Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 965313, 26 pages
doi:10.1155/2010/965313

Research Article

Existence and Asymptotic Behavior of Solutions for Weighted $p(t)$-Laplacian Integrodifferential System Multipoint and Integral Boundary Value Problems in Half Line

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Received 5 February 2010; Revised 13 June 2010; Accepted 28 June 2010

Academic Editor: P. J. Y. Wong

This paper investigates the existence and asymptotic behavior of solutions for weighted $p(t)$-Laplacian integro-differential system with multipoint and integral boundary value condition in half line. When the nonlinearity term f satisfies sub-$(p^- - 1)$ growth condition or general growth condition, we give the existence of solutions via Leray-Schauder degree. Moreover, the existence of nonnegative solutions has been discussed.

1. Introduction

In this paper, we consider the existence and asymptotic behavior of solutions for the following weighted $p(t)$-Laplacian integrodifferential system:

$$-\Delta_{p(t)} u + \delta f\left(t, u, (w(t))^{1/(p(t)-1)} u', S(u), T(u)\right) = 0, \quad t \in (0, +\infty), \tag{1.1}$$

with the following multipoint and integral boundary value condition:

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_0, \qquad \lim_{t \to +\infty} u(t) = \int_0^{+\infty} e(t) u(t) dt, \tag{1.2}$$

where $u : [0,+\infty) \to \mathbb{R}^N$; S and T are linear operators defined by

$$S(u)(t) = \int_0^t \varphi(s,t)u(s)ds, \qquad T(u)(t) = \int_0^{+\infty} \chi(s,t)u(s)ds, \tag{1.3}$$

where $\varphi \in C(D,\mathbb{R})$, $\chi \in C(D,\mathbb{R})$, $D = \{(s,t) \in [0,+\infty) \times [0,+\infty)\}$; $\int_0^{+\infty} |\varphi(s,t)|ds$ and $\int_0^{+\infty} |\chi(s,t)|ds$ are uniformly bounded with t; $p \in C([0,+\infty),\mathbb{R})$, $p(t) > 1$, $\lim_{t\to+\infty} p(t)$ exists and $\lim_{t\to+\infty} p(t) > 1$; $-\Delta_{p(t)} u := -(w(t)|u'|^{p(t)-2}u')'$ is called the weighted $p(t)$-Laplacian; $w \in C([0,+\infty),\mathbb{R})$ satisfies $0 < w(t)$, for all $t \in (0,+\infty)$, and $(w(t))^{-1/(p(t)-1)} \in L^1(0,+\infty)$; $0 < \xi_1 < \cdots < \xi_{m-2} < +\infty$, $\alpha_i \geq 0$, $(i = 1,\ldots,m-2)$ and $0 < \sum_{i=1}^{m-2} \alpha_i < 1$; $e \in L^1(0,+\infty)$ is nonnegative, $\sigma = \int_0^{+\infty} e(t)dt$ and $\sigma \in [0,1]$; $e_0 \in \mathbb{R}^N$; δ is a positive parameter.

The study of differential equations and variational problems with variable exponent growth conditions is a new and interesting topic. Many results have been obtained on these problems, for example, [1–18]. Such problems arise from the study of electrorheological fluids, image processing, and the theory of nonlinear elasticity [2, 10, 18]. Many important models in image processing can be unified to the following variable exponent flow (see [2]):

$$\begin{gathered} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \lambda(u - u_0) = 0, \quad \text{in } \Omega \times [0,T], \\ u(x;t) = g(x), \quad \text{on } \partial\Omega \times [0,T], \\ u(x,t) = u_0. \end{gathered} \tag{1.4}$$

The main benefit of this flow is the manner in which it accommodates the local image information.

If $w(t) \equiv 1$ and $p(t) \equiv p$ (a constant), $-\Delta_{p(t)}$ becomes the well-known p-Laplacian. If $p(t)$ is a general function, $-\Delta_{p(t)}$ represents a nonhomogeneity and possesses more nonlinearity, thus $-\Delta_{p(t)}$ is more complicated than $-\Delta_p$. For example,

(a) if $\Omega \subset \mathbb{R}^n$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (1/p(x))|\nabla u|^{p(x)}dx}{\int_\Omega (1/p(x))|u|^{p(x)}dx} \tag{1.5}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [6]), but the fact that $\lambda_p > 0$ is very important in the study of p-Laplacian problems.

(b) If $w(t) \equiv 1$, $p(t) \equiv p$ (a constant) and $-\Delta_p u > 0$, then u is concave; this property is used extensively in the study of one-dimensional p-Laplacian problems, but it is invalid for $-\Delta_{p(t)}$. It is another difference between $-\Delta_p$ and $-\Delta_{p(t)}$.

There are many results on the existence of solutions for p-Laplacian equation with integral boundary value conditions (see [19–24]). On the existence of solutions for $p(x)$-Laplacian systems boundary value problems, we refer to [4–7, 12–17]. On the p-Laplacian equation multipoint problems, we refer to [25–27] (and the references therein). In [25], under some monotone assumptions, Ahmad and Nieto investigated the existence of solutions for three-point second-order integrodifferential boundary value problems with

p-Laplacian by monotone iterative technique. But results on the existence and asymptotic behavior of solutions for weighted $p(t)$-Laplacian integrodifferential systems with multipoint and integral boundary value conditions are rare. In this paper, when $p(t)$ is a general function, we investigate the existence and asymptotic behavior of solutions for weighted $p(t)$-Laplacian integrodifferential systems with multipoint and integral boundary value conditions. Moreover, we give the existence of nonnegative solutions. This paper do not assume monotone assumptions on f, and f dependent on $(w(t))^{1/(p(t)-1)}u'$, but it should satisfy some growth conditions. Our results partly generalized the results of [25].

Let $N \geq 1$ and $J = [0,+\infty)$; the function $f = (f^1, \ldots, f^N) : J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory, by this we mean that

(i) for almost every $t \in J$, the function $f(t,\cdot,\cdot,\cdot,\cdot)$ is continuous;

(ii) for each $(x,y,z,w) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, the function $f(\cdot,x,y,z,w)$ is measurable on J;

(iii) for each $R > 0$ there is a $\beta_R \in L^1(J,\mathbb{R})$ such that, for almost every $t \in J$ and every $(x,y,z,w) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R$, $|y| \leq R$, $|z| \leq R$, $|w| \leq R$, one has

$$\left|f(t,x,y,z,w)\right| \leq \beta_R(t). \tag{1.6}$$

Throughout the paper, we denote

$$\begin{gathered} w(0)\left|u'\right|^{p(0)-2}u'(0) = \lim_{t\to 0^+} w(t)\left|u'\right|^{p(t)-2}u'(t), \\ w(+\infty)\left|u'\right|^{p(+\infty)-2}u'(+\infty) = \lim_{t\to +\infty} w(t)\left|u'\right|^{p(t)-2}u'(t). \end{gathered} \tag{1.7}$$

The inner product in $\mathbb{R}^N$ will be denoted by $\langle\cdot,\cdot\rangle$; $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. Let $\mathrm{AC}(0,+\infty)$ denote the space of absolutely continuous functions on the interval $(0,+\infty)$. For $N \geq 1$, we set $C = C(J,\mathbb{R}^N), C^1 = \{u \in C \mid u' \in C((0,+\infty),\mathbb{R}^N),\ \lim_{t\to 0^+} w(t)^{1/(p(t)-1)}u'(t)$ exists$\}$. For any $u(t) = (u^1(t),\ldots,u^N(t)) \in C$, we denote $|u^i|_0 = \sup_{t\in(0,+\infty)}|u^i(t)|$, $\|u\|_0 = (\sum_{i=1}^N |u^i|_0^2)^{1/2}$, and $\|u\|_1 = \|u\|_0 + \|(w(t))^{1/(p(t)-1)}u'\|_0$. Spaces C and C^1 will be equipped with the norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. Then $(C,\|\cdot\|_0)$ and $(C^1,\|\cdot\|_1)$ are Banach spaces. Denote $L^1 = L^1(J,\mathbb{R}^N)$ with the norm $\|u\|_{L^1} = [\sum_{i=1}^N(\int_0^\infty |u^i|dt)^2]^{1/2}$.

