Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 401913, 16 pages
doi:10.1155/2011/401913

Research Article

On Certain Subclasses of Meromorphically p-Valent Functions Associated by the Linear Operator D_λ^n

Amin Saif and Adem Kılıçman

Department of Mathematics, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

Correspondence should be addressed to Adem Kılıçman, akilicman@putra.upm.edu.my

Received 26 July 2010; Accepted 28 February 2011

Academic Editor: Jong Kim

The purpose of this paper is to introduce two novel subclasses $\Gamma_\lambda(n,\alpha,\beta)$ and $\Gamma_\lambda^*(n,\alpha,\beta)$ of meromorphic p-valent functions by using the linear operator D_λ^n. Then we prove the necessary and sufficient conditions for a function f in order to be in the new classes. Further we study several important properties such as coefficients inequalities, inclusion properties, the growth and distortion theorems, the radii of meromorphically p-valent starlikeness, convexity, and subordination properties. We also prove that the results are sharp for a certain subclass of functions.

1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^k \quad \left(a_k \geq 0;\ p \in N = \{1,2,\ldots\}\right), \tag{1.1}$$

which are meromorphic and p-valent in the punctured unit disc $U^* = \{z \in C : 0 < |z| < 1\} = U - \{0\}$. For the functions f in the class Σ_p, we define a linear operator D_λ^n by the following form:

$$\begin{aligned}
D_\lambda f(z) &= \left(1+p\lambda\right) f(z) + \lambda z f'(z), \quad (\lambda \geq 0),\\
D_\lambda^0 f(z) &= f(z),\\
D_\lambda^1 f(z) &= D_\lambda f(z),\\
D_\lambda^2 f(z) &= D_\lambda\left(D_\lambda^1 f(z)\right),
\end{aligned} \tag{1.2}$$

and in general for $n = 0, 1, 2, \ldots$, we can write

$$D_\lambda^n f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} (1 + p\lambda + k\lambda)^n a_k z^k, \quad (n \in N_0 = N \cup \{0\};\ p \in N). \tag{1.3}$$

Then we can observe easily that for $f \in \Sigma_p$,

$$z\lambda\left(D_\lambda^n f(z)\right)' = D_\lambda^{n+1} f(z) - (1 + p\lambda) D_\lambda^n f(z), \quad (p \in N;\ n \in N_0). \tag{1.4}$$

Recall [1, 2] that a function $f \in \Sigma_p$ is said to be meromorphically starlike of order α if it is satisfying the following condition:

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in U^*), \tag{1.5}$$

for some α $(0 \leqq \alpha < 1)$. Similarly recall [3] a function $f \in \Sigma_p$ is said to be meromorphically convex of order α if it is satisfying the following condition:

$$\operatorname{Re}\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in U^*) \text{ for some } \alpha\ (0 \le \alpha < 1). \tag{1.6}$$

Let $\Sigma_p(\alpha)$ be a subclass of Σ_p consisting the functions which satisfy the following inequality:

$$\operatorname{Re}\left\{-\frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)}\right\} > p\alpha, \quad (z \in U^*;\ \alpha \ge 0). \tag{1.7}$$

In the following definitions, we will define subclasses $\Gamma_\lambda(n, \alpha, \beta)$ and $\Gamma_\lambda^*(n, \alpha, \beta)$ by using the linear operator D_λ^n.

Definition 1.1. For fixed parameters $\alpha \ge 0,\ 0 \le \beta < 1$, the meromorphically p-valent function $f(z) \in \Sigma_p(\alpha)$ will be in the class $\Gamma_\lambda(n, \alpha, \beta)$ if it satisfies the following inequality:

$$\operatorname{Re}\left\{-\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}\right\} \ge \alpha \left|\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)} + 1\right| + \beta, \quad (n \in N_0). \tag{1.8}$$

