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Research Article

Some Iterative Methods for Solving Equilibrium Problems and Optimization Problems

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We introduce a new iterative scheme for finding a common element of the set of solutions of the equilibrium problems, the set of solutions of variational inequality for a relaxed cocoercive mapping, and the set of fixed points of a nonexpansive mapping. The results presented in this paper extend and improve some recent results of Ceng and Yao (2008), Yao (2007), S. Takahashi and W. Takahashi (2007), Marino and Xu (2006), Iiduka and Takahashi (2005), Su et al. (2008), and many others.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, C is a nonempty closed and convex subset of H, and P_C is the metric projection of H onto C. In the following, we denote by "$\to$" strong convergence, by "$\rightharpoonup$" weak convergence, and by "$\mathbb{R}$" the real number set. Recall that a mapping $S : C \to C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

We denote by $F(S)$ the set of fixed points of the mapping S.

For a given nonlinear operator A, consider the problem of finding $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C, \tag{1.2}$$

which is called the variational inequality. For the recent applications, sensitivity analysis, dynamical systems, numerical methods, and physical formulations of the variational inequalities, see [1–24] and the references therein.

For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C, \tag{1.3}$$

if and only if $u = P_C z$, where P_C is the projection of the Hilbert space onto the closed convex set C.

It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.4}$$

Moreover, $P_C x$ is characterized by the properties $P_C x \in C$ and $\langle x - P_C x, P_C x - y \rangle \geq 0$ for all $y \in C$.

Using characterization of the projection operator, one can easily show that the variational inequality (1.2) is equivalent to finding the fixed point problem of finding $u \in C$ which satisfies the relation

$$u = P_C[u - \lambda A u], \tag{1.5}$$

where $\lambda > 0$ is a constant.

This fixed-point formulation has been used to suggest the following iterative scheme. For a given $u_0 \in C$,

$$u_{n+1} = P_C[u_n - \lambda A u_n], \quad n = 1, 2, \ldots, \tag{1.6}$$

which is known as the projection iterative method for solving the variational inequality (1.2). The convergence of this iterative method requires that the operator A must be strongly monotone and Lipschitz continuous. These strict conditions rule out their applications in many important problems arising in the physical and engineering sciences. To overcome these drawbacks, Noor [2, 3] used the technique of updating the solution to suggest the two-step (or predictor-corrector) method for solving the variational inequality (1.2). For a given $u_0 \in C$,

$$\begin{aligned} w_n &= P_C[u_n - \lambda A u_n], \\ u_{n+1} &= P_C[w_n - \lambda A w_n], \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1.7}$$

which is also known as the modified double-projection method. For the convergence analysis and applications of this method, see the works of Noor [3] and Y. Yao and J.-C. Yao [16].

Numerous problems in physics, optimization, and economics reduce to find a solution of (2.12). Some methods have been proposed to solve the equilibrium problem; see [4, 5]. Combettes and Hirstoaga [4] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Very recently, S. Takahashi and W. Takahashi [6] also introduced a new iterative scheme,

$$F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, \tag{1.8}$$

for approximating a common element of the set of fixed points of a nonexpansive nonself mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see [7–11] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.9}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S, and b is a given point in H. In [10, 11], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n b, \quad n \geq 0, \tag{1.10}$$

converges strongly to the unique solution of the minimization problem (1.9) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [8] introduced a new iterative scheme by the viscosity approximation method [12]:

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.11}$$

They proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C, \tag{1.12}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.13}$$

where C is the fixed point set of a nonexpansive mapping S and h a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

For finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of variational inequalities for α-cocoercive map, Takahashi and Toyoda [13] introduced the following iterative process:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_C(x_n - \lambda_n A x_n), \tag{1.14}$$

for every $n = 0, 1, 2, \ldots$, where A is α-cocoercive, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in (0,1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap \mathrm{VI}(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.14) converges weakly to some $z \in F(S) \cap \mathrm{VI}(C, A)$. Recently, Iiduka and Takahashi [14] proposed another iterative scheme as follows:

$$x_{n+1} = \alpha_n x + (1-\alpha_n) SP_C(x_n - \lambda_n A x_n), \tag{1.15}$$

for every $n = 0, 1, 2, \ldots$, where A is α-cocoercive, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in (0,1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that the sequence $\{x_n\}$ converges strongly to $z \in F(S) \cap \mathrm{VI}(C, A)$.

Recently, Chen et al. [15] studied the following iterative process:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) SP_C(x_n - \lambda_n A x_n) \tag{1.16}$$

and also obtained a strong convergence theorem by viscosity approximation method.

Inspired and motivated by the ideas and techniques of Noor [2, 3] and Y. Yao and J.-C. Yao [16] introduce the following iterative scheme.

Let C be a closed convex subset of real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $z \in F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Suppose that $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ are given by

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n A y_n), \end{aligned} \tag{1.17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are the sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ defined by (1.17) converges strongly to common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings under some parameters controlling conditions.

