Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 961502, 20 pages
doi:10.1155/2010/961502

Research Article

Commutators of Littlewood-Paley Operators on the Generalized Morrey Space

Yanping Chen,[1] Yong Ding,[2] and Xinxia Wang[3]

[1] *Department of Mathematics and Mechanics, Applied Science School, University of Science and Technology Beijing, Beijing 100083, China*
[2] *Laboratory of Mathematics and Complex Systems (BNU), School of Mathematical Sciences, Beijing Normal University, Ministry of Education, Beijing 100875, China*
[3] *The College of Mathematics and System Science, Xinjiang University, Urumqi, Xinjiang 830046, China*

Correspondence should be addressed to Yanping Chen, yanpingch@126.com

Received 6 May 2010; Accepted 11 July 2010

Academic Editor: Shusen Ding

Copyright © 2010 Yanping Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let μ_Ω, μ_S^{ϖ}, and $\mu_\lambda^{*,\varpi}$ denote the Marcinkiewicz integral, the parameterized area integral, and the parameterized Littlewood-Paley g_λ^* function, respectively. In this paper, the authors give a characterization of BMO space by the boundedness of the commutators of μ_Ω, μ_S^{ϖ}, and $\mu_\lambda^{*,\varpi}$ on the generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$.

1. Introduction

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \quad \text{for any } \mu > 0,\ x \in \mathbb{R}^n \setminus \{0\}. \tag{1.1}$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0. \tag{1.2}$$

(c) $\Omega \in \mathrm{Lip}(S^{n-1})$, that is,

$$\left|\Omega(x') - \Omega(y')\right| \leq \left|x' - y'\right|, \quad \text{for any } x', y' \in S^{n-1}. \tag{1.3}$$

In 1958, Stein [1] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{1.4}$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy. \tag{1.5}$$

We refer to see [1, 2] for the properties of μ_Ω.

Let $0 < \varpi < n$ and $\lambda > 1$. The parameterized area integral μ_S^ϖ and the parameterized Littlewood-Paley g_λ^* function $\mu_\lambda^{*,\varpi}$ are defined by

$$\mu_S^\varpi f(x) = \left(\iint_{\Gamma(x)} \left|\frac{1}{t^\varpi}\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \tag{1.6}$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$, and

$$\mu_\lambda^{*,\varpi} f(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^\varpi}\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \tag{1.7}$$

respectively. μ_S^ϖ and $\mu_\lambda^{*,\varpi}$ play very important roles in harmonic analysis and PDE (e.g., see [3–8]).

Before stating our result, let us recall some definitions. For $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, the commutator $[b, \mu_\Omega]$ formed by b and the Marcinkiewicz integral μ_Ω are defined by

$$[b, \mu_\Omega] f(x) = \left(\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x) - b(y)) f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}. \tag{1.8}$$

Let $0 < \varpi < n$ and $\lambda > 1$. The commutator $[b, \mu_S^{\varpi}]$ of μ_S^{ϖ} and the commutator $[b, \mu_\lambda^{*,\varpi}]$ of $\mu_\lambda^{*,\varpi}$ are defined, respectively, by

$$[b, \mu_S^{\varpi}] f(x) = \left(\iint_{\Gamma(x)} \left| \frac{1}{t^{\varpi}} \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \tag{1.9}$$

$$\begin{aligned}[b, \mu_\lambda^{*,\varpi}] f(x) = \Bigg(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \\ \times \left| \frac{1}{t^{\varpi}} \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \Bigg)^{1/2}.\end{aligned} \tag{1.10}$$

Let $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$. It is said that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if

$$\|b\|_* := \sup_{B \subset \mathbb{R}^n} M(b, B) < \infty, \tag{1.11}$$

where $B = B(x, r)$ denotes the ball in $\mathbb{R}^n$ centered at x and with radius r,

$$M(b, B) = \frac{1}{|B|} \int_B |b(x) - b_B| dx, \tag{1.12}$$

and $b_B = (1/|B|) \int_B b(y) dy$.

There are some results about the boundedness of the commutators formed by BMO functions with μ_Ω, μ_S^{ϖ}, and $\mu_\lambda^{*,\varpi}$ (see [7, 9, 10]).

Many important operators gave a characterization of BMO space. In 1976, Coifman et al. [11] gave a characterization of BMO space by the commutator of Riesz transform; in 1982, Chanillo [12] studied the commutator formed by Riesz potential and BMO and gave another characterization of BMO space.

The purpose of this paper is to give a characterization of BMO space by the boundedness of the commutators of μ_Ω, μ_S^{ϖ}, and $\mu_\lambda^{*,\varpi}$ on the generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$.

Definition 1.1. Let $1 < p < \infty$. Suppose that $\varphi : (0, \infty) \to (0, \infty)$ be such that $\varphi(t)$ is nonincreasing and $t^{1/p}\varphi(t)$ is nondecreasing. The generalized Morrey space $L^{p,\varphi}$ is defined by

$$L^{p,\varphi}(\mathbb{R}^n) = \left\{ f \in L_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\varphi}} < \infty \right\}, \tag{1.13}$$

where

$$\|f\|_{L^{p,\varphi}} = \sup_{\substack{x \in \mathbb{R}^n \\ r > 0}} \frac{1}{\varphi(|B(x,r)|)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p}. \tag{1.14}$$

We refer to see [13, 14] for the known results of the generalized Morrey space $L^{p,\varphi}$ for some suitable φ. Noting that $\varphi(t) \equiv t^{-1/p}$, we get the Lebesque space $L^p(\mathbb{R}^n)$. For $\varphi(t) = t^{(\lambda/n-1)/p}$ $(0 < \lambda < n)$, $L^{p,\varphi}(\mathbb{R}^n)$ coincides with the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$.

The main result in this paper is as follows.

Theorem 1.2. *Assume that $\varphi(t)$ is nonincreasing and $t^{1/p}\varphi(t)$ is nondecreasing. Suppose that $[b, \mu_\Omega]$ is defined as* (1.8)*, Ω satisfies* (1.1), (1.2)*, and*

$$\left|\Omega(x') - \Omega(y')\right| \leq \frac{C_1}{\left(\log(2/\left|x' - y'\right|)\right)^{\gamma}}, \quad C_1 > 0,\ \gamma > 1,\ x', y' \in S^{n-1}. \tag{1.15}$$

If $[b, \mu_\Omega]$ is bounded on $L^{p,\varphi}(\mathbb{R}^n)$ for some p $(1 < p < \infty)$, then $b \in BMO(\mathbb{R}^n)$.