Definition 1.2. For fixed parameters $\alpha \ge 1/(2 + \beta);\ 0 \le \beta < 1$, the meromorphically p-valent function $f(z) \in \Sigma_p(\alpha)$ will be in the class $\Gamma_\lambda^*(n, \alpha, \beta)$ if it satisfies the following inequality:

$$\left|\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)} + \alpha + \alpha\beta\right| \le \operatorname{Re}\left\{-\frac{z\left(D_\lambda^n f(z)\right)'}{p\left(D_\lambda^n f(z)\right)}\right\} + \alpha - \alpha\beta, \quad \forall (n \in N_0). \tag{1.9}$$

Meromorphically multivalent functions have been extensively studied by several authors, see for example, Aouf [4–6], Joshi and Srivastava [7], Mogra [8, 9], Owa et al. [10], Srivastava et al. [11], Raina and Srivastava [12], Uralegaddi and Ganigi [13], Uralegaddi and Somanatha [14], and Yang [15]. Similarly, in [16], some new subclasses of meromorphic functions in the punctured unit disk was considered.

In [17], similar results were proved by using the p-valent functions that satisfy the following differential subordinations:

$$\frac{z\left(\mathcal{J}_p(r,\lambda)f(z)\right)^{(j+1)}}{(p-j)\left(\mathcal{J}_p(r,\lambda)f(z)\right)^{(j)}} \prec \frac{a+\left(aB+(A-B)\beta\right)z}{a(1+Bz)} \tag{1.10}$$

and studied the related coefficients inequalities with β complex number.

This paper is organized as follows. It consists of four sections. Sections 2 and 3 investigate the various important properties and characteristics of the classes $\Gamma_\lambda(n,\alpha,\beta)$ and $\Gamma_\lambda^*(n,\alpha,\beta)$ by giving the necessary and sufficient conditions. Further we study the growth and distortion theorems and determine the radii of meromorphically p-valent starlikeness of order μ $(0 \le \mu < p)$ and meromorphically p-valent convexity of order μ $(0 \le \mu < p)$. In Section 4 we give some results related to the subordination properties.

2. Properties of the Class $\Gamma_\lambda(n,\alpha,\beta)$

We begin by giving the necessary and sufficient conditions for functions f in order to be in the class $\Gamma_\lambda(n,\alpha,\beta)$.

Lemma 2.1 (see [2]). *Let*

$$R_a = \begin{cases} a-\dfrac{\alpha+\beta}{1+\alpha}, & \textit{for } a \le 1+\dfrac{1-\beta}{\alpha(1+\alpha)}, \\[2ex] \sqrt{(1-a)^2(1-\alpha^2)-2(1-\beta)(1-a)}, & \textit{for } a \ge 1+\dfrac{1-\beta}{\alpha(1+\alpha)}. \end{cases} \tag{2.1}$$

Then

$$\{w : |w-a| \le R_a\} \subseteq \left\{w : \operatorname{Re}(w) \ge \alpha|w-1|+\beta\right\}. \tag{2.2}$$

Theorem 2.2. *Let $f \in \Sigma_p$. Then f is in the class $\Gamma_\lambda(n,\alpha,\beta)$ if and only if*

$$\sum_{k=p+1}^{\infty}\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n a_k \le p(1-\beta) \qquad \left(\alpha \ge 0;\ 0 \le \beta < 1;\ p \in N;\ n \in N_0\right). \tag{2.3}$$

Proof. Suppose that $f \in \Gamma_\lambda(n,\alpha,\beta)$. Then by the inequalities (1.3) and (1.8), we get that

$$\operatorname{Re}\left\{-\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}\right\} \geq \alpha\left|\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}+1\right|+\beta. \tag{2.4}$$

That is,

$$\begin{aligned}\operatorname{Re}&\left\{\frac{1-\sum_{k=p+1}^{\infty}(k/p)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right\}\\ &\geq \alpha\left|\frac{\sum_{k=p+1}^{\infty}((k/p)+1)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right|+\beta\\ &\geq \operatorname{Re}\left\{\alpha\cdot\frac{\sum_{k=p+1}^{\infty}((k/p)+1)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}+\beta\right\}\\ &= \operatorname{Re}\left\{\frac{\beta+\sum_{k=p+1}^{\infty}[\alpha((k/p)+1)+\beta](k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right\},\end{aligned} \tag{2.5}$$

that is,

$$\operatorname{Re}\left\{\frac{p(1-\beta)-\sum_{k=p+1}^{\infty}(k+k\alpha+p\alpha+p\beta)(k\lambda+p\lambda+1)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k z^{k+p}}\right\} \geq 0. \tag{2.6}$$