We say a function $u : J \to \mathbb{R}^N$ is a solution of (1.1) if $u \in C^1$ with $w(t)|u'|^{p(t)-2}u'$ absolutely continuous on $(0,+\infty)$, which satisfies (1.1) a.e. on J.

In this paper, we always use C_i to denote positive constants if it cannot lead to confusion. Denote

$$z^- = \inf_{t\in J} z(t), \qquad z^+ = \sup_{t\in J} z(t), \quad \text{for any } z \in C(J,\mathbb{R}). \tag{1.8}$$

We say f satisfies sub-$(p^- - 1)$ growth condition if f satisfies

$$\lim_{|x|+|y|+|z|+|w|\to+\infty} \frac{f(t,x,y,z,w)}{(|x|+|y|+|z|+|w|)^{q(t)-1}} = 0, \quad \text{for } t \in J \text{ uniformly}, \tag{1.9}$$

where $q(t) \in C(J,\mathbb{R})$, and $1 < q^- \le q^+ < p^-$. We say f satisfies general growth condition, if f does not satisfy sub-$(p^- - 1)$ growth condition.

We will discuss the existence of solutions of (1.1)-(1.2) in the following two cases

(i) f satisfies sub-$(p^- - 1)$ growth condition;

(ii) f satisfies general growth condition.

This paper is divided into five sections. In the Section 2, we will do some preparation. In Section 3, we will discuss the existence and asymptotic behavior of solutions of (1.1)-(1.2), when f satisfies sub-$(p^- - 1)$ growth condition. In Section 4, when f satisfies general growth condition, we will discuss the existence and asymptotic behavior of solutions of (1.1)-(1.2). Moreover, we discuss the existence of nonnegative solutions. Finally, in Section 5, we give several examples.

2. Preliminary

For any $(t,x) \in J \times \mathbb{R}^N$, denote $\varphi(t,x) = |x|^{p(t)-2}x$. Obviously, φ has the following properties

Lemma 2.1 (see [4]). *φ is a continuous function and satisfies the following conditions.*

(i) *For any $t \in [0,+\infty)$, $\varphi(t,\cdot)$ is strictly monotone, that is*

$$\langle \varphi(t,x_1) - \varphi(t,x_2), x_1 - x_2 \rangle > 0, \quad \text{for any } x_1, x_2 \in \mathbb{R}^N,\ x_1 \ne x_2. \tag{2.1}$$

(ii) *There exists a function $\beta : [0,+\infty) \to [0,+\infty)$, $\beta(s) \to +\infty$ as $s \to +\infty$, such that*

$$\langle \varphi(t,x), x \rangle \ge \beta(|x|)|x|, \quad \forall x \in \mathbb{R}^N. \tag{2.2}$$

It is well known that $\varphi(t,\cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $t \in [0,+\infty)$. For any $t \in J$, denote by $\varphi^{-1}(t,\cdot)$ the inverse operator of $\varphi(t,\cdot)$; then

$$\varphi^{-1}(t,x) = |x|^{(2-p(t))/(p(t)-1)}x, \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \qquad \varphi^{-1}(t,0) = 0. \tag{2.3}$$

It is clear that $\varphi^{-1}(t,\cdot)$ is continuous and sends bounded sets into bounded sets.

Now, let us consider the following problem with boundary value condition (1.2):

$$\left(w(t)\varphi\left(t,u'(t)\right)\right)' = g(t), \ \ t \in (0,+\infty), \ \text{ where } g \in L^1. \tag{2.4}$$

If u is a solution of (2.4) with (1.2), by integrating (2.4) from 0 to t, we find that

$$w(t)\varphi\left(t,u'(t)\right) = w(0)\varphi\left(0,u'(0)\right) + \int_0^t g(s)ds. \tag{2.5}$$

Denote $a = w(0)\varphi(0,u'(0))$. It is easy to see that a is dependent on $g(\cdot)$. Define operator $F : L^1 \to C$ as

$$F(g)(t) = \int_0^t g(s)ds, \qquad \forall t \in J, \ \forall g \in L^1. \tag{2.6}$$

By solving for u' in (2.5) and integrating, we find that

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(a + F(g)\right)\right]\right\}(t), \qquad t \in J. \tag{2.7}$$

From $u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_0$, we have

$$u(0) = \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\left[t, (w(t))^{-1}\left(a + F(g)(t)\right)\right]dt + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.8}$$

Suppose $\sigma \in [0,1)$. From $\lim_{t\to+\infty} u(t) = \int_0^{+\infty} e(t)u(t)dt$, we obtain

$$\begin{aligned} u(0) = &\frac{\int_0^{+\infty}\left\{e(t)\int_0^t \varphi^{-1}\left[r, (w(r))^{-1}\left(a + F(g)(r)\right)\right]dr\right\}dt}{1-\sigma} \\ &- \frac{\int_0^{+\infty} \varphi^{-1}\left[t, (w(t))^{-1}\left(a + F(g)(t)\right)\right]dt}{1-\sigma}. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we have

$$\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left[t,(w(t))^{-1}(a+F(g)(t))\right]dt+e_0}{1-\sum_{i=1}^{m-2}\alpha_i} = \frac{\int_0^{+\infty}\left\{e(t)\int_0^t\varphi^{-1}\left[r,(w(r))^{-1}(a+F(g)(r))\right]dr\right\}dt}{1-\sigma} - \frac{\int_0^{+\infty}\varphi^{-1}\left[t,(w(t))^{-1}(a+F(g)(t))\right]dt}{1-\sigma}. \tag{2.10}$$

For fixed $h\in C$, we denote

$$\Lambda_h(a)=\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left[t,(w(t))^{-1}(a+h(t))\right]dt+e_0}{1-\sum_{i=1}^{m-2}\alpha_i} - \frac{\int_0^{+\infty}\left\{e(t)\int_0^t\varphi^{-1}\left[r,(w(r))^{-1}(a+h(r))\right]dr\right\}dt}{1-\sigma} + \frac{\int_0^{+\infty}\varphi^{-1}\left[t,(w(t))^{-1}(a+h(t))\right]dt}{1-\sigma}. \tag{2.11}$$

Throughout the paper, we denote $E=\int_0^{+\infty}(w(t))^{-1/(p(t)-1)}dt$.

Lemma 2.2. *The function $\Lambda_h(\cdot)$ has the following properties.*

(i) *For any fixed $h\in C$, the equation*

$$\Lambda_h(a)=0 \tag{2.12}$$

has a unique solution $\tilde{a}(h)\in\mathbb{R}^N$.

(ii) *The function $\tilde{a}:C\to\mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover,*

$$|\tilde{a}(h)|\le 3N\left[\frac{2N(E+1)}{\left(1-\sum_{i=1}^{m-2}\alpha_i\right)E}+1\right]^{p^+}\cdot\left[\|h\|_0+(2N)^{p^+}|e_0|^{p^\#-1}\right], \tag{2.13}$$

where the notation $M^{p^\#-1}$ means

$$M^{p^\#-1}=\begin{cases}M^{p^+-1}, & M>1,\\ M^{p^--1}, & M\le 1.\end{cases} \tag{2.14}$$

Proof. (i) Obviously, we have

$$\begin{aligned}
&\int_0^{+\infty}\left\{e(t)\int_0^t \varphi^{-1}\left[r,(w(r))^{-1}(a+h(r))\right]dr\right\}dt \\
&\quad=\int_0^{+\infty}\left\{e(t)\int_0^{+\infty} \varphi^{-1}\left[r,(w(r))^{-1}(a+h(r))\right]dr\right\}dt \\
&\qquad-\int_0^{+\infty}\left\{e(t)\int_t^{+\infty} \varphi^{-1}\left[r,(w(r))^{-1}(a+h(r))\right]dr\right\}dt \\
&\quad=\sigma\int_0^{+\infty} \varphi^{-1}\left[t,(w(t))^{-1}(a+h(t))\right]dt \\
&\qquad-\int_0^{+\infty}\left\{e(t)\int_t^{+\infty} \varphi^{-1}\left[r,(w(r))^{-1}(a+h(r))\right]dr\right\}dt.
\end{aligned} \tag{2.15}$$

