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Research Article

On the Strong Laws for Weighted Sums of ρ^*-Mixing Random Variables

Xing-Cai Zhou,[1,2] Chang-Chun Tan,[3] and Jin-Guan Lin[1]

[1] *Department of Mathematics, Southeast University, Nanjing 210096, China*
[2] *Department of Mathematics and Computer Science, Tongling University, Tongling, Anhui 244000, China*
[3] *School of Mathematics, Heifei University of Technology, Hefei, Anhui 230009, China*

Correspondence should be addressed to Chang-Chun Tan, cctan@ustc.edu.cn

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Complete convergence is studied for linear statistics that are weighted sums of identically distributed ρ^*-mixing random variables under a suitable moment condition. The results obtained generalize and complement some earlier results. A Marcinkiewicz-Zygmund-type strong law is also obtained.

1. Introduction

Suppose that $\{X_n;\ n \geq 1\}$ is a sequence of random variables and S is a subset of the natural number set N. Let $F_S = \sigma(X_i;\ i \in S)$,

$$\rho_n^* = \sup\left\{\operatorname{corr}(f, g) : \forall S \times T \subset N \times N,\ \operatorname{dist}(S, T) \geq n,\ \forall f \in L^2(F_S),\ g \in L^2(F_T)\right\}, \tag{1.1}$$

where

$$\operatorname{corr}(f, g) = \frac{\operatorname{Cov}\{f(X_i;\ i \in S), g(X_j;\ j \in T)\}}{\left[\operatorname{Var}\{f(X_i;\ i \in S)\} \operatorname{Var}\{g(X_j;\ j \in T)\}\right]^{1/2}}. \tag{1.2}$$

Definition 1.1. A random variable sequence $\{X_n;\ n \geq 1\}$ is said to be a ρ^*-mixing random variable sequence if there exists $k \in N$ such that $\rho_k^* < 1$.

The notion of ρ^*-mixing seems to be similar to the notion of ρ-mixing, but they are quite different from each other. Many useful results have been obtained for ρ^*-mixing random variables. For example, Bradley [1] has established the central limit theorem, Byrc and Smoleński [2] and Yang [3] have obtained moment inequalities and the strong law of large numbers, Wu [4, 5], Peligrad and Gut [6], and Gan [7] have studied almost sure convergence, Utev and Peligrad [8] have established maximal inequalities and the invariance principle, An and Yuan [9] have considered the complete convergence and Marcinkiewicz-Zygmund-type strong law of large numbers, and Budsaba et al. [10] have proved the rate of convergence and strong law of large numbers for partial sums of moving average processes based on ρ^--mixing random variables under some moment conditions.

For a sequence $\{X_n;\ n \geq 1\}$ of i.i.d. random variables, Baum and Katz [11] proved the following well-known complete convergence theorem: suppose that $\{X_n;\ n \geq 1\}$ is a sequence of i.i.d. random variables. Then $EX_1 = 0$ and $E|X_1|^{rp} < \infty$ $(1 \leq p < 2, r \geq 1)$ if and only if $\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{n} X_i| > n^{1/p}\varepsilon) < \infty$ for all $\varepsilon > 0$.

Hsu and Robbins [12] and Erdös [13] proved the case $r = 2$ and $p = 1$ of the above theorem. The case $r = 1$ and $p = 1$ of the above theorem was proved by Spitzer [14]. An and Yuan [9] studied the weighted sums of identically distributed ρ^*-mixing sequence and have the following results.