In this paper motivated by the iterative schemes considered in [6, 15, 16], we introduce a general iterative process as follows:

$$\begin{aligned} &F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n) I - \alpha_n A\right) SP_C(I - s_n B) y_n, \end{aligned} \tag{1.18}$$

where A is a linear bounded operator and B is relaxed cocoercive. We prove that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the variational inequalities for a relaxed cocoercive mapping, and the set of solutions of the equilibrium problems (2.12), which solves another variational inequality

$$\langle \gamma f(q) - Aq, q - P \rangle \leq 0, \quad \forall p \in F, \tag{1.19}$$

where $F = F(S) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F)$ and is also the optimality condition for the minimization problem $\min_{x \in F}(1/2)\langle Ax, x \rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). The results obtained in this paper improve and extend the recent ones announced by S. Takahashi and W. Takahashi [6], Iiduka and Takahashi [14], Marino and Xu [8], Chen et al. [15], Y. Yao and J.-C. Yao [16], Ceng and Yao [22], Su et al. [17], and many others.

2. Preliminaries

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Recall the following.

(1) B is called ν-strong monotone if for all $x, y \in C$, we have

$$\langle Bx - By, x - y \rangle \geq \nu \|x - y\|^2, \tag{2.1}$$

for a constant $\nu > 0$. This implies that

$$\|Bx - By\| \geq \nu \|x - y\|, \tag{2.2}$$

that is, B is ν-expansive, and when $\nu = 1$, it is expansive.

(2) B is said to be μ-cocoercive [2, 3] if for all $x, y \in C$, we have

$$\langle Bx - By, x - y \rangle \geq \mu \|Bx - By\|^2, \quad \text{for a constant } \mu > 0. \tag{2.3}$$

Clearly, every μ-cocoercive map B is $1/\mu$-Lipschitz continuous.

(3) B is called $-\mu$-cocoercive if there exists a constant $\mu > 0$ such that

$$\langle Bx - By, x - y \rangle \geq -\mu \|Bx - By\|^2, \quad \forall x, y \in C. \tag{2.4}$$

(4) B is said to be relaxed (μ, ν)-cocoercive if there exists two constants $\mu, \nu > 0$ such that

$$\langle Bx - By, x - y \rangle \geq -\mu \|Bx - By\|^2 + \nu \|x - y\|^2, \quad \forall x, y \in C, \tag{2.5}$$

for $\mu = 0$, B is ν-strongly monotone. This class of maps are more general than the class of strongly monotone maps. It is easy to see that we have the following implication: ν-strongly monotonicity $\Rightarrow$ relaxed (μ, ν)-cocoercivity.

We will give the practical example of the relaxed (μ, ν)-cocoercivity and Lipschitz continuous operator.

Example 2.1. Let $Tx = \kappa x$, for all $x \in C$, for a constant $\kappa > 1$; then, T is relaxed (μ, ν)-cocoercive and Lipschitz continuous. Especially, T is ν-strong monotone.

Proof. 1. Since $Tx = \kappa x$, for all $x \in C$, we have $T : C \to C$. For for all $x, y \in C$, for all $\mu \geq 0$, we also have the below

$$\begin{aligned} \langle Tx - Ty, x - y \rangle &= \kappa \|x - y\|^2 \\ &\geq -\mu \|Tx - Ty\|^2 + (\kappa - 1)\|x - y\|^2. \end{aligned} \tag{2.6}$$

Taking $\nu = \kappa - 1$, it is clear that T is relaxed (μ, ν)-cocoercive.

2. Obviously, for for all $x, y \in C$

$$\|Tx - Ty\| \leq (\kappa + 1)\|x - y\|. \tag{2.7}$$

Then, T is $\kappa + 1$ Lipschitz continuous.

Especially, Taking $\mu = 0$, we observe that

$$\langle Tx - Ty, x - y \rangle \geq (\kappa - 1)\|x - y\|^2. \tag{2.8}$$

Obviously, T is ν-strong monotone.

The proof is completed. □

(5) A mapping $f : H \to H$ is said to be a contraction if there exists a coefficient α $(0 \leq \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in H. \tag{2.9}$$

(6) An operator A is strong positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{2.10}$$

(7) A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \overline{\in} C. \end{cases} \tag{2.11}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, B)$; see [1].

Related to the variational inequality problem (1.2), we consider the equilibrium problem, which was introduced by Blum and Oettli [19] and Noor and Oettli [20] in 1994. To be more precise, let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

For given bifunction $F(\cdot,\cdot) : C \times C \to \mathbb{R}$, we consider the problem of finding $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C \tag{2.12}$$

which is known as the equilibrium problem. The set of solutions of (2.12) is denoted by $\mathrm{EP}(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $x \in \mathrm{EP}(F)$ if and only if $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$, that is, x is a solution of the variational inequality. That is to say, the variational inequality problem is included by equilibrium problem, and the variational inequality problem is the special case of equilibrium problem.

Assume that $\overline{T}$ is a potential function for T (i.e., $\nabla\overline{T}(x) = T(x)$ for all $x \in C$), it is well known that $x \in C$ satisfies the optimality condition $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$ if and only if

$$\text{find a point } x \in C \text{ such that } \overline{T}x = \min_{y \in C} \overline{T}(y). \tag{2.13}$$

We can rewrite the variational inequality $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$ as, for any $\gamma > 0$,

$$\langle x - (x - \gamma Tx), y - x\rangle \geq 0 \quad \forall y \in C. \tag{2.14}$$