Theorem 1.3. *Let $0 < \varpi < n$ and $1 < p < \infty$. Assume that $\varphi(t)$ is nonincreasing and $t^{1/p}\varphi(t)$ is nondecreasing. Suppose that $[b, \mu_S^{\varpi}]$ is defined as* (1.9)*, Ω satisfies* (1.1), (1.2)*, and* (1.15)*. If $[b, \mu_S^{\varpi}]$ is a bounded operator on $L^{p,\varphi}(\mathbb{R}^n)$ for some p $(1 < p < \infty)$, then $b \in BMO(\mathbb{R}^n)$.*

Theorem 1.4. *Let $0 < \varpi < n$, $\lambda > 1$, and $1 < p < \infty$. Assume that $\varphi(t)$ is nonincreasing and $t^{1/p}\varphi(t)$ is nondecreasing. Suppose that $[b, \mu_\lambda^{*,\varphi}]$ is defined as* (1.10)*, Ω satisfies* (1.1), (1.2)*, and* (1.15)*. If $[b, \mu_\lambda^{*,\varpi}]$ is on $L^{p,\varphi}(\mathbb{R}^n)$ for some p $(1 < p < \infty)$, then $b \in BMO(\mathbb{R}^n)$.*

Remark 1.5. It is easy to check that $[b, \mu_S^{\varpi}](f)(x) \leq 2^{\lambda n}[b, \mu_\lambda^{*,\varpi}](f)(x)$ (see, e.g., the proof of (19) in [15, page 89]), we therefore give only the proofs of Theorem 1.2 for $[b, \mu_\Omega]$ and Theorem 1.3 for $[b, \mu_S^{\varpi}]$.

Remark 1.6. It is easy to see that the condition (1.15) is weaker than $\mathrm{Lip}_\beta(S^{n-1})$ for $0 < \beta \leq 1$. In the proof of Theorems 1.2 and 1.3, we will use some ideas in [16]. However, because Marcinkiewicz integral and the parameterized Littlewood-Paley operators are neither the convolution operator nor the linear operators, hence, we need new ideas and nontrivial estimates in the proof.

2. Proof of Theorem 1.2

Let us begin with recalling some known conclusion.

Similar to the proof of [17], we can easily get the following.

Lemma 2.1. *If Ω satisfies conditions* (1.1), (1.2)*, and* (1.15)*, let $\beta > 0$, then for $|x| > 2|y|$, we have*

$$\left|\frac{\Omega(x-y)}{\left|x-y\right|^{\beta}} - \frac{\Omega(x)}{|x|^{\beta}}\right| \leq \frac{C}{|x|^{\beta}\left(\log(|x|/\left|y\right|)\right)^{\gamma}}. \tag{2.1}$$

Now let us return to the proof of Theorem 1.2. Suppose that $[b, \mu_\Omega]$ is a bounded operator on $L^{p,\varphi}(\mathbb{R}^n)$, we are going to prove that $b \in \mathrm{BMO}(\mathbb{R}^n)$.

We may assume that $\|[b,\mu_\Omega]\|_{L^{p,\varphi}\mapsto L^{p,\varphi}} = 1$. We want to prove that, for any $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, the inequality

$$N = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} \left|b(y) - a_0\right| dy \le A(p,\Omega,n,\gamma) \tag{2.2}$$

holds, where $a_0 = |B(x_0,r)|^{-1}\int_{B(x_0,r)} b(y)dy$. Since $[b - a_0, \mu_\Omega] = [b, \mu_\Omega]$, we may assume that $a_0 = 0$. Let

$$f(y) = \left[\operatorname{sgn}(b(y)) - c_0\right]\chi_{B(x_0,r)}(y), \tag{2.3}$$

where $c_0 = (1/|B(x_0,r)|)\int_{B(x_0,r)} \operatorname{sgn}(b(y))dy$. Since $(1/|B(x_0,r)|)\int_{B(x_0,r)} b(y)dy = a_0 = 0$, we can easily get $|c_0| < 1$. Then, f has the following properties:

$$\|f\|_\infty \le 2, \tag{2.4}$$

$$\operatorname{supp} f \subset B(x_0,r), \tag{2.5}$$

$$\int_{\mathbb{R}^n} f(y)dy = 0, \tag{2.6}$$

$$f(y)b(y) > 0, \quad y \in B(x_0,r), \tag{2.7}$$

$$\frac{1}{|B(x_0,r)|}\int_{\mathbb{R}^n} f(y)b(y)dy = N. \tag{2.8}$$

In this proof for $j = 1,\ldots,15$, A_j is a positive constant depending only on Ω, p, n, γ, and A_i $(1 \le i < j)$. Since Ω satisfies (1.2), then there exists an A_1 such that $0 < A_1 < 1$ and

$$\sigma\left(\left\{x' \in S^{n-1} : \Omega(x') \ge \frac{2C_1}{(\log(2/A_1))^\gamma}\right\}\right) > 0, \tag{2.9}$$

where σ is the measure on S^{n-1} which is induced from the Lebesgue measure on $\mathbb{R}^n$. By the condition (1.15), it is easy to see that

$$\Lambda := \left\{x' \in S^{n-1} : \Omega(x') \ge \frac{2C_1}{(\log(2/A_1))^\gamma}\right\} \tag{2.10}$$

is a closed set. We claim that

$$\text{if } x' \in \Lambda \text{ and } y' \in S^{n-1}, \text{ satisfying } \left|x' - y'\right| \le A_1, \text{ then } \Omega(y') \ge \frac{C_1}{(\log(2/A_1))^\gamma}. \tag{2.11}$$

In fact, since $|\Omega(x') - \Omega(y')| \le C_1/(\log(2/|x'-y'|))^\gamma \le C_1/(\log(2/A_1))^\gamma$, note that $\Omega(x') \ge 2(C_1/(\log(2/A_1))^\gamma)$, we can get $\Omega(y') \ge C_1/(\log(2/A_1))^\gamma$. Taking $A_2 > 3/A_1$, let

$$G = \left\{x \in \mathbb{R}^n : |x - x_0| \ge A_2 r,\ (x - x_0)' \in \Lambda\right\}. \tag{2.12}$$