It is easy to see that

$$\begin{aligned}
\Lambda_h(a)=&\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left[t,(w(t))^{-1}(a+h(t))\right]dt+e_0}{1-\sum_{i=1}^{m-2}\alpha_i} \\
&+\frac{\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,(w(r))^{-1}(a+h(r))\right]dr\right\}dt}{1-\sigma} \\
&+\int_0^{+\infty}\varphi^{-1}\left[t,(w(t))^{-1}(a+h(t))\right]dt.
\end{aligned} \tag{2.16}$$

From Lemma 2.1, it is immediate that

$$\langle \Lambda_h(a_1)-\Lambda_h(a_2),a_1-a_2\rangle>0,\quad \text{for } a_1\neq a_2, \tag{2.17}$$

and hence, if (2.12) has a solution, then it is unique.

Let $t_0=3N[2N(E+1)/(1-\sum_{i=1}^{m-2}\alpha_i)E+1]^{p^+}\cdot[\|h\|_0+(2N)^{p^+}|e_0|^{p^\#-1}]$. If $|a|>t_0$, since $(w(t))^{-1/(p(t)-1)}\in L^1(0,+\infty)$ and $h\in C$, it is easy to see that there exists an $i\in\{1,\dots,N\}$ such that the ith component a^i of a satisfies

$$\left|a^i\right|\geq\frac{|a|}{N}>3\left[\frac{2N(E+1)}{\left(1-\sum_{i=1}^{m-2}\alpha_i\right)E}+1\right]^{p^+}\cdot\left[\|h\|_0+(2N)^{p^+}|e_0|^{p^\#-1}\right]. \tag{2.18}$$

Thus $(a^i + h^i(t))$ keeps sign on J and

$$\begin{aligned} \left|a^i + h^i(t)\right| &\geq \left|a^i\right| - \|h\|_0 \\ &\geq \frac{2|a|}{3N} > 2\left[\frac{2N(E+1)}{\left(1-\sum_{i=1}^{m-2}\alpha_i\right)E} + 1\right]^{p^+} \cdot \left[\|h\|_0 + (2N)^{p^+}|e_0|^{p^\#-1}\right], \quad \forall t \in J. \end{aligned} \tag{2.19}$$

Obviously, $|a + h(t)| \leq 4|a|/3 \leq 2N|a^i + h^i(t)|$, then

$$\begin{aligned} |a + h(t)|^{(2-p(t))/(p(t)-1)}\left|a^i + h^i(t)\right| &> \frac{1}{2N}\left|a^i + h^i(t)\right|^{1/(p(t)-1)} \\ &> \frac{E+1}{\left(1-\sum_{i=1}^{m-2}\alpha_i\right)E}|e_0|, \quad \forall t \in J. \end{aligned} \tag{2.20}$$

Thus the ith component $\Lambda_h^i(a)$ of $\Lambda_h(a)$ is nonzero and keeps sign, and then we have

$$\Lambda_h(a) \neq 0. \tag{2.21}$$

Let us consider the equation

$$\lambda\Lambda_h(a) + (1-\lambda)a = 0, \quad \lambda \in [0,1]. \tag{2.22}$$

It is easy to see that all the solutions of (2.22) belong to $b(t_0+1) = \{x \in \mathbb{R}^N \mid |x| < t_0+1\}$. So, we have

$$d_B[\Lambda_h(a), b(t_0+1), 0] = d_B[I, b(t_0+1), 0] \neq 0, \tag{2.23}$$

and it shows the existence of solutions of $\Lambda_h(a) = 0$.

In this way, we define a function $\widetilde{a}(h) : C[0,+\infty) \to \mathbb{R}^N$, which satisfies

$$\Lambda_h(\widetilde{a}(h)) = 0. \tag{2.24}$$

(ii) By the proof of (i), we also obtain that $\widetilde{a}$ sends bounded sets to bounded sets, and

$$|\widetilde{a}(h)| \leq 3N\left[\frac{2N(E+1)}{\left(1-\sum_{i=1}^{m-2}\alpha_i\right)E} + 1\right]^{p^+} \cdot \left[\|h\|_0 + (2N)^{p^+}|e_0|^{p^\#-1}\right]. \tag{2.25}$$

It only remains to prove the continuity of $\widetilde{a}$. Let $\{u_n\}$ be a convergent sequence in C and $u_n \to u$ as $n \to +\infty$. Since $\{\widetilde{a}(u_n)\}$ is a bounded sequence, then it contains a convergent subsequence $\{\tilde{a}(u_{n_j})\}$. Let $\tilde{a}(u_{n_j}) \to a_0$ as $j \to +\infty$. Since $\Lambda_{u_{n_j}}(\tilde{a}(u_{n_j})) = 0$, letting $j \to +\infty$, we have $\Lambda_u(a_0) = 0$. From (i), we get $a_0 = \tilde{a}(u)$; it means that $\tilde{a}$ is continuous.

This completes the proof. □

Similarly, if u is a solution of (2.4) with (1.2) when $\sigma = 1$ we have

$$u(t) = u(0) + F\Big\{\varphi^{-1}\Big[t, (w(t))^{-1}\big(a^* + F(g)(t)\big)\Big]\Big\}(t), \quad t \in J, \tag{2.26}$$

where $a^* = w(0)\varphi(0, u'(0))$, then a^* is dependent on $g(\cdot)$.

The boundary value condition (1.2) implies that

$$\begin{aligned} u(0) &= \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\Big[t, (w(t))^{-1}\big(a^* + F(g)(t)\big)\Big]dt + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i}, \\ &\int_0^{+\infty} \Big\{e(t) \int_t^{+\infty} \varphi^{-1}\Big[r, (w(r))^{-1}\big(a^* + F(g)(r)\big)\Big]dr\Big\}dt = 0. \end{aligned} \tag{2.27}$$

For fixed $h \in C$, we denote

$$\Theta_h(a^*) = \int_0^{+\infty} \Big\{e(t) \int_t^{+\infty} \varphi^{-1}\Big[r, (w(r))^{-1}(a^* + h(r))\Big]dr\Big\}dt. \tag{2.28}$$

Lemma 2.3. *The function $\Theta_h(\cdot)$ has the following properties.*

(i) *For any fixed $h \in C$, the equation*

$$\Theta_h(a^*) = 0 \tag{2.29}$$

has a unique solution $\widetilde{a^*}(h) \in \mathbb{R}^N$.

(ii) *The function $\widetilde{a^*} : C \to \mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover,*

$$\left|\widetilde{a^*}(h)\right| \le 3N\|h\|_0. \tag{2.30}$$

Proof. It is similar to the proof of Lemma 2.2, we omit it here. □

Now, we define $a : L^1 \to \mathbb{R}^N$ as

$$a(u) = \tilde{a}(F(u)), \tag{2.31}$$

and define $a^* : L^1 \to \mathbb{R}^N$ as

$$a^*(u) = \widetilde{a^*}(F(u)). \tag{2.32}$$

It is also clear that $a(\cdot)$ and $a^*(\cdot)$ are continuous and they send bounded sets of L^1 into bounded sets of $\mathbb{R}^N$, and hence they are compact continuous.