Taking z to be real and putting $z \to 1^-$ through real values, then the inequality (2.6) yields

$$\frac{p(1-\beta)-\sum_{k=p+1}^{\infty}(k+k\alpha+p\alpha+p\beta)(k\lambda+p\lambda+1)^n a_k}{1+\sum_{k=p+1}^{\infty}(k\lambda+p\lambda+1)^n a_k} \geq 0, \tag{2.7}$$

which leads us at once to (2.3).

In order to prove the converse, suppose that the inequality (2.3) holds true. In Lemma 2.1, since $1 \leq 1+((1-\beta)/\alpha(1+\alpha))$, put $a=1$. Then for $p \in N$ and $n \in N_0$, let $w_{np} = -z(D_\lambda^n f(z))'/p(D_\lambda^n f(z))$. If we let $z \in \partial U^* = \{z \in C : |z|=1\}$, we get from the inequalities (1.3) and (2.3) that $|w_{np}-1| \leq R_1$. Thus by Lemma 2.1 above, we ge that

$$\begin{aligned}\operatorname{Re}\left\{-\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}-1\right\} &= \operatorname{Re}\{w_{np}\} \geq \alpha|w_{np}-1|+\beta = \alpha\left|-\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}-1\right|+\beta\\ &= \alpha\left|\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}+1\right|+\beta, \quad (\alpha \geq 0;\ 0 \leq \beta < 1;\ p \in N;\ n \in N_0).\end{aligned} \tag{2.8}$$

Therefore by the maximum modulus theorem, we obtain $f \in \Gamma_\lambda(n,\alpha,\beta)$. □

Corollary 2.3. *If $f \in \Gamma_\lambda(n,\alpha,\beta)$, then*

$$a_k \leq \frac{p(1-\beta)}{[p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n}, \quad (\alpha \geq 0;\ 0 \leq \beta < 1;\ p \in N;\ n \in N_0). \tag{2.9}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \frac{p(1-\beta)}{[p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n} z^k, \quad (\alpha \geq 0;\ 0 \leq \beta < 1;\ p \in N;\ n \in N_0). \tag{2.10}$$

Theorem 2.4. *The class $\Gamma_\lambda(n,\alpha,\beta)$ is closed under convex linear combinations.*

Proof. Suppose the function

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^{k,j} \quad (a_{k,j} \geq 0;\ j = 1,2;\ p \in N), \tag{2.11}$$

be in the class $\Gamma_\lambda(n,\alpha,\beta)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = (1-\delta) f_1(z) + \delta f_2(z) \quad (0 \leq \delta \leq 1), \tag{2.12}$$

is also in the class $\Gamma_\lambda(n,\alpha,\beta)$. Since

$$h(z) = z^{-p} + \sum_{k=p+1}^{\infty} [(1-\delta)a_{k,1} + \delta a_{k,2}] z^{k,j}, \quad (0 \leq \delta \leq 1), \tag{2.13}$$

and by Theorem 2.2, we get that

$$\begin{aligned} &\sum_{k=p+1}^{\infty} [p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n[(1-\delta)a_{k,1}+\delta a_{k,2}] \\ &\quad = \sum_{k=p+1}^{\infty} (1-\delta)[p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n a_{k,1} \\ &\qquad + \sum_{k=p+1}^{\infty} \delta[p(\alpha+\beta)+k(1+\alpha)](k\lambda+p\lambda+1)^n a_{k,2} \\ &\quad \leq (1-\delta)p(1-\beta) + \delta p(1-\beta) = p(1-\beta), \quad (\alpha \geq 0;\ 0 \leq \beta < 1;\ p \in N;\ n \in N_0). \end{aligned} \tag{2.14}$$

Hence $f \in \Gamma_\lambda(n,\alpha,\beta)$. □

The following are the growth and distortion theorems for the class $\Gamma_\lambda(n,\alpha,\beta)$.