For $x \in G$, we have

$$\begin{aligned}
\left|[b,\mu_\Omega]f(x)\right| &\ge \left|\mu_\Omega(bf)(x)\right| - |b(x)|\left|\mu_\Omega f(x)\right| \\
&= \left\{\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}} b(y) f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\
&\quad - |b(x)|\left\{\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}} f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{1/2} \\
&:= I_1 - I_2.
\end{aligned} \tag{2.13}$$

For I_1, noting that if $y \in B(x_0, r)$, then $|x - x_0| > A_2|y - x_0|$ for $x \in G$. Thus, we have

$$\left|(x-y)' - (x-x_0)'\right| \le 2\frac{|y-x_0|}{|x-x_0|} \le \frac{2}{A_2} < A_1. \tag{2.14}$$

Using (2.11), we get $\Omega((x-y)') \ge C_1/(\log(2/A_1))^\gamma$. Noting that $|x - x_0| \simeq |x - y|$, it follows from (2.5), (2.7), (2.8), and Hölder's inequality that

$$\begin{aligned}
I_1 &\ge \left\{\int_{|x-x_0|}^\infty \left(\int_{B(x_0,r)} \frac{\Omega\left((x-y)'\right) b(y) f(y)}{|x-y|^{n-1}} \chi_{\{|x-y|\le t\}}(y)\,dy\right)^2 \frac{dt}{t^3}\right\}^{1/2} \\
&\ge \left(\int_{|x-x_0|}^\infty \int_{B(x_0,r)} \frac{\Omega\left((x-y)'\right) b(y) f(y)}{|x-y|^{n-1}} \chi_{\{|x-y|\le t\}}\,dy \frac{dt}{t^3}\right)\left(\int_{|x-x_0|}^\infty \frac{dt}{t^3}\right)^{-1/2} \\
&\ge \frac{C_1}{(\log(2/A_1))^\gamma}|x-x_0| \int_{B(x_0,r)} |x-y|^{-n+1} b(y) f(y) \int_{\substack{|x-x_0|\le t \\ |x-y|\le t}} \frac{dt}{t^3}\,dy \\
&\ge \frac{C}{(\log(2/A_1))^\gamma}|x-x_0|^{-n} \int_{B(x_0,r)} b(y) f(y)\,dy \\
&= A_3 N r^n |x-x_0|^{-n}.
\end{aligned} \tag{2.15}$$

For $x \in G$, by $\Omega \in L^\infty(S^{n-1})$, (2.4), (2.5), (2.6), the Minkowski inequality, and Lemma 2.1, we obtain

$$\begin{aligned}
I_2 &= |b(x)| \left\{ \int_0^\infty \left| \int_{\mathbb{R}^n} f(y) \left(\frac{\Omega(x-y)}{|x-y|^{n-1}} \chi_{\{|x-y|\le t\}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \chi_{\{|x-x_0|\le t\}} \right) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2} \\
&\le |b(x)| \left\{ \left(\int_0^\infty \left(\int_{|x-y|\le t<|x-x_0|} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \right. \\
&\quad + \left(\int_0^\infty \left(\int_{|x-x_0|\le t<|x-y|} \frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}} |f(y)| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad \left. + \left(\int_0^\infty \left(\int_{\substack{|x-x_0|\le t \\ |x-y|\le t}} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| |f(y)| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \right\} \\
&\le |b(x)| \left\{ \int_{B(x_0,r)} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \left(\int_{|x-y|\le t<|x-x_0|} \frac{dt}{t^3} \right)^{1/2} dy \right. \\
&\quad + \int_{B(x_0,r)} \frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}} |f(y)| \left(\int_{|x-y|>t\ge|x-x_0|} \frac{dt}{t^3} \right)^{1/2} dy \\
&\quad \left. + \int_{B(x_0,r)} |f(y)| \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \left(\int_{\substack{|x-y|\le t \\ |x-x_0|\le t}} \frac{dt}{t^3} \right)^{1/2} dy \right\} \\
&\le C|b(x)| \left(r^{1/2} \int_{B(x_0,r)} \frac{|f(y)|}{|x-x_0|^{n+1/2}} dy + \int_{B(x_0,r)} \frac{|f(y)|}{|x-x_0|^n (\log(|x-x_0|/r))^\gamma} dy \right) \\
&\le A_4 |b(x)| r^n |x-x_0|^{-n} \left(\log \frac{|x-x_0|}{r} \right)^{-\gamma}.
\end{aligned} \tag{2.16}$$

Let

$$F = \left\{ x \in G : |b(x)| > \frac{A_3 N}{2A_4} \left(\log \frac{|x-x_0|}{r} \right)^\gamma,\ |x-x_0| < N^{1/n} r \right\}. \tag{2.17}$$

Without loss of generality, we may assume that $N > A_2 > 1$, otherwise, we get the desired result. Since $\varphi(t)$ is nonincreasing, it follows that $\varphi(|B(x_0, N^{1/n}r)|) \le \varphi(|B(x_0, r)|) = \varphi(r^n)$. By (2.13), (2.15), and (2.16), we have

$$\begin{aligned}
\|f\|_{L^{p,\varphi}}^p &\ge \|[b,\mu_\Omega]f\|_{L^{p,\varphi}}^p \\
&\ge \frac{1}{(\varphi(|B(x_0,N^{1/n}r)|))^p |B(x_0,N^{1/n}r)|} \int_{|x-x_0|<N^{1/n}r} |[b,\mu_\Omega]f(x)|^p dx \\
&\ge \frac{1}{(\varphi(r^n))^p N r^n} \int_{(G\setminus F)\cap\{|x-x_0|<N^{1/n}r\}} \left(\frac{1}{2}A_3 N r^n |x-x_0|^{-n}\right)^p dx \\
&\ge \frac{1}{(\varphi(r^n))^p N r^n} \int_{\{A_5(|F|+(A_2 r)^n)^{1/n}<|x-x_0|<N^{1/n}r\}\cap G} \left(\frac{1}{2}A_3 N r^n |x-x_0|^{-n}\right)^p dx \\
&= \frac{\omega_{n-1}}{(\varphi(r^n))^p N r^n} \left(\frac{A_3 N r^n}{2}\right)^p \int_{A_5(|F|+(A_2 r)^n)^{1/n}}^{N^{1/n}r} t^{-pn+n-1} dt \\
&\ge \frac{\omega_{n-1}}{(\varphi(r^n))^p} (N r^n)^{p-1} \frac{(A_3/2)^p}{n-np} \left(N^{1-p} r^{n(1-p)} - A_5^{(1-p)n} (|F| + (A_2 r)^n)^{1-p}\right).
\end{aligned} \tag{2.18}$$

Thus,

$$\left(|F| + (A_2 r)^n\right)^{1-p} \le A_6 N^{1-p} r^{n(1-p)} \left(1 + \varphi(r^n)^p \|f\|_{L^{p,\varphi}}^p\right). \tag{2.19}$$

Now, we claim that

$$\|f\|_{L^{p,\varphi}} \le \frac{C}{\varphi(r^n)}, \tag{2.20}$$

where C is independent of r. In fact,

$$\|f\|_{L^{p,\varphi}} = \sup_{\substack{x\in\mathbb{R}^n \\ t>0}} \frac{1}{\varphi(|B(x,t)|)} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p}. \tag{2.21}$$

Now, we consider the $L^{p,\varphi}$ norm of f in the following two cases.