If we introduce the nearest point projection P_C from H onto C,

$$P_C x = \arg\min_{u \in C} \frac{1}{2}\|x - u\|^2, \quad x \in H, \tag{2.15}$$

which is characterized by the inequality

$$C \ni \hat{x} = P_C x \Longleftrightarrow \langle x - \hat{x}, y - \hat{x}\rangle \leq 0, \quad \forall y \in C, \tag{2.16}$$

then we see from the above (2.14) that the minimization (2.13) is equivalent to the fixed point problem

$$P_C(x - \gamma Tx) = x. \tag{2.17}$$

Therefore, they have a relation as follows:

$$\text{finding } x \in C, \quad x \in \mathrm{EP}(F)$$
$$\Updownarrow$$
$$\text{Finding } x \in C, \quad F(x,y) \geq 0, \quad \forall y \in C, \quad \text{let } F(x,y) = \langle \gamma T x, y - x \rangle \geq 0, \quad \forall \gamma > 0, \ \forall y \in C.$$
$$\Updownarrow$$
$$\min_{y \in C} \overline{T}(y), \quad \text{where } \nabla \overline{T}(x) = T(x), \ \forall x \in C.$$
$$\Updownarrow$$
$$x \in \mathrm{Fix}(P_C(I - \gamma T)). \tag{2.18}$$

In addition to this, based on the result (3) of Lemma 2.7, $\mathrm{Fix}(T_r) = \mathrm{EP}(F)$, we know if the element $x \in F := \mathrm{Fix}(S) \cap \mathrm{EP}(F) \cap \mathrm{VI}(C, B)$, we have x is the solution of the nonlinear equation

$$x - SP_C(I - \gamma B)T_r x = 0, \quad \forall \gamma > 0, \tag{2.19}$$

where T_r is defined as in Lemma 2.7. Once we have the solutions of the equation (2.19), then it simultaneously solves the fixed points problems, equilibrium points problems, and variational inequalities problems. Therefore, the constrained set $F := \mathrm{Fix}(S) \cap \mathrm{EP}(F) \cap \mathrm{VI}(C, B)$ is very important and applicable.

We now recall some well-known concepts and results. It is well-known that for all $x, y \in H$ and $\lambda \in [0, 1]$ there holds

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.20}$$

A space X is said to satisfy Opial's condition [18] if for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x. \tag{2.21}$$

Lemma 2.2 (see [9, 10]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \tag{2.22}$$

where γ_n is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n \to \infty} \delta_n / \gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} \alpha_n = 0$.

Lemma 2.3. *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.23}$$

Lemma 2.4 (Marino and Xu [8]). *Assume that B is a strong positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.5 (see [21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.6 (Blum and Oettli [19]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.24}$$

Lemma 2.7 (Combettes and Hirstoaga [4]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\} \tag{2.25}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$ which satisfies (A1)–(A4), let S be a nonexpansive mapping of C into H, and let B be a λ-Lipschitzian, relaxed (μ, ν)-cocoercive map of C into H such that $F = F(S) \cap EP(F) \cap VI(C, B) \neq \emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{gathered} F(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \quad \forall \eta \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right) S P_C(I - s_n B) y_n \end{gathered} \tag{3.1}$$

for all n, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\}, \{s_n\} \subset [0,\infty)$ satisfy

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C5) $\liminf_{n\to\infty} r_n > 0$;

(C6) $\{s_n\} \in [a,b]$ *for some* a, b *with* $0 \leq a \leq b \leq 2(\nu - \mu\lambda^2)/\lambda^2$.

Then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))(q)$, which solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F. \tag{3.2}$$

Proof. Note that from the condition (C1), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on H, then

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}. \tag{3.3}$$

observe that

$$\begin{aligned}\langle ((1-\beta_n)I - \alpha_n A)x, x\rangle &= 1 - \beta_n - \alpha_n\langle Ax, x\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \\ &\geq 0,\end{aligned} \tag{3.4}$$

that is to say $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle ((1-\beta_n)I - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned} \tag{3.5}$$

First, we show that $I - s_nB$ is nonexpansive. Indeed, from the relaxed (μ, ν)-cocoercive and λ-Lipschitzian definition on B and condition (C6), we have

$$\begin{aligned}\|(I - s_nB)x - (I - s_nB)y\|^2 &= \|(x - y) - s_n(Bx - By)\|^2 \\ &= \|x - y\|^2 - 2s_n\langle x - y, Bx - By\rangle + s_n^2\|Bx - By\|^2 \\ &\leq \|x - y\|^2 - 2s_n\left[-\mu\|Bx - By\|^2 + \nu\|x - y\|^2\right] + s_n^2\|Bx - By\|^2 \\ &\leq \|x - y\|^2 + 2s_n\lambda^2\mu\|x - y\|^2 - 2s_n\nu\|x - y\|^2 + \lambda^2 s_n^2\|x - y\|^2 \\ &= \left(1 + 2s_n\lambda^2\mu - 2s_n\nu + \lambda^2 s_n^2\right)\|x - y\|^2 \\ &\leq \|x - y\|^2,\end{aligned} \tag{3.6}$$

which implies that the mapping $I - s_nB$ is nonexpansive.

Now, we observe that $\{x_n\}$ is bounded. Indeed, take $p \in F$, since $y_n = T_{r_n}x_n$, we have

$$\|y_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|. \tag{3.7}$$

Put $\rho_n = P_C(I - s_nB)y_n$, since $p \in \text{VI}(C, B)$, we have $p = P_C(I - s_nB)p$. Therefore, we have

$$\begin{aligned}\|\rho_n - p\| &= \|\text{P}_C(I - s_nB)y_n - P_C(I - s_nB)p\| \\ &\leq \|(I - s_nB)y_n - (I - s_nB)p\| \\ &\leq \|y_n - p\| \leq \|x_n - p\|.\end{aligned} \tag{3.8}$$

Due to (3.5), it follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_nA)(S\rho_n - p)\| \\ &\leq (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \alpha_n\|\gamma f(x_n) - Ap\| \\ &\leq (1 - \alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| \\ &\leq (1 - \alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\ &= (1 - (\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|.\end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}\right\}, \quad n \geq 0. \tag{3.10}$$

Hence, $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, y_n, and ρ_n.