If u is a solution of (2.4) with (1.2), when $\sigma \in [0,1)$, we have

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(a(g) + F(g)(t)\right)\right]\right\}(t), \quad \forall t \in [0,+\infty),$$
$$u(0) = \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\left[t, (w(t))^{-1}\left(a + F(g)(t)\right)\right]dt + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.33}$$

When $\sigma = 1$, we have

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(a^*(g) + F(g)(t)\right)\right]\right\}(t), \quad \forall t \in [0,+\infty),$$
$$u(0) = \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\left[t, (w(t))^{-1}\left(a^* + F(g)(t)\right)\right]dt + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.34}$$

We denote

$$K_1(h)(t) := (K_1 \circ h)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}(a(h) + F(h))\right]\right\}(t), \quad \forall t \in (0,+\infty),$$
$$K_2(h)(t) := (K_2 \circ h)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}(a^*(h) + F(h))\right]\right\}(t), \quad \forall t \in (0,+\infty). \tag{2.35}$$

Lemma 2.4. *The operators K_i $(i = 1,2)$ are continuous and they send equi-integrable sets in L^1 to relatively compact sets in C^1.*

Proof. we only prove that the operator K_1 is continuous and sends equi-integrable sets in L^1 to relatively compact sets in C^1; the rest is similar.

It is easy to check that $K_1(h)(t) \in C^1$, for all $h \in L^1$. Since $(w(t))^{-1/(p(t)-1)} \in L^1$ and

$$K_1(h)'(t) = \varphi^{-1}\left[t, (w(t))^{-1}(a(h) + F(h))\right], \quad \forall t \in [0,+\infty), \tag{2.36}$$

it is easy to check that K_1 is a continuous operator from L^1 to C^1.

Let now U be an equi-integrable set in L^1; then there exists $\rho_* \in L^1$, such that

$$|u(t)| \le \rho_*(t) \quad \text{a.e. in } J, \quad \text{for any } u \in L^1. \tag{2.37}$$

We want to show that $\overline{K_1(U)} \subset C^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $K_1(U)$; then there exists a sequence $\{h_n\} \in U$ such that $u_n = K_1(h_n)$. For any $t_1, t_2 \in J$, we have

$$\begin{aligned}|F(h_n)(t_1) - F(h_n)(t_2)| &= \left|\int_0^{t_1} h_n(t)dt - \int_0^{t_2} h_n(t)dt\right| \\ &= \left|\int_{t_1}^{t_2} h_n(t)dt\right| \leq \left|\int_{t_1}^{t_2} \rho_*(t)dt\right|.\end{aligned} \tag{2.38}$$

Hence the sequence $\{F(h_n)\}$ is uniformly bounded and equi-continuous. By Ascoli-Arzela Theorem, there exists a subsequence of $\{F(h_n)\}$ (which we rename the same) being convergent in C. According to the bounded continuous of the operator a, we can choose a subsequence of $\{a(h_n) + F(h_n)\}$ (which we still denote $\{a(h_n) + F(h_n)\}$) which is convergent in C, then $w(t)\varphi(t, K_1(h_n)'(t)) = a(h_n) + F(h_n)$ is convergent in C.

Since

$$K_1(h_n)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}(a(h_n) + F(h_n))\right]\right\}(t), \quad \forall t \in [0, +\infty), \tag{2.39}$$

it follows from the continuity of φ^{-1} and the integrability of $w(t)^{-1/(p(t)-1)}$ in L^1 that $K_1(h_n)$ is convergent in C. Thus $\{u_n\}$ is convergent in C^1. This completes the proof. □

Let us define $P, Q : C^1 \to C^1$ as

$$P(h) = \frac{\sum_{i=1}^{m-2} \alpha_i (K_1 \circ h)(\xi_i) + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i}, \qquad Q(h) = \frac{\sum_{i=1}^{m-2} \alpha_i (K_2 \circ h)(\xi_i) + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.40}$$

It is easy to see that P and Q are both compact continuous.

We denote by $N_f(u) : [0, +\infty) \times C^1 \to L^1$ the Nemytski operator associated to f defined by

$$N_f(u)(t) = f\left(t, u(t), (w(t))^{1/(p(t)-1)} u'(t), S(u)(t), T(u)(t)\right), \quad \text{a.e. on } J. \tag{2.41}$$

Lemma 2.5. (i) *When* $\sigma \in [0, 1)$, u *is a solution of* (1.1)-(1.2) *if and only if* u *is a solution of the following abstract equation:*

$$u = P(\delta N_f(u)) + K_1(\delta N_f(u)). \tag{2.42}$$

(ii) *When* $\sigma = 1$, u *is a solution of* (1.1)-(1.2) *if and only if* u *is a solution of the following abstract equation:*

$$u = Q(\delta N_f(u)) + K_2(\delta N_f(u)). \tag{2.43}$$

Proof. (i) If u is a solution of (1.1)-(1.2) when $\sigma \in [0,1)$, by integrating (1.1) from 0 to t, we find that

$$w(t)\varphi(t, u'(t)) = a(\delta N_f(u)) + F(\delta N_f(u))(t), \quad \forall t \in (0, +\infty). \tag{2.44}$$

From (2.44), we have

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(a(\delta N_f(u)) + F(\delta N_f(u))(t)\right)\right]\right\}(t), \quad \forall t \in [0, +\infty). \tag{2.45}$$

Since

$$u(0) = \sum_{i=1}^{m-2} \alpha_i \left[u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(a(\delta N_f(u)) + F(\delta N_f(u))(t)\right)\right]\right\}(\xi_i)\right] + e_0, \tag{2.46}$$

we have

$$\begin{aligned} u(0) &= \frac{\sum_{i=1}^{m-2} \alpha_i F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(a(\delta N_f(u)) + F(\delta N_f(u))(t)\right)\right]\right\}(\xi_i) + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i} \\ &= \frac{\sum_{i=1}^{m-2} \alpha_i K_1(\delta N_f(u))(\xi_i) + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i} = P(\delta N_f(u)). \end{aligned} \tag{2.47}$$

So we have

$$u = P(\delta N_f(u)) + K_1(\delta N_f(u)). \tag{2.48}$$

Conversely, if u is a solution of (2.42), then

$$u(0) = P(\delta N_f(u)) + K_1(\delta N_f(u))(0) = P(\delta N_f(u)) = \frac{\sum_{i=1}^{m-2} \alpha_i K_1(\delta N_f(u))(\xi_i) + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i}, \tag{2.49}$$

and then

$$u(0) = \sum_{i=1}^{m-2} \alpha_i \left[u(0) + K_1(\delta N_f(u))(\xi_i)\right] + e_0 = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_0. \tag{2.50}$$

It follows from (2.42) that

$$u(+\infty) = P(\delta N_f(u)) + K_1(\delta N_f(u))(+\infty). \tag{2.51}$$

By the condition of the mapping a, we have

$$u(0) = \frac{\sum_{i=1}^{m-2} \alpha_i K_1(\delta N_f(u))(\xi_i) + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i}$$
$$= -\frac{\int_0^{+\infty} \left\{ e(t) \int_t^{+\infty} \varphi^{-1}\left[r, (w(r))^{-1}\left(a + F(\delta N_f(u))(r)\right)\right] dr \right\} dt}{1-\sigma} \tag{2.52}$$
$$- \int_0^{+\infty} \varphi^{-1}\left[t, (w(t))^{-1}\left(a + F(\delta N_f(u))(t)\right)\right] dt,$$

and then

$$u(+\infty) = -\frac{\int_0^{+\infty} \left\{ e(t) \int_t^{+\infty} \varphi^{-1}\left[r, (w(r))^{-1}\left(a + F(\delta N_f(u))(r)\right)\right] dr \right\} dt}{1-\sigma}$$
$$= -\frac{\int_0^{+\infty} \left\{ e(t) \int_0^{+\infty} \varphi^{-1}\left[r, (w(r))^{-1}\left(a + F(\delta N_f(u))(r)\right)\right] dr \right\} dt}{1-\sigma}$$
$$+ \frac{\int_0^{+\infty} \left\{ e(t) \int_0^{t} \varphi^{-1}\left[r, (w(r))^{-1}\left(a + F(\delta N_f(u))(r)\right)\right] dr \right\} dt}{1-\sigma} \tag{2.53}$$
$$= \frac{\int_0^{+\infty} e(t)[u(+\infty) - u(0)]dt}{\sigma - 1} - \frac{\int_0^{+\infty} e(t)[u(t) - u(0)]dt}{\sigma - 1}$$
$$= \frac{\sigma u(+\infty) - \int_0^{+\infty} e(t)u(t)dt}{\sigma - 1},$$

thus

$$u(+\infty) = \int_0^{+\infty} e(t)u(t)dt. \tag{2.54}$$

From (2.50) and (2.54), we obtain (1.2).