Theorem 2.5. *If $f \in \Gamma_\lambda(n,\alpha,\beta)$, then*

$$\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(1-\beta)}{(2\alpha+\beta+1)(p+1)^{n-1}} \cdot \frac{p!2^{-n}}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \leq \left|f^{(m)}(z)\right|$$
$$\leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(1-\beta)}{(2\alpha+\beta+1)(p+1)^{n-1}} \cdot \frac{p!2^{-n}}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \tag{2.15}$$
$$(0<|z|=r<1;\ \alpha \geq 0;\ 0 \leq \beta < 1;\ p \in N;\ n,m \in N_0;\ p>m).$$

The result is sharp for the function f given by

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \frac{(1-\beta)}{(2\alpha+\beta+1)(2p+2)^n} z^p, \quad (n \in N_0;\ p \in N). \tag{2.16}$$

Proof. From Theorem 2.2, we get that

$$\frac{p(2\alpha+\beta+1)(2p+2)^n}{(p+1)!} \sum_{k=p+1}^{\infty} k!a_k \leq \sum_{k=p+1}^{\infty} \left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n a_k$$
$$\leq p(1-\beta), \tag{2.17}$$

that is,

$$\sum_{k=p+1}^{\infty} k!a_k \leq \frac{p(1-\beta)(p+1)!}{p(2\alpha+\beta+1)(2p+2)^n} = \frac{(1-\beta)p!2^{-n}}{(2\alpha+\beta+1)(p+1)^{n-1}}. \tag{2.18}$$

By the differentiating the function f in the form (1.1) m times with respect to z, we get that

$$f^m(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-(p+m)} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m}, \quad (m \in N_0;\ p \in N) \tag{2.19}$$

and Theorem 2.5 follows easily from (2.18) and (2.19). Finally, it is easy to see that the bounds in (2.15) are attained for the function f given by (2.18). □

Next we determine the radii of meromorphically p-valent starlikeness of order μ $(0 \leq \mu < p)$ and meromorphically p-valent convexity of order μ $(0 \leq \mu < p)$ for the class $\Gamma_\lambda(n,\alpha,\beta)$.

Theorem 2.6. *If $f \in \Gamma_\lambda(n,\alpha,\beta)$, then f is meromorphically p-valent starlike of order $\mu(0 \leq \mu < 1)$ in the disk $|z| < r_1$, that is,*

$$\mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \mu \quad (0 \leq \mu < p;\ |z| < r_1;\ p \in N), \tag{2.20}$$

where

$$r_1 = \inf_{k\geq p+1}\left\{\frac{(p-\mu)\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{p(k+\mu)(1-\beta)}\right\}^{1/(k+p)}. \tag{2.21}$$

Proof. By the form (1.1), we get that

$$\begin{aligned}\left|\frac{(zf'(z)/f(z))+p}{(zf'(z)/f(z))-p+2\mu}\right| &= \left|\frac{\sum_{k=p+1}^{\infty}(k+p)a_k z^k}{2(p-\mu)z^{-p}+\sum_{k=p+1}^{\infty}(k-p+2\mu)a_k z^k}\right| \\ &\leq \frac{\sum_{k=p+1}^{\infty}(k+p)|z|^k}{2(p-\mu)a_k|z|^{-p}+\sum_{k=p+1}^{\infty}(k-p+2\mu)a_k|z|^k} \\ &= \frac{\sum_{k=p+1}^{\infty}(k+p)a_k|z|^{k+p}}{2(p-\mu)+\sum_{k=p+1}^{\infty}(k-p+2\mu)a_k|z|^{k+p}}.\end{aligned} \tag{2.22}$$

Then the following incurability

$$\left|\frac{(zf'(z)/f(z))+p}{(zf'(z)/f(z))-p+2\mu}\right| \leq 1, \quad (0\leq\mu<p;\ p\in N) \tag{2.23}$$

also holds if

$$\sum_{k=p+1}^{\infty}\frac{(k+\mu)}{(p-\mu)}a_k|z|^{k+p} \leq 1, \quad (0\leq\mu<p;\ p\in N). \tag{2.24}$$