Case 1 ($t > r$). Since $s^{1/p}\varphi(s)$ is nondecreasing in s, then

$$\frac{1}{\varphi(|B(x,t)|)} \frac{1}{|B(x,t)|^{1/p}} \le \frac{1}{\varphi(r^n)} \frac{1}{r^{n/p}}. \tag{2.22}$$

Thus,

$$\begin{aligned}\|f\|_{L^{p,\varphi}} &\leq \sup_{\substack{x\in\mathbb{R}^n\\t>0}} \frac{1}{\varphi(r^n)}\ \frac{1}{r^{n/p}}\left(\int_{B(x,t)}|f(y)|^p dy\right)^{1/p}\\ &= \sup_{\substack{x\in\mathbb{R}^n\\t>0}} \frac{1}{\varphi(r^n)}\frac{1}{r^{n/p}}\left(\int_{B(x,t)\cap B(x_0,r)}|f(y)|^p dy\right)^{1/p}\\ &\leq \frac{C}{\varphi(r^n)}.\end{aligned}\tag{2.23}$$

Case 2 ($t \leq r$). Since $\varphi(s)$ is nonincreasing in s, then

$$\frac{1}{\varphi(|B(x,t)|)} \leq \frac{1}{\varphi(r^n)}.\tag{2.24}$$

Thus,

$$\begin{aligned}\|f\|_{L^{p,\varphi}} &\leq \sup_{\substack{x\in\mathbb{R}^n\\t>0}} \frac{1}{\varphi(r^n)}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}|f(y)|^p dy\right)^{1/p}\\ &\leq \frac{C}{\varphi(r^n)}.\end{aligned}\tag{2.25}$$

Now, (2.20) is established. Then, by (2.19) and (2.20), we get

$$|F| + (A_2 r)^n \geq A_7 N r^n.\tag{2.26}$$

If $N \leq 2A_7^{-1}A_2^n$, then Theorem 1.2 is proved. If $N > 2A_7^{-1}A_2^n$, then

$$|F| \geq \frac{A_7}{2} N r^n.\tag{2.27}$$

Let $g(y) = \chi_{B(x_0,r)}(y)$. For $x \in F$, we have

$$\begin{aligned}\left|[b,\mu_\Omega]g(x)\right| &\geq |b(x)|\left\{\int_0^\infty \left|\int_{|x-y|\leq t}\frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}}g(y)dy\right|^2\frac{dt}{t^3}\right\}^{1/2}\\ &\quad - \left\{\int_0^\infty \left|\int_{|x-y|\leq t}\frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}}b(y)g(y)dy\right|^2\frac{dt}{t^3}\right\}^{1/2}\\ &:= K_1 - K_2.\end{aligned}\tag{2.28}$$

Noting that if $y \in B(x_0, r)$ and $x \in F$, we get $|(x-y)' - (x-x_0)'| \le A_1$. Applying (2.11), we have $\Omega((x-y)') \ge C_1/(\log(2/A_1))^\gamma$. Since $|x-y| \simeq |x-x_0|$ when $y \in B(x_0, r)$ and $x \in F$, it follows that

$$\begin{aligned}
K_1 &\ge |b(x)| \left\{ \int_{|x-x_0|}^{\infty} \left(\int_{B(x_0,r)} \frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}} \chi_{\{|x-y|\le t\}}(y)\, dy \right)^2 \frac{dt}{t^3} \right\}^{1/2} \\
&\ge |b(x)| \left(\int_{|x-x_0|}^{\infty} \int_{B(x_0,r)} \frac{\Omega\left((x-y)'\right)}{|x-y|^{n-1}} \chi_{\{|x-y|\le t\}}\, dy \frac{dt}{t^3} \right) \left(\int_{|x-x_0|}^{\infty} \frac{dt}{t^3} \right)^{-1/2} \\
&\ge \frac{C_1 |b(x)|}{(\log(2/A_1))^\gamma} |x-x_0| \int_{B(x_0,r)} |x-y|^{-n+1} \int_{\substack{|x-x_0|\le t \\ |x-y|\le t}} \frac{dt}{t^3}\, dy \\
&\ge A_8 |b(x)| |x-x_0|^{-n} \int_{B(x_0,r)} dy \\
&= A_8 r^n |b(x)| |x-x_0|^{-n}.
\end{aligned} \tag{2.29}$$

By $\Omega \in L^\infty(S^{n-1})$, $|x-x_0| \simeq |x-y|$ when $y \in B(x_0, r)$ and $x \in F$ and the Minkowski inequality, we have

$$\begin{aligned}
K_2 &\le C \int_{B(x_0,r)} \frac{|b(y)|}{|x-y|^n}\, dy \\
&\le A_9 |x-x_0|^{-n} \int_{B(x_0,r)} |b(y)|\, dy \\
&= A_9 N r^n |x-x_0|^{-n}.
\end{aligned} \tag{2.30}$$

Thus, by (2.28), (2.29), and (2.30), we get, for $x \in F$,

$$\left| [b, \mu_\Omega] g(x) \right| \ge A_8 r^n |b(x)| |x-x_0|^{-n} - A_9 N r^n |x-x_0|^{-n}. \tag{2.31}$$

Similar to the proof of (2.20), we can easily get $\|g\|_{L^{p,\varphi}} \le C/\varphi(r^n)$. Thus, by (2.31),