From (2.42), we have

$$u'(t) = \varphi^{-1}\left[t, (w(t))^{-1}\left(a + F(\delta N_f(u))(t)\right)\right], \tag{2.55}$$

and then

$$\left(w(t)\varphi(t, u')\right)' = \delta N_f(u)(t). \tag{2.56}$$

Hence u is a solution of (1.1)-(1.2) when $\sigma \in [0, 1)$.

(ii) It is similar to the proof of (i).

This completes the proof. □

3. When f Satisfies Sub-$(p^- - 1)$ Growth Condition

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions for (1.1)-(1.2), when f satisfies sub-$(p^- - 1)$ growth condition. Moreover, the asymptotic behavior has been discussed.

Theorem 3.1. *If f satisfies sub-$(p^- - 1)$ growth condition, then problem* (1.1)-(1.2) *has at least a solution for any fixed parameter δ when $\sigma \in [0,1)$.*

Proof. Denote $\Psi_f(u,\lambda) := P(\lambda\delta N_f(u)) + K_1(\lambda\delta N_f(u))$, where $N_f(u)$ is defined in (2.41). When $\sigma \in [0,1)$, we know that (1.1)-(1.2) has the same solution of

$$u = \Psi_f(u,\lambda), \tag{3.1}$$

when $\lambda = 1$.

It is easy to see that the operator P is compact continuous. According to Lemmas 2.2 and 2.4, we can see that $\Psi_f(\cdot,\lambda)$ is compact continuous from C^1 to C^1 for any $\lambda \in [0,1]$.

We claim that all the solutions of (3.1) are uniformly bounded for $\lambda \in [0,1]$. In fact, if it is false, we can find a sequence of solutions $\{(u_n,\lambda_n)\}$ for (3.1) such that $\|u_n\|_1 \to +\infty$ as $n \to +\infty$, and $\|u_n\|_1 > 1$ for any $n = 1,2,\ldots$.

From Lemma 2.2, we have

$$\begin{aligned} \left|a\left(\lambda_n\delta N_f(u_n)\right)\right| &\le C_1\left(\left\|N_f(u_n)\right\|_0 + 2N|e_0|^{p^\#-1}\right) \\ &\le C_2\left(\|u_n\|_1^{q^+-1} + 1\right), \end{aligned} \tag{3.2}$$

which together with the sub-$(p^- - 1)$ growth condition of f implies that

$$\left|a\left(\lambda_n\delta N_f(u_n)\right) + F\left(\lambda_n\delta N_f(u_n)\right)\right| \le \left|a\left(\lambda_n\delta N_f(u_n)\right)\right| + \left|F\left(\lambda_n\delta N_f(u_n)\right)\right| \le C_3\|u_n\|_1^{q^+-1}. \tag{3.3}$$

From (3.1), we have

$$w(t)\left|u_n'(t)\right|^{p(t)-2}u_n'(t) = a\left(\lambda_n\delta N_f(u_n)\right) + F\left(\lambda_n\delta N_f(u_n)\right), \quad t \in J, \tag{3.4}$$

then

$$w(t)\left|u_n'(t)\right|^{p(t)-1} \le \left|a\left(\lambda_n\delta N_f(u_n)\right)\right| + \left|F\left(\lambda_n\delta N_f(u_n)\right)\right| \le C_4\|u_n\|_1^{q^+-1}. \tag{3.5}$$

Denote $\alpha = (q^+ - 1)/(p^- - 1)$; we have

$$\left\|(w(t))^{1/(p(t)-1)}u_n'(t)\right\|_0 \le C_5\|u_n\|_1^{\alpha}. \tag{3.6}$$

Combining (2.47) and (3.3), we have

$$|u_n(0)| \le C_6 \|u_n\|_1^{\alpha}, \quad \text{where } \alpha = \frac{q^+ - 1}{p^- - 1}. \tag{3.7}$$

For any $j = 1, \dots, N$, since

$$\begin{aligned} \left|u_n^j(t)\right| &= \left|u_n^j(0) + \int_0^t \left(u_n^j\right)'(r)dr\right| \\ &\le \left|u_n^j(0)\right| + \left|\int_0^t (w(r))^{-1/(p(r)-1)} \sup_{t\in(0,+\infty)} \left|(w(t))^{1/(p(t)-1)} \left(u_n^j\right)'(t)\right| dr\right| \\ &\le [C_7 + C_5 E] \|u_n\|_1^{\alpha} \le C_8 \|u_n\|_1^{\alpha}, \end{aligned} \tag{3.8}$$

we have

$$\left|u_n^j\right|_0 \le C_9 \|u_n\|_1^{\alpha}, \quad j = 1, \dots, N; \; n = 1, 2, \dots. \tag{3.9}$$

Thus

$$\|u_n\|_0 \le C_{10} \|u_n\|_1^{\alpha}, \quad n = 1, 2, \dots. \tag{3.10}$$

Combining (3.6) and (3.10), we obtain that $\{\|u_n\|_1\}$ is bounded.

Thus, we can choose a large enough $R_0 > 0$ such that all the solutions of (3.1) belong to $B(R_0) = \{u \in C^1 \mid \|u\|_1 < R_0\}$. Thus, the Leray-Schauder degree $d_{LS}[I - \Psi_f(\cdot, \lambda), B(R_0), 0]$ is well defined for each $\lambda \in [0, 1]$, and

$$d_{LS}\left[I - \Psi_f(\cdot, 1), B(R_0), 0\right] = d_{LS}\left[I - \Psi_f(\cdot, 0), B(R_0), 0\right]. \tag{3.11}$$

Let

$$u_0 = \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\left[t, (w(t))^{-1} a(0)\right] dt + e_0}{1 - \sum_{i=1}^{m-2} \alpha_i} + \int_0^r \varphi^{-1}\left[t, (w(t))^{-1} a(0)\right] dt, \tag{3.12}$$

where $a(0)$ is defined in (2.31); thus u_0 is the unique solution of $u = \Psi_f(u, 0)$.

It is easy to see that u is a solution of $u = \Psi_f(u, 0)$ if and only if u is a solution of the following system:

$$\begin{gathered} -\Delta_{p(t)} u = 0, \quad t \in (0, +\infty), \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_0, \qquad \lim_{t\to+\infty} u(t) = \int_0^{+\infty} e(t) u(t) dt. \end{gathered} \tag{I}$$

Obviously, system (I) possesses a unique solution u_0. Note that $u_0 \in B(R_0)$, we have

$$d_{\mathrm{LS}}\left[I - \Psi_f(\cdot, 1), B(R_0), 0\right] = d_{\mathrm{LS}}\left[I - \Psi_f(\cdot, 0), B(R_0), 0\right] \neq 0, \tag{3.13}$$

Therefore (1.1)-(1.2) has at least one solution when $\sigma \in [0, 1)$. This completes the proof. □

Similarly, we have the following theorem.

Theorem 3.2. *If f satisfies sub-$(p^- - 1)$ growth condition, then for any fixed parameter δ, problem* (1.1)-(1.2) *has at least a solution when $\sigma = 1$.*

Now let us consider the boundary asymptotic behavior of solutions of system (1.1)-(1.2).

Theorem 3.3. *If u is a solution of* (1.1)-(1.2) *which is given in Theorem 3.1 or Theorem 3.2, then*

(i) $|u'(t)| \le C_1/(w(t))^{1/(p(t)-1)}$, $t \in (0, +\infty)$;

(ii) $|u(+\infty) - u(r)| \le \int_r^{+\infty} (C_2/(w(t))^{1/(p(t)-1)})dt$, *as* $r \to +\infty$;

(iii) $|u(r) - u(0)| \le \int_0^r (C_3/(w(t))^{1/(p(t)-1)})dt$, *as* $r \to 0^+$.