Theorem B. *Let $\{X_n;\ n \geq 1\}$ be a ρ^*-mixing sequence of identically distributed random variables, $\alpha p > 1$, $\alpha > 1/2$, and suppose that $EX_1 = 0$ for $\alpha \leq 1$. Assume that $\{a_{ni};\ 1 \leq i \leq n\}$ is an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^p = O(\delta), \quad 0 < \delta < 1, \tag{1.3}$$

$$\sharp A_{nk} = \sharp\left\{1 \leq i \leq n : |a_{ni}|^p > (k+1)^{-1}\right\} \geq n e^{-1/k}. \tag{1.4}$$

If $E|X_1|^p < \infty$, then

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} a_{ni} X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{1.5}$$

Theorem C. *Let $\{X_n;\ n \geq 1\}$ be a ρ^*-mixing sequence of identically distributed random variables, $\alpha p > 1$, $\alpha > 1/2$, and $EX_1 = 0$ for $\alpha \leq 1$. Assume that $\{a_{ni};\ 1 \leq i \leq n\}$ is array of real numbers satisfying* (1.3). *Then*

$$n^{-1/p} \sum_{i=1}^{n} a_{ni} X_i \longrightarrow 0 \ a.s. \ (n \longrightarrow \infty). \tag{1.6}$$

Recently, Sung [15] obtained the following complete convergence results for weighted sums of identically distributed NA random variables.

Theorem D. *Let $\{X, X_n;\ n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni};\ 1 \leq i \leq n,\ n \geq 1\}$ be an array of constants satisfying*

$$A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n} = \sum_{i=1}^{n} \frac{|a_{ni}|^\alpha}{n} \tag{1.7}$$

for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ where $1 < \alpha \leq 2$. If

$$\begin{gathered} E|X|^\alpha < \infty, \quad \text{for } \alpha > \gamma, \\ E|X|^\alpha \log|X| < \infty, \quad \text{for } \alpha = \gamma, \\ E|X|^\gamma < \infty, \quad \text{for } \alpha < \gamma, \end{gathered} \tag{1.8}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > b_n \varepsilon \right) < \infty \quad \forall \varepsilon > 0. \tag{1.9}$$

We find that the proof of Theorem C is mistakenly based on the fact that (1.5) holds for $\alpha p = 1$. Hence, the Marcinkiewicz-Zygmund-type strong laws for ρ^*-mixing sequence have not been established.

In this paper, we shall not only partially generalize Theorem D to ρ^*-mixing case, but also extend Theorem B to the case $\alpha p = 1$. The main purpose is to establish the Marcinkiewicz-Zygmund strong laws for linear statistics of ρ^*-mixing random variables under some suitable conditions.

We have the following results.

Theorem 1.2. *Let $\{X, X_n; n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables, and let $\{a_{ni}; 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying*

$$A_\beta = \limsup_{n\to\infty} A_{\beta,n} < \infty, \quad A_{\beta,n} = \sum_{i=1}^{n} \frac{|a_{ni}|^\beta}{n}, \tag{1.10}$$

where $\beta = \max(\alpha, \gamma)$ for some $0 < \alpha \leq 2$ and $\gamma > 0$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. If $EX = 0$ for $1 < \alpha \leq 2$ and (1.8) for $\alpha \neq \gamma$, then (1.9) holds.

Remark 1.3. The proof of Theorem D was based on Theorem 1 of Chen et al. [16], which gave sufficient conditions about complete convergence for NA random variables. So far, it is not known whether the result of Chen et al. [16] holds for ρ^*-mixing sequence. Hence, we use different methods from those of Sung [15]. We only extend the case $\alpha \neq \gamma$ of Theorem D to ρ^*-mixing random variables. It is still open question whether the result of Theorem D about the case $\alpha = \gamma$ holds for ρ^*-mixing sequence.

Theorem 1.4. *Under the conditions of Theorem 1.2, the assumptions $EX = 0$ for $1 < \alpha \le 2$ and (1.8) for $\alpha \ne \gamma$ imply the following Marcinkiewicz-Zygmund strong law:*

$$b_n^{-1}\sum_{i=1}^{n} a_{ni}X_i \longrightarrow 0 \text{ a.s. } (n \longrightarrow \infty). \tag{1.11}$$

2. Proof of the Main Result

Throughout this paper, the symbol C represents a positive constant though its value may change from one appearance to next. It proves convenient to define $\log x = \max(1, \ln x)$, where $\ln x$ denotes the natural logarithm.

To obtain our results, the following lemmas are needed.