$\varphi(Nr^n) \leq \varphi(r^n)$, and $|b(x)| > (NA_3/2A_4)(\log(|x-x_0|/r))^\gamma$ when $x \in F$, we have

$$
\begin{aligned}
\frac{A_{10}}{\varphi(r^n)} &\geq \|g\|_{L^{p,\varphi}} \geq \|[b,\mu_\Omega]g\|_{L^{p,\varphi}} \\
&\geq \frac{1}{\varphi(Nr^n)(Nr^n)^{1/p}}\left(\int_{|x-x_0|<N^{1/n}r}\left|[b,\mu_\Omega]g(x)\right|^p dx\right)^{1/p} \\
&\geq \frac{1}{\varphi(r^n)(Nr^n)^{1/p}}\int_{|x-x_0|<N^{1/n}r}\left|[b,\mu_\Omega]g(x)\right|dx\left(\int_{|x-x_0|<N^{1/n}r}dx\right)^{-1/p'} \\
&\geq \frac{1}{\varphi(r^n)Nr^n}\int_F\left|[b,\mu_\Omega]g(x)\right|dx \\
&\geq \frac{A_8 r^n}{\varphi(r^n)Nr^n}\int_F|b(x)||x-x_0|^{-n}dx - \frac{A_9 Nr^n}{\varphi(r^n)Nr^n}\int_F|x-x_0|^{-n}dx \\
&\geq \frac{A_{11}}{\varphi(r^n)}\int_F\left(\log\frac{|x-x_0|}{r}\right)^\gamma|x-x_0|^{-n}dx \\
&\quad - \frac{A_9}{\varphi(r^n)}\int_F|x-x_0|^{-n}dx \\
&:= L_1 - L_2.
\end{aligned}
\tag{2.32}
$$

We first estimate L_2. Since $A_2 r < |x-x_0| < N^{1/n}r$ for $x \in F$, we have

$$
L_2 \leq \frac{A_9\omega_{n-1}}{\varphi(r^n)}\int_{A_2 r}^{N^{1/n}r}\rho^{-1}d\rho \leq \frac{A_{12}}{\varphi(r^n)}\log N. \tag{2.33}
$$

Now, the estimate of L_1 is divided into two cases, namely, 1: $\gamma \geq n$; 2: $1 < \gamma < n$.

Case 1 ($\gamma \geq n$). Since the function $\log s/s$ is decreasing for $s \geq 3$ and $3r < A_2 r < |x-x_0| < N^{1/n}r$ for $x \in F$, by (2.27), we get

$$
\begin{aligned}
L_1 &= \frac{A_{11}r^{-n}}{\varphi(r^n)}\int_F\left(\frac{\log(|x-x_0|/r)}{|x-x_0|/r}\right)^n\left(\log\frac{|x-x_0|}{r}\right)^{\gamma-n}dx \\
&\geq \frac{A_7 A_{11}}{2\varphi(r^n)}(\log A_2)^{\gamma-n}N\left(\frac{\log N^{1/n}}{N^{1/n}}\right)^n \\
&\geq \frac{A_{13}}{\varphi(r^n)}(\log N)^n.
\end{aligned}
\tag{2.34}
$$

Case 2 $(1 < \gamma < n)$. Since the function $(\log s)^{\gamma}/s^{n}$ is decreasing for $s \geq 3$ and $3r < A_2 r < |x - x_0| < N^{1/n} r$ for $x \in F$, by (2.27), we have

$$L_1 = \frac{A_{11} r^{-n}}{\varphi(r^n)} \int_F \frac{(\log(|x - x_0|/r))^{\gamma}}{(|x - x_0|/r)^n} dx \geq \frac{A_7 A_{11}}{2\varphi(r^n)} N \frac{(\log N^{1/n})^{\gamma}}{N} \geq \frac{A_{14}}{\varphi(r^n)} (\log N)^{\gamma}. \tag{2.35}$$

From Cases 1 and 2, we know that there exists a constant $\tau > 1$ such that

$$L_1 \geq \frac{A_{15}}{\varphi(r^n)} (\log N)^{\tau}. \tag{2.36}$$

So by (2.32), (2.33), and (2.36), we get

$$A_{10} \geq A_{15} (\log N)^{\tau} - A_{12} \log N. \tag{2.37}$$

Then, $N \leq A(\Omega, p, n, \gamma)$. Theorem 1.2 is proved.

3. Proof of Theorem 1.3

Similar to the proof of Theorem 1.2, we only give the outline.

Suppose that $[b, \mu_S^{\varpi}]$ is a bounded operator on $L^{p,\varphi}(\mathbb{R}^n)$, we are going to prove that $b \in \mathrm{BMO}(\mathbb{R}^n)$.

We may assume that $\|[b, \mu_S^{\varpi}]\|_{L^{p,\varphi} \mapsto L^{p,\varphi}} = 1$. We want to prove that, for any $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, the inequality

$$N = \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} \left| b(y) - a_0 \right| dy \leq B(\Omega, p, n, \varpi) \tag{3.1}$$

holds, where $a_0 = |B(x_0, r)|^{-1} \int_{B(x_0,r)} b(y) dy$. Since $[b - a_0, \mu_S^{\varpi}] = [b, \mu_S^{\varpi}]$, we may assume that $a_0 = 0$. Let $f(y)$ be as (2.3), then (2.4)–(2.8) hold. In this proof for $j = 1, \ldots, 13$, B_j is a positive constant depending only on Ω, p, n, ϖ, and B_i $(1 \leq i < j)$. Since Ω satisfies (1.2), then there exists a B_1 such that $0 < B_1 < 1$ and

$$\sigma\left(\left\{ x' \in S^{n-1} : \Omega(x') \geq \frac{2C_1}{(\log(2/B_1))^{\gamma}} \right\} \right) > 0, \tag{3.2}$$

where σ is the measure on S^{n-1} which is induced from the Lebesgue measure on $\mathbb{R}^n$. By the condition (1.15), it is easy to see that

$$\Lambda := \left\{ x' \in S^{n-1} : \Omega(x') \geq \frac{2C_1}{(\log(2/B_1))^{\gamma}} \right\} \tag{3.3}$$

is a closed set. As the proof of (2.11), we can get the following:

$$\text{if } x' \in \Lambda \text{ and } y' \in S^{n-1}, \text{ satisfying } \left|x' - y'\right| \leq B_1, \text{ then } \Omega(y') \geq \frac{C_1}{(\log(2/B_1))^{\gamma}}. \tag{3.4}$$

Taking $B_2 > 3/B_1 + 1$, let

$$G = \left\{ x \in \mathbb{R}^n : |x - x_0| \geq B_2 r, \ (x - x_0)' \in \Lambda \right\}. \tag{3.5}$$