Next, we show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.11}$$

Observing that $y_n=T_{r_n}x_n$ and $y_{n+1}=T_{r_{n+1}}x_{n+1}$, we have

$$F(y_n,\eta)+\frac{1}{r_n}\langle \eta-y_n, y_n-x_n\rangle\geq 0,\quad \forall\eta\in C, \tag{3.12}$$

$$F(y_{n+1},\eta)+\frac{1}{r_{n+1}}\langle \eta-y_{n+1}, y_{n+1}-x_{n+1}\rangle\geq 0,\quad \forall\eta\in C. \tag{3.13}$$

Putting $\eta=y_{n+1}$ in (3.12) and $\eta=y_n$ in (3.13), we have

$$\begin{aligned}F(y_n,y_{n+1})+\frac{1}{r_n}\langle y_{n+1}-y_n, y_n-x_n\rangle\geq 0,\quad \forall\eta\in C,\\ F(y_{n+1},y_n)+\frac{1}{r_{n+1}}\langle y_n-y_{n+1}, y_{n+1}-x_{n+1}\rangle\geq 0,\quad \forall\eta\in C.\end{aligned} \tag{3.14}$$

It follows from (A2) that

$$\left\langle y_{n+1}-y_n, \frac{y_n-x_n}{r_n}-\frac{y_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle\geq 0. \tag{3.15}$$

That is,

$$\left\langle y_{n+1}-y_n, y_n-y_{n+1}+y_{n+1}-x_n-\frac{r_n}{r_{n+1}}(y_{n+1}-x_{n+1})\right\rangle\geq 0. \tag{3.16}$$

Without loss of generality, let us assume that there exists a real number m such that $r_n>m>0$ for all n. It follows that

$$\|y_{n+1}-y_n\|^2\leq\|y_{n+1}-y_n\|\left(\|x_{n+1}-x_n\|+\left(1-\frac{r_n}{r_{n+1}}\right)\|(y_{n+1}-x_{n+1})\|\right). \tag{3.17}$$

It follows that

$$\begin{aligned}\|y_{n+1}-y_n\|&\leq\|x_{n+1}-x_n\|+\left(1-\frac{r_n}{r_{n+1}}\right)\|(y_{n+1}-x_{n+1})\|\\ &\leq\|x_{n+1}-x_n\|+\frac{M_1}{m}|r_{n+1}-r_{n+1}|,\end{aligned} \tag{3.18}$$

where M_1 is an appropriate constant such that $\sup_{n\geq 1}\|y_n - x_n\| \leq M_1$. Note that

$$\begin{aligned}\|\rho_{n+1} - \rho_n\| &= \|P_C(I - s_{n+1}B)y_{n+1} - P_C(I - s_nB)y_n\| \\ &\leq \|(I - s_{n+1}B)y_{n+1} - (I - s_nB)y_n\| \\ &= \|(I - s_{n+1}B)y_{n+1} - (I - s_{n+1}B)y_n + (s_n - s_{n+1})By_n\| \\ &\leq \|y_{n+1} - y_n\| + |s_n - s_{n+1}|\|By_n\|.\end{aligned} \tag{3.19}$$

Substituting (3.18) into (3.19) yields that

$$\|\rho_{n+1} - \rho_n\| \leq \|x_{n+1} - x_n\| + M_2(|r_{n+1} - r_n| + |s_{n+1} - s_n|), \tag{3.20}$$

where M_2 is an appropriate constant such that $M_2 = \max\{\sup_{n\geq 1}\|By_n\|, M_1/m\}$.

Define

$$x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n, \quad n \geq 0. \tag{3.21}$$

Observe that from the definition of y_n, we obtain

$$\begin{aligned}z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1 - \beta_{n+1})I - \alpha_{n+1}A)S\rho_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(x_n) + ((1 - \beta_n)I - \alpha_n A)S\rho_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\gamma f(x_{n+1}) - \frac{\alpha_n}{1 - \beta_n}\gamma f(x_n) + \frac{\alpha_n}{1 - \beta_n}AS\rho_n \\ &\quad - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}AS\rho_{n+1} + S\rho_{n+1} - S\rho_n \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\gamma f(x_{n+1}) - AS\rho_{n+1}) + \frac{\alpha_n}{1 - \beta_n}(AS\rho_n - \gamma f(x_n))S\rho_{n+1} \\ &\quad - S\rho_n.\end{aligned} \tag{3.22}$$

It follows that with

$$\begin{aligned}\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\|+\|AS\rho_{n+1}\|\right)+\frac{\alpha_n}{1-\beta_n}\left(\|\gamma f(x_n)\|+\|AS\rho_n\|\right)\\ &\quad+\|\rho_{n+1}-\rho_n\|-\|x_{n+1}-x_n\|\\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\|+\|AS\rho_{n+1}\|\right)+\frac{\alpha_n}{1-\beta_n}\left(\|\gamma f(x_n)\|+\|AS\rho_n\|\right)\\ &\quad+M_2\left(|r_{n+1}-r_n|+|s_{n+1}-s_n|\right).\end{aligned} \tag{3.23}$$

This together with (C1), (C3), and (C4) implies that

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\right)\leq 0. \tag{3.24}$$

Hence, by Lemma 2.5, we obtain $\|z_n-x_n\|\to 0$ as $n\to\infty$.