Proof. Since $\lim_{r\to+\infty} p(r)$ exists, $\lim_{r\to+\infty} p(r) > 1$, and $u \in C^1$, we have $|(w(t))^{1/(p(t)-1)} u'(t)| \le C$, for all $t \in [0, +\infty)$. Thus

(i) $|u'(t)| \le C_1/(w(t))^{1/(p(t)-1)}$, $t \in (0, +\infty)$;

(ii) $|u(+\infty) - u(r)| = |\int_r^{+\infty} u'(t)dt| \le \int_r^{+\infty} (C_2/(w(t))^{1/(p(t)-1)})dt$, as $r \to +\infty$;

(iii) $|u(r) - u(0)| = |\int_0^r u'(t)dt| \le \int_0^r (C_3/(w(t))^{1/(p(t)-1)})dt$, as $r \to 0^+$.

This completes the proof. □

Corollary 3.4. *Assume that $\lim_{r\to+\infty} p(r)$ exists, $\lim_{r\to+\infty} p(r) > 1$, and*

$$\begin{aligned} &C_4 \le \frac{w(t)}{t^{\alpha}} \le C_5, \quad \alpha > p(t) - 1 \quad \text{as } t \longrightarrow +\infty, \\ &C_6 \le \frac{w(t)}{t^{\varpi}} \le C_7, \quad \varpi < p(t) - 1 \quad \text{as } t \longrightarrow 0^+, \end{aligned} \tag{3.14}$$

then

(i) $|u'(t)| \le C_8/t^{\alpha/(p(t)-1)}$, $t \in (1, +\infty)$ *and* $|u'(t)| \le C_9/t^{\varpi/(p(t)-1)}$, $t \in (0, 1)$;

(ii) $|u(+\infty) - u(r)| \le \int_r^{+\infty} (C_{10}/t^{\alpha/(p(t)-1)})dt$, *as* $r \to +\infty$;

(iii) $|u(r) - u(0)| \le \int_0^r (C_{11}/t^{\varpi/(p(t)-1)})dt$, *as* $r \to 0^+$.

4. When f Satisfies General Growth Condition

In the following, we will investigate the existence and asymptotic behavior of solutions for $p(t)$-Laplacian ordinary system, when f satisfies general growth condition. Moreover, we will give the existence of nonnegative solutions.

Denote

$$\Omega_\varepsilon = \left\{u \in C^1 \mid \max_{1\le i\le N}\left(\left|u^i\right|_0 + \left|(w(t))^{1/(p(t)-1)}\left(u^i\right)'\right|_0\right) < \varepsilon\right\}, \qquad \theta = \frac{\varepsilon}{2+1/E}. \tag{4.1}$$

Assume

(A$_1$) Let positive constant ε be such that $u_0 \in \Omega_\varepsilon$, $|P(0)| < \theta$, and $|a(0)| < (1/N(2E + 2))\inf_{t\in J}|\varepsilon/2(E+1)|^{p(t)-1}$, where u_0 is defined in (3.12), and $a(\cdot)$ is defined in (2.31).

It is easy to see that Ω_ε is an open bounded domain in C^1. We have the following theorem

Theorem 4.1. *Assume that (A$_1$) is satisfied. If positive parameter δ is small enough, then the problem* (1.1)-(1.2) *has at least one solution on $\overline{\Omega_\varepsilon}$ when $\sigma \in [0,1)$.*

Proof. Denote $\Psi_f(u,\lambda) = P(\lambda\delta N_f(u)) + K_1(\lambda\delta N_f(u))$. According to Lemma 2.5, u is a solution of

$$-\Delta_{p(t)}u + \lambda\delta f\left(t, u, (w(t))^{1/(p(t)-1)}u', S(u), T(u)\right) = 0, \quad t \in (0,+\infty), \tag{4.2}$$

with (1.2) if and only if u is a solution of the following abstract equation:

$$u = \Psi_f(u,\lambda). \tag{4.3}$$

From Lemmas 2.2 and 2.4, we can see that $\Psi_f(\cdot,\lambda)$ is compact continuous from C^1 to C^1 for any $\lambda \in [0,1]$. According to Leray-Schauder's degree theory, we only need to prove that

(1°) $u = \Psi_f(u,\lambda)$ has no solution on $\partial\Omega_\varepsilon$ for any $\lambda \in [0,1)$;

(2°) $d_{LS}[I - \Psi_f(\cdot,0), \Omega_\varepsilon, 0] \neq 0$;

then we can conclude that the system (1.1)-(1.2) has a solution on $\overline{\Omega_\varepsilon}$.

(1°) If there exists a $\lambda \in [0,1)$ and $u \in \partial\Omega_\varepsilon$ is a solution of (4.2) with (1.2), then (λ,u) satisfies

$$w(t)\varphi(t,u'(t)) = a(\lambda\delta N_f(u)) + \lambda\delta F(N_f(u))(t), \quad t \in (0,+\infty). \tag{4.4}$$

Since $u \in \partial\Omega_\varepsilon$, there exists an i such that $|u^i|_0 + |(w(t))^{1/(p(t)-1)}(u^i)'|_0 = \varepsilon$.

(i) Suppose that $|u^i|_0 > 2\theta$, then $|(w(t))^{1/(p(t)-1)}(u^i)'|_0 < \varepsilon - 2\theta = \theta/E$. On the other hand, for any $t, t' \in J$, we have

$$\left|u^i(t) - u^i(t')\right| = \left|\int_{t'}^{t}\left(u^i\right)'(r)dr\right| \le \int_0^{+\infty}(w(r))^{-1/(p(r)-1)}\left|(w(r))^{1/(p(r)-1)}\left(u^i\right)'(r)\right|dr < \theta. \tag{4.5}$$

This implies that $|u^i(t)| > \theta$ for each $t \in J$.

Note that $u \in \overline{\Omega_\varepsilon}$, then

$$\left| f\left(t, u, (w(t))^{1/(p(t)-1)} u', S(u), T(u)\right) \right| \le \beta_{C_* \varepsilon}(t), \tag{4.6}$$

where

$$C_* := N + \sup_{t \in J} \int_0^{+\infty} \left|\varphi(s,t)\right| ds + \sup_{t \in J} \int_0^{+\infty} \left|\chi(s,t)\right| ds, \tag{4.7}$$

holding $|F(N_f)| \le \int_0^{+\infty} \beta_{C_* \varepsilon}(t) dt$. Since $P(\cdot)$ is continuous, when $0 < \delta$ is small enough, from (A_1), we have

$$|u(0)| = \left| P\left(\lambda \delta N_f(u)\right) \right| < \theta. \tag{4.8}$$

It is a contradiction to $|u^i(t)| > \theta$ for each $t \in J$.