For $x \in G$, we have

$$\begin{aligned}
\left|[b, \mu_S^{\varpi}] f(x)\right| &= \left(\int_0^{\infty} \int_{|x-y|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dydt}{t^{n+1+2\varpi}} \right)^{1/2} \\
&\geq \left(\int_{4|x-x_0|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \right. \\
&\qquad \times \left. \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} (b(x) - b(z)) f(z) dz \right|^2 \frac{dydt}{t^{n+1+2\varpi}} \right)^{1/2} \\
&\geq \left(\int_{4|x-x_0|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|,\, (y-x_0)' \in \Lambda}} \right. \\
&\qquad \times \left. \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} b(z) f(z) dz \right|^2 \frac{dydt}{t^{n+1+2\varpi}} \right)^{1/2} \\
&\quad - |b(x)| \left(\int_{4|x-x_0|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} f(z) dz \right|^2 \frac{dydt}{t^{n+1+2\varpi}} \right)^{1/2} \\
&:= I_1 - I_2.
\end{aligned} \tag{3.6}$$

For I_1, noting that if $|z-x_0|<r$, $|x-x_0|>B_2|z-x_0|$, and $|y-x_0|>2B_2|z-x_0|$, then we get

$$\left|(y-z)'-(y-x_0)'\right|\le 2\frac{|z-x_0|}{|y-x_0|}\le\frac{1}{B_2}<B_1. \tag{3.7}$$

Then by (3.4), we get $\Omega((y-z)')\ge C_1/(\log(2/B_1))^{\gamma}$. Since $4|x-x_0|>|y-x_0|+|z-x_0|\ge|y-z|\ge|y-x_0|-|z-x_0|>2|x-x_0|-|x-x_0|/2=3|x-x_0|/2$ and $4|x-x_0|>|x-y|\ge|y-x_0|-|x-x_0|>|x-x_0|$, we get $4|x-x_0|\ge|y-z|\ge 3|x-x_0|/2$ and $4|x-x_0|>|x-y|>|x-x_0|$. Thus, by (2.5), (2.7), (2.8), and the Hölder inequality, we get

$$\begin{aligned}
I_1&\ge C\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\int_{B(x_0,r)}\frac{\Omega(y-z)}{|y-z|^{n-\varpi}}b(z)f(z)\chi_{\{(|y-z|)<t\}}dz\frac{dy\,dt}{t^{n+1+2\varpi}}\\
&\quad\times\left(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\frac{dy\,dt}{t^{n+1+2\varpi}}\right)^{-1/2}\\
&\ge C|x-x_0|^{2\varpi-n}\int_{B(x_0,r)}b(z)f(z)\int_{\substack{(y-x_0)'\in\Lambda\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\int_{\substack{4|x-x_0|<t,\,|x-y|<t\\ |y-z|<t}}\frac{dt\,dy}{t^{n+1+2\varpi}}dz\\
&=C|x-x_0|^{2\varpi-n}\int_{B(x_0,r)}b(z)f(z)\int_{\substack{(y-x_0)'\in\Lambda\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\int_{4|x-x_0|<t}\frac{dt\,dy}{t^{n+1+2\varpi}}dz\\
&\ge C|x-x_0|^{-n}\int_{B(x_0,r)}b(z)f(z)dz\\
&=B_3Nr^n|x-x_0|^{-n}.
\end{aligned}\tag{3.8}$$

By (2.5) and (2.6), we have

$$\begin{aligned}
I_2=|b(x)|\Bigg(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\\
\times\left|\int_{\mathbb{R}^n}\left(\frac{\Omega(y-z)}{|y-z|^{n-\varpi}}\chi_{\{|y-z|<t\}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\varpi}}\chi_{\{|y-x_0|<t\}}\right)f(z)dz\right|^2\frac{dy\,dt}{t^{n+1+2\varpi}}\Bigg)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le |b(x)|\Bigg(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\\
&\qquad\times\left|\int_{\substack{|y-z|<t\\ |y-x_0|<t}}\left(\frac{\Omega(y-z)}{|y-z|^{n-\varpi}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\varpi}}\right)f(z)dz\right|^2\frac{dydt}{t^{n+1+2\varpi}}\Bigg)^{1/2}\\
&\quad+|b(x)|\left(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\left|\int_{\substack{|y-z|<t\\ |y-x_0|\ge t}}\frac{\Omega(y-z)}{|y-z|^{n-\varpi}}f(z)dz\right|^2\frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2}\\
&\quad+|b(x)|\left(\int_{4|x-x_0|}^{\infty}\int_{\substack{|x-y|<t\\ 2|x-x_0|<|y-x_0|<3|x-x_0|}}\left|\int_{\substack{|y-z|\ge t\\ |y-x_0|<t}}\frac{\Omega(y-x_0)}{|y-x_0|^{n-\varpi}}f(z)dz\right|^2\frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2}\\
&:= I_2^1+I_2^2+I_2^3.
\end{aligned}
\tag{3.9}
$$

In I_2^2, we have $t \le |y-x_0| < 3|x-x_0|$ and $t \ge 4|x-x_0|$. In I_2^3, we get $t \le |y-z| < 4|x-x_0|$ and $t \ge 4|x-x_0|$. It is easy to see that $I_2^2 = I_2^3 = 0$. Now, we estimate I_2^1, by $\Omega \in L^\infty(S^{n-1})$, the Minkowski inequality, Lemma 2.1 for $|y-x_0| > 2|z-x_0|$, and (2.4), we get

$$
\begin{aligned}
I_2^1 &\le C|b(x)|\int_{B(x_0,r)}|f(z)|dz\Bigg(\int_{2|x-x_0|<|y-x_0|<3|x-x_0|}\int_{\substack{4|x-x_0|\le t,\,|y-z|<t\\ |y-x_0|<t,\,|x-y|\le t}}\\
&\qquad\times\frac{1}{|y-x_0|^{2(n-\varpi)}(\log(|y-x_0|/r))^{2\gamma}}\frac{dtdy}{t^{n+1+2\varpi}}\Bigg)^{1/2}\\
&\le B_4|b(x)|r^n|x-x_0|^{-n}\left(\log\frac{|x-x_0|}{r}\right)^{-\gamma}.
\end{aligned}
\tag{3.10}
$$

From (3.9) and (3.10), we get

$$
I_2 \le B_4|b(x)|r^n|x-x_0|^{-n}\left(\log\frac{|x-x_0|}{r}\right)^{-\gamma}. \tag{3.11}
$$