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{3.25}$$

Note that

$$\begin{aligned}\|x_{n+1}-x_n\|&=\|\alpha_n(\gamma f(x_n)-Ax_n)+((1-\beta_n)I-\alpha_n A)(S\rho_n-x_n)\|\\ &\leq \alpha_n\|\gamma f(x_n)-Ax_n\|+(1-\beta_n-\alpha_n\overline{\gamma})\|S\rho_n-x_n\|.\end{aligned} \tag{3.26}$$

This together with (3.25) implies that

$$\|S\rho_n-x_n\|\longrightarrow 0. \tag{3.27}$$

For $p\in F$, we have

$$\begin{aligned}\|y_n-p\|^2&=\|T_{r_n}x_n-T_{r_n}p\|^2\\ &\leq\langle T_{r_n}x_n-T_{r_n}p,x_n-p\rangle\\ &=\langle y_n-p,x_n-p\rangle\\ &=\frac{1}{2}\left(\|y_n-p\|^2+\|x_n-p\|^2-\|x_n-y_n\|^2\right)\end{aligned} \tag{3.28}$$

and hence

$$\|y_n-p\|^2\leq\|x_n-p\|^2-\|x_n-y_n\|^2. \tag{3.29}$$

Set $\lambda > 0$ as a constant such that

$$\lambda > \sup_k \{ \|\gamma f(x_k) - AS\rho_k\|, \|x_k - p\| \}. \tag{3.30}$$

By (3.29) and (3.30), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)S\rho_n - p\|^2 \\
&= \|[(1-\beta_n)I - \alpha_n A](S\rho_n - p) + \beta_n(x_n - p) + \alpha_n(\gamma f(x_n) - Ap)\|^2 \\
&= \|(1-\beta_n)(S\rho_n - p) - \alpha_n A(S\rho_n - p) + \beta_n(x_n - p) + \alpha_n(\gamma f(x_n) - Ap)\|^2 \\
&= \|(1-\beta_n)(S\rho_n - p) + \beta_n(x_n - p) + \alpha_n(\gamma f(x_n) - AS\rho_n)\|^2 \\
&\le \|(1-\beta_n)(S\rho_n - p) + \beta_n(x_n - p)\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - AS\rho_n, x_{n+1} - p\rangle \\
&\le \|(1-\beta_n)(S\rho_n - p) + \beta_n(x_n - p)\|^2 + 2\alpha_n\lambda^2 \\
&\le (1-\beta_n)\|S\rho_n - p\|^2 + \beta_n\|x_n - p\|^2 + 2\alpha_n\lambda^2 \\
&\le (1-\beta_n)\|\rho_n - p\|^2 + \beta_n\|x_n - p\|^2 + 2\alpha_n\lambda^2 \\
&\le (1-\beta_n)\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + 2\alpha_n\lambda^2 \\
&\le \|x_n - p\|^2 - (1-\beta_n)\|y_n - x_n\|^2 + 2\alpha_n\lambda^2.
\end{aligned} \tag{3.31}$$

It follows that

$$\begin{aligned}
\|y_n - x_n\|^2 &\le \frac{1}{1-\beta_n}\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\lambda^2\right) \\
&= \frac{1}{1-\beta_n}\left((\|x_n - p\| - \|x_{n+1} - p\|)(\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n\lambda^2\right) \\
&\le \frac{1}{1-\beta_n}\left((\|x_n - x_{n+1}\|)(\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n\lambda^2\right).
\end{aligned} \tag{3.32}$$

By $\|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, as $n \to \infty$, and $\{x_n\}$ is bounded, we obtain that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.33}$$

For $p \in F$, we have

$$\begin{aligned}\|\rho_n - p\|^2 &= \|P_C(I - s_nB)y_n - P_C(I - S_nB)p\|^2 \\ &\leq \|(y_n - p) - s_n(By_n - Bp)\|^2 \\ &= \|y_n - p\|^2 - 2s_n\langle y_n - p, By_n - Bp\rangle + s_n^2\|By_n - Bp\|^2 \\ &\leq \|x_n - p\|^2 - 2s_n\left[-\mu\|By_n - Bp\|^2 + \nu\|y_n - p\|^2\right] + s_n^2\|By_n - Bp\|^2 \\ &\leq \|x_n - p\|^2 + 2s_n\mu\|By_n - Bp\|^2 - 2s_n\nu\|y_n - p\|^2 + s_n^2\|By_n - Bp\|^2 \\ &\leq \|x_n - p\|^2 + \left(2s_n\mu + s_n^2 - \frac{2s_n\nu}{\lambda^2}\right)\|By_n - Bp\|^2.\end{aligned} \tag{3.34}$$

Observe (3.31) that

$$\|x_{n+1} - p\|^2 \leq (1 - \beta_n)\|\rho_n - p\|^2 + \beta_n\|x_n - p\|^2 + 2\alpha_n\lambda^2. \tag{3.35}$$

Substituting (3.34) into (3.35), we have

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + \left(2s_n\mu + s_n^2 - \frac{2s_n\nu}{\lambda^2}\right)\|By_n - Bp\|^2 + 2\alpha_n\lambda^2. \tag{3.36}$$