Lemma 2.1 (Utev and Peligrad [8]). *Suppose N is a positive integer, $0 \le r < 1$, and $q \ge 2$. Then there exists a positive constant $D = D(N, r, q)$ such that the following statement holds.*

If $\{X_i;\ i \ge 1\}$ is a sequence of random variables such that $\rho_N^* \le r$ with $EX_i = 0$ and $E|X_i|^q < \infty$ for every $i \ge 1$, then for all $n \ge 1$,

$$E\left(\max_{1\le i\le n}|S_i|^q\right) \le D\left(\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right), \tag{2.1}$$

where $S_i = \sum_{j=1}^{i} X_j$.

Lemma 2.2. *Let X be a random variable and $\{a_{ni};\ 1 \le i \le n,\ n \ge 1\}$ be an array of constants satisfying (1.10), $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. Then*

$$\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P(|a_{ni}X| > b_n) \le \begin{cases} CE|X|^{\alpha} & \text{for } \alpha > \gamma, \\ CE|X|^{\gamma} & \text{for } \alpha < \gamma. \end{cases} \tag{2.2}$$

Proof. If $\gamma > \alpha$, by $\sum_{i=1}^{n} |a_{ni}|^{\gamma} = O(n)$ and Lyapounov's inequality, then

$$\frac{1}{n}\sum_{i=1}^{n}|a_{ni}|^{\alpha} \le \left(\frac{1}{n}\sum_{i=1}^{n}|a_{ni}|^{\gamma}\right)^{\alpha/\gamma} = O(1). \tag{2.3}$$

Hence, (1.7) is satisfied. From the proof of (2.1) of Sung [15], we obtain easily that the result holds. □

Proof of Theorem 1.2. Let $X_{ni} = a_{ni}X_i I(|a_{ni}X_i| \le b_n)$. For all $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right| > \varepsilon b_n\right) \le \sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|a_{nj}X_j\right| > b_n\right) + \sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_{ni}\right| > \varepsilon b_n\right)$$
$$:= I_1 + I_2. \tag{2.4}$$

To obtain (1.9), we need only to prove that $I_1 < \infty$ and $I_2 < \infty$.

By Lemma 2.2, one gets

$$I_1 \le \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n}P\left(\left|a_{nj}X_j\right| > b_n\right) = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n}P\left(\left|a_{nj}X\right| > b_n\right) < \infty. \tag{2.5}$$

Before the proof of $I_2 < \infty$, we prove firstly

$$b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}X_i I(|a_{ni}X_i| \le b_n)\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.6}$$

For $0 < \alpha \le 1$,

$$b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}X_i I(|a_{ni}X_i| \le b_n)\right| \le b_n^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|a_{ni}X_i| \le b_n) \le b_n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|^{\alpha}E|X|^{\alpha}$$
$$\le C(\log n)^{-\alpha/\gamma}E|X|^{\alpha} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.7}$$

For $1 < \alpha \le 2$,

$$b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}X_i I(|a_{ni}X_i| \le b_n)\right| = b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}X_i I(|a_{ni}X_i| > b_n)\right| (EX_i = 0)$$
$$\le b_n^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|a_{ni}X_i| > b_n) \le b_n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|^{\alpha}E|X|^{\alpha} \tag{2.8}$$
$$\le C(\log n)^{-\alpha/\gamma}E|X|^{\alpha} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Thus (2.6) holds. So, to prove $I_2 < \infty$, it is enough to show that

$$I_3 = \sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_{ni} - EX_{ni}\right| > \varepsilon b_n\right) < \infty, \quad \forall \varepsilon > 0. \tag{2.9}$$

By the Chebyshev inequality and Lemma 2.1, for $q \ge \max\{2, \gamma\}$, we have

$$\begin{aligned} I_3 &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-q} E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_{ni} - EX_{ni}\right|^q\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E|a_{ni}X_i|^q I(|a_{ni}X_i| \le b_n) \\ &\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-q}\left[\sum_{i=1}^{n} E(a_{ni}X_i)^2 I(|a_{ni}X_i| \le b_n)\right]^{q/2} \\ &:= I_{31} + I_{32}. \end{aligned} \tag{2.10}$$

For I_{31}, we consider the following two cases.