Let

$$
F = \left\{x\in G : |b(x)| > \frac{B_3N}{2B_4}\left(\log\frac{|x-x_0|}{r}\right)^{\gamma},\ |x-x_0| < N^{1/n}r\right\}. \tag{3.12}
$$

Without loss of generality, we may assume that $N > B_2 > 1$, otherwise, we get the desired result. Since $\varphi(t)$ is nonincreasing, we have $\varphi(|B(x_0, N^{1/n}r)|) \le \varphi(|B(x_0, r)|) = \varphi(r^n)$. Then by, (3.6), (3.8), and (3.11), we get

$$\begin{aligned}
\|f\|_{L^{p,\varphi}}^p &\ge \left\|[b,\mu_S^{\varpi}]f\right\|_{L^{p,\varphi}}^p \\
&\ge \frac{1}{(\varphi(|B(x_0,N^{1/n}r)|))^p|B(x_0,N^{1/n}r)|}\int_{|x-x_0|<N^{1/n}r}\left|[b,\mu_S^{\varpi}]f(x)\right|^p dx \\
&\ge \frac{1}{(\varphi(r^n))^p N r^n}\int_{(G\setminus F)\cap\{|x-x_0|<N^{1/n}r\}}\left(\frac{1}{2}B_3 N r^n|x-x_0|^{-n}\right)^p dx \\
&\ge \frac{1}{(\varphi(r^n))^p N r^n}\int_{\{B_5(|F|+(B_2r)^n)^{1/n}<|x-x_0|<N^{1/n}r\}\cap G}\left(\frac{1}{2}B_3 N r^n|x-x_0|^{-n}\right)^p dx \\
&= \frac{1}{(\varphi(r^n))^p N r^n}\left(\frac{B_3 N r^n}{2}\right)^p\int_{B_5(|F|+(B_2r)^n)^{1/n}}^{N^{1/n}r} t^{-pn+n-1}dt\int_{\Lambda}J(x')d\sigma(x') \\
&\ge \frac{1}{(\varphi(r^n))^p}\sigma(\Lambda)(Nr^n)^{p-1}\frac{(B_3/2)^p}{n-np}\left(N^{1-p}r^{n(1-p)} - B_5^{(1-p)n}\left(|F|+(B_2r)^n\right)^{1-p}\right).
\end{aligned} \tag{3.13}$$

Thus,

$$\left(|F|+(B_2r)^n\right)^{1-p} \le B_6 N^{1-p} r^{n(1-p)}\left(1+(\varphi(r^n))^p\|f\|_{L^{p,\lambda}}^p\right). \tag{3.14}$$

Then, by (2.20) and (3.14), we get

$$|F| + (B_2r)^n \ge B_7 N r^n. \tag{3.15}$$

If $N \le 2B_7^{-1}B_2^n$, then Theorem 1.3 is proved. If $N > 2B_7^{-1}B_2^n$, then

$$|F| \ge \frac{B_7}{2}Nr^n. \tag{3.16}$$

Let $g(y) = \chi_{B(x_0,r)}(y)$. For $x \in F$, we have

$$\begin{aligned}
\left|[b,\mu_S^{\varpi}]g(x)\right| &= \left(\int_0^\infty \int_{|x-y|<t} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}(b(x)-b(z))g(z)dz\right|^2 \frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2} \\
&\geq \left(\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}(b(x)-b(z))g(z)dz\right|^2 \frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2} \\
&\geq |b(x)| \left(\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|,\,(y-x_0)'\in\Lambda}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}g(z)dz\right|^2 \frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2} \\
&\quad - \left(\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left|\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}b(z)g(z)dz\right|^2 \frac{dydt}{t^{n+1+2\varpi}}\right)^{1/2} \\
&:= K_1 - K_2.
\end{aligned} \tag{3.17}$$

For K_1, as above mentioned, we have $\Omega((y-z)') \geq C_1/(\log(2/B_1))^{\gamma}$. Since $4|x-x_0| \geq |y-z| \geq 3|x-x_0|/2$ and $4|x-x_0| > |x-y| > |x-x_0|$, it follows the Hölder inequality that

$$\begin{aligned}
K_1 &= |b(x)| \left\{\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left(\int_{B(x_0,r)} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}\chi_{\{(|y-z|)<t\}}dz\right)^2 \frac{dydt}{t^{n+1+2\varpi}}\right\}^{1/2} \\
&\geq |b(x)| \int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \int_{B(x_0,r)} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}}\chi_{\{(|y-z|)<t\}}dz \frac{dydt}{t^{n+1+2\varpi}} \\
&\quad \times \left(\int_{4|x-x_0|}^\infty \int_{\substack{|x-y|<t,\,(y-x_0)'\in\Lambda \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \frac{dydt}{t^{n+1+2\varpi}}\right)^{-1/2} \\
&\geq C|b(x)||x-x_0|^{2\varpi-n} \int_{B(x_0,r)} \int_{\substack{(y-x_0)'\in\Lambda \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \int_{4|x-x_0|<t} \frac{dtdy}{t^{n+1+2\varpi}}dz \\
&\geq B_8 N|x-x_0|^{-n} \int_{B(x_0,r)} dz \\
&= B_8 N r^n |x-x_0|^{-n}.
\end{aligned} \tag{3.18}$$

By $\Omega \in L^\infty(S^{n-1})$, the Minkowski inequality, and $|x-x_0| \simeq |y-z|$ for $2|x-x_0| < |y-x_0| < 3|x-x_0|$ and $z \in B(x_0,r)$, we get

$$\begin{aligned} K_2 &= \left(\int_{4|x-x_0|}^{\infty} \int_{\substack{|x-y|<t \\ 2|x-x_0|<|y-x_0|<3|x-x_0|}} \left|\int_{B(x_0,r)} \frac{\Omega(y-z)}{|y-z|^{n-\varpi}} b(z) \chi_{\{|y-z|<t\}} dz\right|^2 \frac{dy\,dt}{t^{n+1+2\varpi}}\right)^{1/2} \\ &\leq \frac{C}{|x-x_0|^{n-\varpi}} \int_{B(x_0,r)} |b(z)| dz \left(\int_{2|x-x_0|<|y-x_0|<3|x-x_0|} \int_{4|x-x_0|}^{\infty} \frac{dt\,dy}{t^{n+1+2\varpi}}\right)^{1/2} \\ &\leq B_9 |x-x_0|^{-n} \int_{B(x_0,r)} |b(z)| dz \\ &= B_9 N r^n |x-x_0|^{-n}. \end{aligned} \tag{3.19}$$