It follows from condition (C6) that

$$\begin{aligned}\left(\frac{2a\nu}{\lambda^2} - 2b\mu - b^2\right)\|By_n - Bp\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\lambda^2 \\ &\leq \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n\lambda^2.\end{aligned} \tag{3.37}$$

From condition (C1) and (3.25), we have that

$$\lim_{n\to\infty}\|By_n - Bp\| = 0. \tag{3.38}$$

On the other hand, we have

$$\begin{aligned}
\|\rho_n - p\|^2 &= \|P_C(I - s_nB)y_n - P_C(I - S_nB)p\|^2 \\
&\le \langle (I - s_nB)y_n - (I - S_nB)p, \rho_n - p\rangle \\
&= \frac{1}{2}\Big\{\|(I - s_nB)y_n - (I - S_nB)p\|^2 + \|\rho_n - p\|^2 \\
&\qquad -\|(I - s_nB)y_n - (I - S_nB)p - (\rho_n - p)\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|y_n - p\|^2 + \|\rho_n - p\|^2 - \|(y_n - \rho_n) - s_n(By_n - Bp)\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|y_n - \rho_n\|^2 - s_n^2\|By_n - Bp\|^2 \\
&\qquad +2s_n\langle y_n - \rho_n, Ay_n - Ap\rangle\Big\},
\end{aligned} \tag{3.39}$$

which yields that

$$\|\rho_n - p\|^2 \le \|x_n - p\|^2 - \|y_n - \rho_n\|^2 + 2s_n\|y_n - \rho_n\|\|By_n - Bp\|. \tag{3.40}$$

Substituting (3.40) into (3.35) yields that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|x_n - p\|^2 - (1 - \beta_n)\|y_n - \rho_n\|^2 \\
&\qquad + 2s_n\|y_n - \rho_n\|\|By_n - Bp\| + 2\alpha_n\lambda^2.
\end{aligned} \tag{3.41}$$

It follows that

$$\begin{aligned}
\|y_n - \rho_n\|^2 &\le \frac{1}{1-\beta_n}\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right) \\
&\qquad + \frac{2s_n}{1-\beta_n}\|y_n - \rho_n\|\|By_n - Bp\| + \frac{2\alpha_n\lambda^2}{1-\beta_n} \\
&\le \frac{1}{1-\beta_n}\|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\qquad + \frac{2s_n}{1-\beta_n}\|y_n - \rho_n\|\|By_n - Bp\| + \frac{2\alpha_n\lambda^2}{1-\beta_n}.
\end{aligned} \tag{3.42}$$

From condition (C1), (3.25), and (3.38), we have that

$$\lim_{n\to\infty}\|y_n - \rho_n\| = 0. \tag{3.43}$$

Observe that

$$\begin{aligned}\|y_n - Sy_n\| &\leq \|y_n - x_n\| + \|x_n - S\rho_n\| + \|S\rho_n - Sy_n\| \\ &\leq \|y_n - x_n\| + \|x_n - S\rho_n\| + \|\rho_n - y_n\|.\end{aligned} \tag{3.44}$$

From (3.27), (3.33), and (3.43), we have

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \tag{3.45}$$

Observe that $P_F(\gamma f + (I - A))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}\|P_F(\gamma f + (I - A))x - P_F(\gamma f + (I - A))y\| &\leq \|(\gamma f + (I - A))x - (\gamma f + (I - A))y\| \\ &\leq \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\ &\leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\ &= [1 - (\overline{\gamma} - \gamma\alpha)]\|x - y\|.\end{aligned} \tag{3.46}$$

Banach's Contraction Mapping Principle guarantees that $P_F(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$, that is, $q = P_F(\gamma f + (I - A))q$.

Next, we show that

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle \leq 0. \tag{3.47}$$

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle = \limsup_{i\to\infty} \langle \gamma f(q) - Aq, x_{n_i} - q\rangle. \tag{3.48}$$

Correspondingly, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup w$.

Next, we show that $w \in F$. First, we prove $w \in \mathrm{EP}(F)$. Since $y_n = T_{r_n} x_n$, we have

$$F(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \quad \forall \eta \in C. \tag{3.49}$$

It follows from (A2) that,

$$\left\langle \eta - y_n, \frac{y_n - x_n}{r_n} \right\rangle \geq F(\eta, y_n). \tag{3.50}$$

It follows that

$$\left\langle \eta - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(\eta, y_{n_i}). \tag{3.51}$$

Since $(y_{n_i} - x_{n_i})/r_{n_i} \to 0$, $y_{n_i} \rightharpoonup w$, and (A4), we have $F(\eta, w) \leq 0$ for all $\eta \in C$. For t with $0 < t \leq 1$ and $\eta \in C$, let $\eta_t = t\eta + (1-t)w$. Since $\eta \in C$ and $w \in C$, we have $\eta_t \in C$ and hence $F(\eta_t, w) \leq 0$. So, from (A1) and (A4), we have

$$0 = F(\eta_t, \eta_t) \leq tF(\eta_t, \eta) + (1-t)F(\eta_t, w) \leq tF(\eta_t, \eta). \tag{3.52}$$

That is, $F(\eta_t, \eta) \leq 0$. It follows from (A3) that $F(w, \eta) \geq 0$ for all $\eta \in C$ and hence $w \in \mathrm{EP}(F)$. Since Hilbert spaces satisfy Opial's condition, from (3.43), suppose $w \neq Sw$; we have

$$\begin{aligned}
\liminf_{i \to \infty} \|y_{n_i} - w\| &< \liminf_{i \to \infty} \|y_{n_i} - Sw\| \\
&= \liminf_{i \to \infty} \|y_{n_i} - Sy_{n_i} + Sy_{n_i} - Sw\| \\
&\leq \liminf_{i \to \infty} \|Sy_{n_i} - Sw\| \\
&< \liminf_{i \to \infty} \|y_{n_i} - w\|
\end{aligned} \tag{3.53}$$

which is a contradiction. Thus, we have $w \in F(S)$.