(ii) Suppose that $|u^i|_0 \le 2\theta$; then $\theta / E \le |(w(t))^{1/(p(t)-1)} (u^i)'|_0 \le \varepsilon$. This implies that

$$\left| (w(t_2))^{1/(p(t_2)-1)} \left(u^i\right)'(t_2) \right| > \frac{\varepsilon}{2(E+1)} \quad \text{for some } t_2 \in J. \tag{4.9}$$

Since $u \in \overline{\Omega_\varepsilon}$, it is easy to see that

$$\left| (w(t_2))^{1/(p(t_2)-1)} \left(u^i\right)'(t_2) \right| > \frac{\varepsilon}{2(E+1)} = \frac{N\varepsilon}{N(2E+2)} \ge \frac{\left| (w(t_2))^{1/(p(t_2)-1)} u'(t_2) \right|}{N(2E+2)}. \tag{4.10}$$

Combining (4.4) and (4.10), we have

$$\begin{aligned} \frac{|\varepsilon/2(E+1)|^{p(t_2)-1}}{N(2E+2)} &< \frac{1}{N(2E+2)} w(t_2) \left| \left(u^i\right)'(t_2) \right|^{p(t_2)-1} \\ &\le \frac{1}{N(2E+2)} w(t_2) \left| u'(t_2) \right|^{p(t_2)-1} \\ &\le w(t_2) \left| u'(t_2) \right|^{p(t_2)-2} \left| \left(u^i\right)'(t_2) \right| \\ &\le \left| a\left(\lambda \delta N_f\right) \right| + \lambda \left| \delta F\left(N_f\right)(t_2) \right|. \end{aligned} \tag{4.11}$$

Since $u \in \overline{\Omega_\varepsilon}$ and f is Caratheodory, it is easy to see that

$$\left| f\left(t, u, (w(t))^{1/(p(t)-1)} u', S(u), T(u)\right) \right| \le \beta_{C_* \varepsilon}(t). \tag{4.12}$$

Thus

$$\left|\delta F\left(N_f(u)\right)\right| \le \delta \int_0^{+\infty} \beta_{C_*\varepsilon}(t)dt. \tag{4.13}$$

According to Lemma 2.2, $a(\cdot)$ is continuous; then we have

$$\left|a\left(\lambda\delta N_f(u)\right)\right| \longrightarrow |a(0)| \quad \text{as } \delta \longrightarrow 0. \tag{4.14}$$

When $0 < \delta$ is small enough, from (A_1) and (4.11), we can conclude that

$$\begin{aligned} &\frac{|\varepsilon/(2(E+1))|^{p(t_2)-1}}{N(2E+2)} \\ &\quad < \left|a\left(\lambda\delta N_f(u)\right)\right| + \lambda\left|\delta F\left(N_f(u)\right)(t)\right| < \frac{1}{N(2E+2)} \inf_{t\in J}\left|\frac{\varepsilon}{2(E+1)}\right|^{p(t)-1}. \end{aligned} \tag{4.15}$$

It is a contradiction.

Summarizing this argument, for each $\lambda \in [0,1)$, the problem (4.2) with (1.2) has no solution on $\partial\Omega_\varepsilon$

(2°) Since u_0 (where u_0 is defined in (3.12)) is the unique solution of $u = \Psi_f(u,0)$, and (A_1) holds $u_0 \in \Omega_\varepsilon$, we can see that the Leray-Schauder degree

$$d_{\mathrm{LS}}\left[I - \Psi_f(\cdot,0), \Omega_\varepsilon, 0\right] \ne 0. \tag{4.16}$$

This completes the proof. □

Assume the following.

(A_2) Let positive constant ε be such that $u_0^* \in \Omega_\varepsilon$, $|Q(0)| < \theta$ and $|a^*(0)| < (1/N(2E+2))\inf_{t\in J}|\varepsilon/2(E+1)|^{p(t)-1}$, where $a^*(\cdot)$ is defined in (2.32) and

$$u_0^* = \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi^{-1}\left[t,(w(t))^{-1}a^*(0)\right]dt + e_0}{1-\sum_{i=1}^{m-2}\alpha_i} + \int_0^r \varphi^{-1}\left[t,(w(t))^{-1}a^*(0)\right]dt. \tag{4.17}$$

Theorem 4.2. *Assume that (A_2) is satisfied. If positive parameter δ is small enough, then the problem* (1.1)-(1.2) *has at least one solution on $\overline{\Omega_\varepsilon}$ when $\sigma = 1$.*

Proof. Similar to the proof of Theorem 4.1, we omit it here. □

Note. If u is a solution of (1.1)-(1.2) which is given in Theorem 4.1 or Theorem 4.2, then the conclusions of Theorem 3.3 and Corollary 3.4 are valid.

In the following, we will deal with the existence of nonnegative solutions of (1.1)-(1.2) when $\sigma \in [0,1]$. For any $x = (x^1,\ldots,x^N) \in \mathbb{R}^N$, the notation $x \ge 0$ $(x > 0)$ means $x^j \ge 0$ $(x^j > 0)$ for any $j = 1,\ldots,N$. For any $x, y \in \mathbb{R}^N$, and the notation $x \ge y$ means $x - y \ge 0$, the notation $x > y$ means $x - y > 0$.

Theorem 4.3. *We assume that*

(1^0) $\delta f(t,x,y,z,w) \leq 0$, *for all* $(t,x,y,z,w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$;

(2^0) $e_0 = 0$.

Then every solution of (1.1)-(1.2) *is nonnegative when* $\sigma \in [0,1)$.

Proof. Let u be a solution of (1.1)-(1.2). From Lemma 2.5, we have

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(t)\right)\right]\right\}(t), \quad \forall t \in J. \tag{4.18}$$

We claim that $a(\delta N_f(u)) \geq 0$. If it is false, then there exists some $j \in \{1,\ldots,N\}$ such that $a^j(\delta N_f(u)) < 0$. Combining conditions (1^0), we have

$$\left[a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(t)\right]^j < 0, \quad \forall t \in J. \tag{4.19}$$

Similar to the proof before Lemma 2.2, the boundary value conditions and (2^0) imply that

$$\begin{aligned} 0 = {} & \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} \varphi^{-1}\left[t,(w(t))^{-1}\left(a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(t)\right)\right]dt}{1-\sum_{i=1}^{m-2}\alpha_i} \\ & + \frac{\int_0^{+\infty}\left\{e(t)\int_t^{+\infty} \varphi^{-1}\left[r,(w(r))^{-1}\left(a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(r)\right)\right]dr\right\}dt}{1-\sigma} \\ & + \int_0^{+\infty} \varphi^{-1}\left[t,(w(t))^{-1}\left(a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(t)\right)\right]dt. \end{aligned} \tag{4.20}$$

From (4.19), we get a contradiction to (4.20). Thus $a(\delta N_f(u)) \geq 0$.

We claim that

$$a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(+\infty) \leq 0. \tag{4.21}$$

If it is false, then there exists some $j \in \{1,\ldots,N\}$ such that

$$\left[a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(+\infty)\right]^j > 0. \tag{4.22}$$

It follows from (1^0) and (4.22) that

$$\left[a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(t)\right]^j > 0, \quad \forall t \in J. \tag{4.23}$$

From (4.23), we get a contradiction to (4.20). Thus (4.21) is valid.

Denote

$$\Gamma(t) = a\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(t), \quad \forall t \in J. \tag{4.24}$$

Obviously, $\Gamma(0) = a(\delta N_f(u)) \geq 0$, $\Gamma(+\infty) \leq 0$, and $\Gamma(t)$ is decreasing, that is, $\Gamma(t') \leq \Gamma(t'')$ for any $t', t'' \in J$ with $t' \geq t''$. For any $j = 1, \ldots, N$, there exist $\zeta_j \in J$ such that

$$\Gamma^j(t) \geq 0, \quad \forall t \in (0, \zeta_j), \qquad \Gamma^j(t) \leq 0, \quad \forall t \in (\zeta_j, +\infty), \tag{4.25}$$

which implies that $u^j(t)$ is increasing on $[0, \zeta_j]$, and $u^j(t)$ is decreasing on $(\zeta_j, +\infty)$. Thus

$$\min\left\{u^j(0), u^j(+\infty)\right\} = \inf_{t \in J} u^j(t), \quad j = 1, \ldots, N. \tag{4.26}$$

For any fixed $j \in \{1, \ldots, N\}$, if

$$u^j(0) = \inf_{t \in J} u^j(t), \tag{4.27}$$

which together with (2^0) and (1.2) implies that

$$u^j(0) = \sum_{i=1}^{m-2} \alpha_i u^j(\xi_i) \geq \sum_{i=1}^{m-2} \alpha_i u^j(0), \tag{4.28}$$

then

$$u^j(0) \geq 0. \tag{4.29}$$

If

$$u^j(+\infty) = \inf_{t \in J} u^j(t), \tag{4.30}$$

and from (1.2) and (4.30), we have

$$u^j(+\infty) = \int_0^{+\infty} e^j(t) u^j(t) dt \geq \int_0^{+\infty} e^j(t) u^j(+\infty) dt = \sigma u^j(+\infty), \tag{4.31}$$

then

$$u^j(+\infty) \geq 0. \tag{4.32}$$

Thus $u(t) \geq 0$, for all $t \in [0, +\infty)$. The proof is completed. □

Corollary 4.4. *We assume*

(1^0) $\delta f(t, x, y, z, w) \leq 0$, *for all* $(t, x, y, z, w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ *with* $x, z, w \geq 0$;

(2^0) $\psi(s, t) \geq 0, \chi(s, t) \geq 0$, *for all* $(s, t) \in D$;

(3^0) $e_0 = 0$.