Thus, by (3.17), (3.18), and (3.19), we get, for $x \in F$,

$$\left|[b,\mu_S^\varpi] g(x)\right| \geq B_8 r^n |b(x)| |x-x_0|^{-n} - B_9 N r^n |x-x_0|^{-n}. \tag{3.20}$$

Thus, by (3.20), $\varphi(Nr^n) \leq \varphi(r^n)$, $|b(x)| > (NB_3/2B_4)(\log(|x-x_0|/r))^\gamma$ when $x \in F$ and the Hölder inequality, we have

$$\begin{aligned} \frac{B_{10}}{\varphi(r^n)} &\geq \|g\|_{L^{p,\varphi}} \geq \left\|[b,\mu_S^\varpi] g\right\|_{L^{p,\varphi}} \\ &\geq \frac{1}{\varphi(Nr^n)(Nr^n)^{1/p}} \left(\int_{|x-x_0|<N^{1/n}r} \left|[b,\mu_S^\varpi] g(x)\right|^p dx\right)^{1/p} \\ &\geq \frac{1}{\varphi(r^n)(Nr^n)^{1/p}} \int_{|x-x_0|<N^{1/n}r} \left|[b,\mu_S^\varpi] g(x)\right| dx \left(\int_{|x-x_0|<N^{1/n}r} dx\right)^{-1/p'} \\ &\geq \frac{1}{\varphi(r^n) N r^n} \int_F \left|[b,\mu_S^\varpi] g(x)\right| dx \\ &\geq \frac{B_8 r^n}{\varphi(r^n) N r^n} \int_F |b(x)| |x-x_0|^{-n} dx - \frac{B_9 N r^n}{\varphi(r^n) N r^n} \int_F |x-x_0|^{-n} dx \\ &\geq \frac{B_{11}}{\varphi(r^n)} \int_F \left(\log \frac{|x-x_0|}{r}\right)^\gamma |x-x_0|^{-n} dx \\ &\quad - \frac{B_9}{\varphi(r^n)} \int_F |x-x_0|^{-n} dx \\ &:= L_1 - L_2. \end{aligned} \tag{3.21}$$

As the proof of (2.33) and (2.36), we can get that there exists a constant $\tau > 1$ such that

$$L_1 \geq \frac{B_{12}}{\varphi(r^n)} (\log N)^{\tau}, \qquad L_2 \leq \frac{B_{13}}{\varphi(r^n)} \log N. \tag{3.22}$$

So, by (3.21) and (3.22), we get

$$B_{10} \geq B_{12} (\log N)^{\tau} - B_{13} \log N. \tag{3.23}$$

Then, $N \leq B(\Omega, p, n, \varpi)$. Theorem 1.3 is proved.

Acknowledgments

The authors wish to express their gratitude to the referee for his/her valuable comments and suggestions. The research was supported by NSF of China (Grant nos.: 10901017, 10931001), SRFDP of China (Grant no.: 20090003110018), and NSF of Zhenjiang (Grant no.: Y7080325).

References

[1] E. M. Stein, “On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz,” *Transactions of the American Mathematical Society*, vol. 88, pp. 430–466, 1958.

[2] Y. Ding, D. Fan, and Y. Pan, “Weighted boundedness for a class of rough Marcinkiewicz integrals,” *Indiana University Mathematics Journal*, vol. 48, no. 3, pp. 1037–1055, 1999.

[3] S.-Y. A. Chang, J. M. Wilson, and T. H. Wolff, “Some weighted norm inequalities concerning the Schrödinger operators,” *Commentarii Mathematici Helvetici*, vol. 60, no. 2, pp. 217–246, 1985.

[4] C. E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, vol. 83 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Washington, DC, USA, 1994.

[5] Q. Xue and Y. Ding, “Weighted L^p boundedness for parametrized Littlewood-Paley operators,” *Taiwanese Journal of Mathematics*, vol. 11, no. 4, pp. 1143–1165, 2007.

[6] M. Sakamoto and K. Yabuta, “Boundedness of Marcinkiewicz functions,” *Studia Mathematica*, vol. 135, no. 2, pp. 103–142, 1999.

[7] A. Torchinsky and S. L. Wang, “A note on the Marcinkiewicz integral,” *Colloquium Mathematicum*, vol. 60-61, no. 1, pp. 235–243, 1990.

[8] E. M. Stein, “The development of square functions in the work of A. Zygmund,” *Bulletin of the American Mathematical Society*, vol. 7, no. 2, pp. 359–376, 1982.

[9] Y. Ding, S. Lu, and K. Yabuta, “On commutators of Marcinkiewicz integrals with rough kernel,” *Journal of Mathematical Analysis and Applications*, vol. 275, no. 1, pp. 60–68, 2002.

[10] Y. Ding and Q. Xue, “Endpoint estimates for commutators of a class of Littlewood-Paley operators,” *Hokkaido Mathematical Journal*, vol. 36, no. 2, pp. 245–282, 2007.

[11] R. Coifman, R. Rochberg, and G. Weiss, “Factorization theorems for Hardy spaces in several variables,” *The Annals of Mathematics*, vol. 103, no. 3, pp. 611–635, 1976.

[12] S. Chanillo, “A note on commutators,” *Indiana University Mathematics Journal*, vol. 31, no. 1, pp. 7–16, 1982.

[13] T. Mizuhara, “Commutators of singular integral operators on Morrey spaces with general growth functions,” *Sūrikaisekikenkyūsho Kōkyūroku*, no. 1102, pp. 49–63, 1999, Proceedings of the Coference on Harmonic Analysis and Nonlinear Partial Differential Equations, Kyoto, Japan, 1998.

[14] Y. Komori and T. Mizuhara, “Factorization of functions in $H^1(\mathbb{R}^n)$ and generalized Morrey spaces,” *Mathematische Nachrichten*, vol. 279, no. 5-6, pp. 619–624, 2006.

[15] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, no. 30, Princeton University Press, Princeton, NJ, USA, 1970.
[16] A. Uchiyama, “On the compactness of operators of Hankel type,” *Tôhoku Mathematical Journal*, vol. 30, no. 1, pp. 163–171, 1978.
[17] Y. Ding, “A note on end properties of Marcinkiewicz integral,” *Journal of the Korean Mathematical Society*, vol. 42, no. 5, pp. 1087–1100, 2005.