Next, let us show that $w \in \mathrm{VI}(C, B)$. Put

$$Tw_1 = \begin{cases} Bw_1 + N_C w_1, & w_1 \in C, \\ \emptyset, & w_1 \overline{\in} C. \end{cases} \tag{3.54}$$

Since B is relaxed (μ, ν)-cocoercive and from condition (C6), we have

$$\langle Bx - By, x - y \rangle \geq (-\mu)\|Bx - By\|^2 + \nu\|x - y\|^2 \geq \left(\nu - \mu\lambda^2\right)\|x - y\|^2 \geq 0, \tag{3.55}$$

which yields that B is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Bw_1 \in N_C w_1$ and $\rho_n \in C$, we have

$$\langle w_1 - \rho_n, w_2 - Bw_1 \rangle \geq 0. \tag{3.56}$$

On the other hand, from $\rho_n = P_C(I - s_n B)y_n$, we have

$$\langle w_1 - \rho_n, \rho_n - (I - s_n B)y_n \rangle \geq 0. \tag{3.57}$$

and hence

$$\left\langle w_1 - \rho_n, \frac{\rho_n - y_n}{s_n} + By_n \right\rangle \geq 0. \tag{3.58}$$

It follows that

$$\begin{aligned}
\langle w_1 - \rho_{n_i}, w_2 \rangle &\geq \langle w_1 - \rho_{n_i}, Bw_1 \rangle \\
&\geq \langle w_1 - \rho_{n_i}, Bw_1 \rangle - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} + By_{n_i} \right\rangle \\
&= \left\langle w_1 - \rho_{n_i}, Bw_1 - \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} - By_{n_i} \right\rangle \\
&= \langle w_1 - \rho_{n_i}, Bw_1 - B\rho_{n_i} \rangle + \langle w_1 - \rho_{n_i}, B\rho_{n_i} - By_{n_i} \rangle \\
&\quad - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} \right\rangle \\
&\geq \langle w_1 - \rho_{n_i}, B\rho_{n_i} - By_{n_i} \rangle - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} \right\rangle,
\end{aligned} \tag{3.59}$$

which implies that $\langle w_1 - w, w_2 \rangle \geq 0$, We have $w \in T^{-1}0$ and hence $w \in \mathrm{VI}(C, B)$. That is, $w \in F$.

Since $q = P_F(\gamma f + (I - A))q$, we have

$$\begin{aligned}
\limsup_{n \to \infty} \langle \gamma f(q) - Aq, x_n - q \rangle &= \limsup_{i \to \infty} \langle \gamma f(q) - Aq, x_{n_i} - q \rangle \\
&= \langle \gamma f(q) - Aq, w - q \rangle \leq 0.
\end{aligned} \tag{3.60}$$

That is, (3.47) holds.

Finally, we show that $x_n \to q$, where $q = P_F(\gamma f + (I - A))q$, which solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F. \tag{3.61}$$

We consider

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|((1-\beta_n)I-\alpha_n A)(S\rho_n - q)+\beta_n(x_n-q)+\alpha_n(\gamma f(x_n)-Aq)\|^2 \\
&= \|((1-\beta_n)I-\alpha_n A)(S\rho_n - q)+\beta_n(x_n-q)\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad +2\beta_n\alpha_n\langle x_n-q,\gamma f(x_n)-Aq\rangle \\
&\quad +2\alpha_n\langle((1-\beta_n)I-\alpha_n A)(S\rho_n-q),\gamma f(x_n)-Aq\rangle \\
&\le ((1-\beta_n I-\alpha_n\overline{\gamma})\|S\rho_n-q\|+\beta_n\|x_n-q\|)^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad +2\beta_n\gamma\alpha_n\langle x_n-q,f(x_n)-f(q)\rangle+2\beta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle \\
&\quad +2(1-\beta_n)\gamma\alpha_n\langle S\rho_n-q,f(x_n)-f(q)\rangle \\
&\quad +2(1-\beta_n)\alpha_n\langle S\rho_n-q,\gamma f(q)-Aq\rangle-2\alpha_n^2\langle A(S\rho_n-q),\gamma f(q)-Aq\rangle,
\end{aligned} \tag{3.62}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \left[(1-\alpha_n\overline{\gamma})^2+2\alpha\beta_n\gamma\alpha_n+2\alpha(1-\beta_n)\gamma\alpha_n\right]\|x_n-q\|^2 \\
&\quad +\alpha_n^2\|\gamma f(x_n)-Aq\|^2+2\beta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle \\
&\quad +2(1-\beta_n)\alpha_n\langle S\rho_n-q,\gamma f(q)-Aq\rangle-2\alpha_n^2\langle A(S\rho_n-q),\gamma f(q)-Aq\rangle \\
&\le [1-2(\overline{\gamma}-\alpha\gamma)\alpha_n]\|x_n-q\|^2+\overline{\gamma}^2\alpha_n^2\|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad +2\beta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\beta_n)\alpha_n\langle S\rho_n-q,\gamma f(q)-Aq\rangle \\
&\quad -2\alpha_n^2\|A(S\rho_n-q)\|\cdot\|\gamma f(q)-Aq\| \\
&= [1-2(\overline{\gamma}-\alpha\gamma)\alpha_n]\|x_n-q\|^2 \\
&\quad +\alpha_n\Big\{\alpha_n\Big(\overline{\gamma}^2\|x_n-q\|^2+\|\gamma f(x_n)-Aq\|^2+2\|A(S\rho_n-q)\|\cdot\|\gamma f(q)-Aq\|\Big) \\
&\qquad +2\beta_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\beta_n)\langle S\rho_n-q,\gamma f(q)-Aq\rangle\Big\}.
\end{aligned} \tag{3.63}$$