Then we have

(a) *On the conditions of Theorem 3.1, then* (1.1)-(1.2) *has at least a nonnegative solution* u *when* $\sigma \in [0,1)$.

(b) *On the conditions of Theorem 4.1, then* (1.1)-(1.2) *has at least a nonnegative solution* u *when* $\sigma \in [0,1)$.

Proof. (a) Define

$$L(u) = \left(L_*\left(u^1\right), \dots, L_*\left(u^N\right)\right), \tag{4.33}$$

where

$$L_*(t) = \begin{cases} t, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{4.34}$$

Denote

$$\widetilde{f}(t,u,v,S(u),T(u)) = f(t,L(u),v,S(L(u)),T(L(u))), \quad \forall (t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N. \tag{4.35}$$

Then $\widetilde{f}(t,u,v,S(u),T(u))$ satisfies Caratheodory condition, and $\widetilde{f}(t,u,v,S(u),T(u)) \leq 0$ for any $(t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N$.

Obviously, we have

(A_3) $\lim_{|u|+|v| \to +\infty} (\widetilde{f}(t,u,v,S(u),T(u))/(|u|+|v|)^{q(t)-1}) = 0$, for $t \in J$ uniformly, where $q(t) \in C(J,\mathbb{R})$, and $1 < q^- \leq q^+ < p^-$.

Then $\widetilde{f}(t,\cdot,\cdot,\cdot,\cdot)$ satisfies sub-$(p^- - 1)$ growth condition.

Let us consider the existence of solutions of the following system:

$$-\Delta_{p(t)} u + \delta \widetilde{f}\left(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u)\right) = 0, \quad t \in (0,+\infty), \tag{4.36}$$

with boundary value condition (1.2). According to Theorem 3.1, (4.36) with (1.2) has at least a solution u. From Theorem 4.3, we can see that u is nonnegative. Thus, u is a nonnegative solution of (1.1)-(1.2) when $\sigma \in [0,1)$.

(b) It is similar to the proof of (a).

This completes the proof. □

Theorem 4.5. *We assume that*

(1^0) $\delta f(t,x,y,z,w) \leq 0$, *for all* $(t,x,y,z,w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$;

(2^0) $e(t) > 0$, *for almost every* $t \in J$;

(3^0) $e_0 = 0$.

Then every solution of (1.1)-(1.2) *is nonnegative when* $\sigma = 1$.

Proof. Similar to the proof of Theorem 4.3, we can find that there exist $\zeta_j \in J$ for any $j = 1, \ldots, N$ such that $u^j(t)$ is increasing on $[0, \zeta_j]$, and $u^j(t)$ is decreasing on $(\zeta_j, +\infty)$. Thus

$$\min\left\{u^j(0), u^j(+\infty)\right\} = \inf_{t\in J} u^j(t), \quad j = 1, \ldots, N. \tag{4.37}$$

For any fixed $j \in \{1, \ldots, N\}$, if

$$u^j(0) = \inf_{t\in J} u^j(t), \tag{4.38}$$

which together with (3^0) and (1.2) implies that

$$u^j(0) = \sum_{i=1}^{m-2} \alpha_i u^j(\xi_i) \geq \sum_{i=1}^{m-2} \alpha_i u^j(0), \tag{4.39}$$

then

$$u^j(0) \geq 0. \tag{4.40}$$

If

$$u^j(+\infty) = \inf_{t\in J} u^j(t), \tag{4.41}$$

from (1.2), we have

$$(1-\sigma)u^j(+\infty) = \int_0^{+\infty} e^j(t)\left(u^j(t) - u^j(+\infty)\right)dt. \tag{4.42}$$

Since $\sigma = 1$ and $e(t) > 0$, we have

$$u^j(t) \equiv u^j(+\infty). \tag{4.43}$$

It follows from (4.43), (1.2), and (3^0) that

$$u^j(+\infty) = 0. \tag{4.44}$$

Thus $u(t) \geq 0$, for all $t \in [0, +\infty)$. The proof is completed. □

Corollary 4.6. *We assume that*

(1^0) $\delta f(t, x, y, z, w) \leq 0$, *for all* $(t, x, y, z, w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$;

(2^0) $e(t) > 0$, *for almost every* $t \in J$;

(3^0) $\psi(s, t) \geq 0, \chi(s, t) \geq 0$, *for all* $(s, t) \in D$;

(4^0) $e_0 = 0$.

Then we have the following.

(a) *On the conditions of Theorem 3.2, then* (1.1)-(1.2) *has at least a nonnegative solution* u *when* $\sigma = 1$.

(b) *On the conditions of Theorem 4.2, then* (1.1)-(1.2) *has at least a nonnegative solution* u *when* $\sigma = 1$.

Proof. It is similar to the proof of Theorem 4.5. □

5. Examples

Example 5.1. Consider the following problem:

$$\begin{gathered}
-\Delta_{p(t)} u - |u|^{q(t)-2} u - S(u)(t) - (t+1)^{-2} = 0, \quad t \in (0, +\infty), \\
u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) \ , \qquad \lim_{t \to +\infty} u(t) = \int_0^{+\infty} e^{-2t} u(t)\, dt,
\end{gathered} \tag{S_1}$$

where $p(t) = 6 + e^{-t} \sin t$, and $q(t) = 3 + 2^{-t} \cos t$, $S(u)(t) = \int_0^\infty e^{-2s} (\sin st + 1) u(s) ds$.

Obviously, $|u|^{q(t)-2} u + S(u)(t) + (t+1)^{-2}$ is Caratheodory, $q(t) \leq 4 < 5 \leq p(t)$, and the conditions of Theorems 3.1 and 4.3 are satisfied; then (S_1) has a nonnegative solution.

Example 5.2. Consider the following problem:

$$\begin{gathered}
-\Delta_{p(t)} u + f\Big(r, u, (w(r))^{1/(p(r)-1)} u', S(u)\Big) + \delta h\Big(r, u, (w(r))^{1/(p(r)-1)} u', S(u)\Big) + e^{-t} \\
= 0, \quad t \in (0, +\infty), \\
u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_0, \qquad \lim_{t \to +\infty} u(t) = \int_0^{+\infty} e^{-t} u(t)\, dt,
\end{gathered} \tag{S_2}$$

where h is Caratheodory and

$$f\Big(r, u, (w(r))^{1/(p(r)-1)} u', S(u)\Big) = |u|^{q(t)-2} u + w(t) \left|u'\right|^{q(t)-2} u' + S(u)(t), \tag{5.1}$$

$p(t) = 7 + 3^{-t} \cos 3t$, $q(t) = 4 + e^{-2t} \sin 2t$, and $S(u)(t) = \int_0^\infty e^{-s} (\cos st + 1) u(s) ds$.

Obviously, $|u|^{q(t)-2} u + w(t) |u'|^{q(t)-2} u' + S(u)(t)$ is Caratheodory, $q(t) \leq 5 < 6 \leq p(t)$, and the conditions of Theorem 4.1 are satisfied; then (S_2) has a solution when δ is small enough.

Acknowledgments

This paper was partly supported by the National Science Foundation of China (10701066 and 10926075 and 10971087), and China Postdoctoral Science Foundation funded project (20090460969), and the Natural Science Foundation of Henan Education Committee (2008-755-65).

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