If $\alpha < \gamma$, note that $E|X|^\gamma < \infty$. We have

$$I_{31} \le C\sum_{n=1}^{\infty} n^{-1} b_n^{-\gamma} \sum_{i=1}^{n} |a_{ni}|^\gamma E|X|^\gamma \le C\sum_{n=1}^{\infty} n^{-\frac{\gamma}{\alpha}} (\log n)^{-1} < \infty. \tag{2.11}$$

If $\alpha > \gamma$, note that $E|X|^\alpha < \infty$. we have

$$I_{31} \le C\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^\alpha E|X|^\alpha \le C\sum_{n=1}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} < \infty. \tag{2.12}$$

Next, we prove $I_{32} < \infty$ in the following two cases.

If $\alpha < \gamma \le 2$ or $\gamma < \alpha \le 2$, take $q > \max(2, 2\gamma/\alpha)$. Noting that $E|X|^\alpha < \infty$, we have

$$\begin{aligned} I_{32} &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha q/2}\left[\sum_{i=1}^{n} |a_{ni}|^\alpha E|X|^\alpha\right]^{q/2} \\ &\le C\sum_{n=1}^{\infty} n^{-1} (\log n)^{-\alpha q/(2\gamma)} < \infty. \end{aligned} \tag{2.13}$$

If $\gamma > 2 \ge \alpha$ or $\gamma \ge 2 > \alpha$, one gets $E|X|^2 < \infty$. Since $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$, it implies $\max_{1\le i\le n} |a_{ni}|^\alpha \le Cn$. Therefore, we have

$$\sum_{i=1}^{n} |a_{ni}|^k = \sum_{i=1}^{n} |a_{ni}|^\alpha |a_{ni}|^{k-\alpha} \le Cnn^{(k-\alpha)/\alpha} = Cn^{k/\alpha} \tag{2.14}$$

for all $k \geq \alpha$. Hence, $\sum_{i=1}^{n} |a_{ni}|^2 = O(n^{2/\alpha})$. Taking $q > \gamma$, we have

$$
\begin{aligned}
I_{32} &\leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-q} \left[\sum_{i=1}^{n} |a_{ni}|^2 \right]^{q/2} \\
&\leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-q} n^{q/\alpha} = C \sum_{n=1}^{\infty} n^{-1} (\log n)^{-q/\gamma} < \infty.
\end{aligned}
\tag{2.15}
$$

□

Proof of Theorem 1.4. By (1.9), a standard computation (see page 120 of Baum and Katz [11] or page 1472 of An and Yuan [9]), and the Borel-Cantelli Lemma, we have

$$
\frac{\max_{1 \leq j \leq 2^i} \left| \sum_{i=1}^{j} a_{ni} X_i \right|}{2^{(i+1)/\alpha} \left(\log 2^{i+1}\right)^{1/\gamma}} \longrightarrow 0 \text{ a.s. } (i \longrightarrow \infty). \tag{2.16}
$$

For any $n \geq 1$, there exists an integer i such that $2^{i-1} \leq n < 2^i$. So

$$
\max_{2^{i-1} \leq n < 2^i} \frac{\left| \sum_{j=1}^{n} a_{nj} X_j \right|}{b_n} \leq \frac{\max_{1 \leq j \leq 2^i} \left| \sum_{i=1}^{j} a_{nj} X_j \right|}{2^{(i-1)/\alpha} \left(\log 2^{i-1}\right)^{1/\gamma}} = 2^{2/\alpha} \frac{\max_{1 \leq j \leq 2^i} \left| \sum_{j=1}^{n} a_{nj} X_j \right|}{2^{(i+1)/\alpha} \left(\log 2^{i+1}\right)^{1/\gamma}} \left(\frac{i+1}{i-1} \right)^{1/\gamma}. \tag{2.17}
$$

From (2.16) and (2.17), we have

$$
\lim_{n \to \infty} b_n^{-1} \sum_{i=1}^{n} a_{ni} X_i = 0 \text{ a.s.} \tag{2.18}
$$

□

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