Since $\{x_n\}$, $\{f(x_n)\}$, and $\{S\rho_n\}$ are bounded, we can take a constant $M_2>0$ such that

$$M_2 \ge \overline{\gamma}^2\|x_n-q\|^2+\|\gamma f(x_n)-Aq\|^2+2\|A(S\rho_n-q)\|\cdot\|\gamma f(q)-Aq\| \tag{3.64}$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq \left[1 - 2(\overline{\gamma} - \alpha\gamma)\alpha_n\right] \|x_n - q\|^2 + \alpha_n \xi_n, \tag{3.65}$$

where

$$\xi_n = 2\beta_n \langle x_n - q, \gamma f(q) - Aq \rangle + 2(1 - \beta_n) \langle S\rho_n - q, \gamma f(q) - Aq \rangle + \alpha_n M_2. \tag{3.66}$$

From (3.27) and (3.47), we also have

$$\begin{aligned}
\limsup_{n\to\infty} \langle \gamma f(q) - Aq, S\rho_n - q \rangle &= \limsup_{n\to\infty} \langle \gamma f(q) - Aq, S\rho_n - x_n \rangle + \limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q \rangle \\
&\leq \limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q \rangle \\
&\leq 0.
\end{aligned} \tag{3.67}$$

By (C1), (3.47), and (3.67), we get $\limsup_{n\to\infty} \xi_n \leq 0$. Now applying Lemma 2.2 to (3.65) concludes that $x_n \to q$ $(n \to \infty)$.

This completes the proof. □

Remark 3.2. Some iterative algorithms were presented in Yamada [11], Combettes [24], and Iiduka-Yamada [25], for example, the steepest descent method, the hybrid steepest descent method, and the conjugate gradient methods; these methods have common form

$$x_{n+1} = x_n + \omega_n d_n, \tag{3.68}$$

where x_n is the nth approximation to the solution, $\omega_n > 0$ is a step size, and d_n is a search direction. In this paper, We define $T := SP_C(I - sB)T_r$; the method (3.1) will be changed as

$$\begin{aligned}
x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) SP_C(I - s_n B) T x_n \\
&= x_n + (1 - \beta_n)(-x_n + T x_n) + \alpha_n \left[\gamma f(x_n) - AT x_n\right].
\end{aligned} \tag{3.69}$$

Take $\omega_n d_n = (1 - \beta_n)(-x_n + T x_n) + \alpha_n [\gamma f(x_n) - AT x_n]$, the method (3.1) will be changed as (3.68).

Remark 3.3. The computational possibility of the resolvent, T_r, of F in Lemma 2.7 and Theorem 3.1 is well defined mathematically, but, in general, the computation of T_r is very difficult in large-scale finite spaces and infinite spaces.

4. Applications

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$ which satisfies (A1)–(A4); let S be a nonexpansive mapping of C into H such that $F = F(S) \cap \mathrm{EP}(F) \cap \mathrm{VI}(C, B) \neq \emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let $\{x_n\}$ and $\{Y_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{gathered} F(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n \rangle \geq 0, \quad \forall \eta \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)Sy_n, \end{gathered} \tag{4.1}$$

for all n, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\}, \{s_n\} \subset [0, \infty)$ satisfy

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$;

(C5) $\liminf_{n\to\infty}\gamma_n > 0$.

Then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))(q)$, which solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F. \tag{4.2}$$

Proof. Taking $\{s_n\} = 0$ in Theorem 3.1, we can get the desired conclusion easily. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H, let S be a nonexpansive mapping of C into H, and let B be a λ-Lipschitzian, relaxed (μ, ν)-cocoercive map of C into H such that $F = F(S) \cap VI(C, B) \neq \emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)SP_C(I - s_n B)P_C x_n, \tag{4.3}$$

for all n, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\}, \{s_n\} \subset [0, \infty)$ satisfy

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C5) $\liminf_{n\to\infty}\gamma_n > 0$;

(C6) $\{s_n\} \in [a, b]$ *for some a, b with* $0 \leq a \leq b \leq 2(\nu - \mu\lambda^2)/\lambda^2$.

Then, both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in F$, *where* $q = P_F(\gamma f + (I - A))(q)$, *which solves the following variational inequality:*

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F. \tag{4.4}$$

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $\gamma_n = 1$ for all n in Theorem 3.1. Then we have $y_n = P_C x_n$. we can obtain the desired conclusion easily. □

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