)} - 1\right)}\epsilon\|x\|^{p+q}, \\
\|k(x) - T(x)\| &\leq \frac{3}{\left(2^{(p+q+2)} - 1\right)}\epsilon\|x\|^{p+q}.
\end{aligned} \tag{2.81}$$

Finally, if we put $\phi(x,y) = \epsilon(\|x\|^p\|y\|^q + \|x\|^{p+q} + \|y\|^{p+q})$, then, there exists a unique solution $T : E_1 \to E_2$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{9}{2\left(2^{(p+q+2)} - 1\right)}\epsilon\|x\|^{p+q}, \\
\|g(x) - T(x)\| &\leq \frac{9}{\left(2^{(p+q+2)} - 1\right)}\epsilon\|x\|^{p+q}, \\
\|h(x) - T(x)\| &\leq \frac{9}{\left(2^{(p+q+2)} - 1\right)}\epsilon\|x\|^{p+q}, \\
\|k(x) - T(x)\| &\leq \frac{9}{\left(2^{(p+q+2)} - 1\right)}\epsilon\|x\|^{p+q}
\end{aligned} \tag{2.82}$$

for all $x \in E_1$.

Remark 2.6. The methods of proofs, used in this paper, can be also applied to the problem of stability of (1.6) on the restricted domain, analogously as in the papers [18, 46]. Also, this corresponds to the results in [38, 41–43], where the Pexiderized equations' stability on restricted domains has been investigated.

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