Then by Corollary 2.3 the inequality (2.24) will be true if

$$\frac{(k+\mu)}{(p-\mu)}|z|^{k+p} \leq \frac{\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{p(1-\beta)}, \quad (0\leq\mu<p;\ p\in N), \tag{2.25}$$

that is,

$$|z|^{k+p} \leq \frac{(p-\mu)\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{p(k+\mu)(1-\beta)}, \quad (0\leq\mu<p;\ p\in N). \tag{2.26}$$

Therefore the inequality (2.26) leads us to the disc $|z| < r_1$, where r_1 is given by the form (2.21). □

Theorem 2.7. *If $f \in \Gamma_\lambda(n,\alpha,\beta)$, then f is meromorphically p-valent convex of order μ $(0\leq\mu<1)$ in the disk $|z|<r_2$, that is,*

$$\operatorname{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\} > \mu \quad (0\leq\mu<p;\ |z|<r_2;\ p\in N), \tag{2.27}$$

where

$$r_2 = \inf_{k\geq p+1}\left\{\frac{(p-\mu)\left[(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{k(k+\mu)(1-\beta)}\right\}^{1/(k+p)}. \tag{2.28}$$

Proof. By the form (1.1), we get that

$$\begin{aligned}\left|\frac{1+(zf''(z)/f'(z))+p}{1+(zf''(z)/f'(z))-p+2\mu}\right| &= \left|\frac{\sum_{k=p+1}^{\infty}k(k+p)a_k z^k}{2p(p-\mu)z^{-p}+\sum_{k=p+1}^{\infty}k(k-p+2\mu)a_k z^k}\right| \\ &\leq \frac{\sum_{k=p+1}^{\infty}k(k+p)|z|^k}{2p(p-\mu)a_k|z|^{-p}+\sum_{k=p+1}^{\infty}k(k-p+2\mu)a_k|z|^k} \\ &= \frac{\sum_{k=p+1}^{\infty}k(k+p)a_k|z|^{k+p}}{2p(p-\mu)+\sum_{k=p+1}^{\infty}k(k-p+2\mu)a_k|z|^{k+p}}.\end{aligned} \tag{2.29}$$

Then the following incurability:

$$\left|\frac{1+(zf''(z)/f'(z))+p}{1+(zf''(z)/f'(z))-p+2\mu}\right| \leq 1, \quad (0\leq\mu<p;\ p\in N) \tag{2.30}$$

will hold if

$$\sum_{k=p+1}^{\infty}\frac{k(k+\mu)}{p(p-\mu)}a_k|z|^{k+p} \leq 1, \quad (0\leq\mu<p;\ p\in N). \tag{2.31}$$

Then by Corollary 2.3 the inequality (2.31) will be true if

$$\frac{k(k+\mu)}{p(p-\mu)}|z|^{k+p} \leq \frac{\left[p(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{p(1-\beta)}, \quad (0\leq\mu<p;\ p\in N), \tag{2.32}$$

that is,

$$|z|^{k+p} \leq \frac{(p-\mu)\left[(\alpha+\beta)+k(1+\alpha)\right](k\lambda+p\lambda+1)^n}{k(k+\mu)(1-\beta)}, \quad (0\leq\mu<p;\ p\in N). \tag{2.33}$$

Therefore the inequality (2.33) leads us to the disc $|z| < r_2$, where r_2 is given by the form (2.28). □

3. Properties of the Class $\Gamma_\lambda^*(n,\alpha,\beta)$

We first give the necessary and sufficient conditions for functions f in order to be in the class $\Gamma_\lambda^*(n,\alpha,\beta)$.

Lemma 3.1 (see [2]). *Let $\mu > \delta$ and*

$$R_a = \begin{cases} a-\delta, & \text{for } a \le 2\mu+\delta, \\ 2\sqrt{\mu(a-\mu-\delta)}, & \text{for } a \ge 2\mu+\delta. \end{cases} \tag{3.1}$$

Then

$$\{w : |w-a| \le R_a\} \subseteq \{w : |w-(\mu+\delta)| \le \operatorname{Re}\{w+\mu-\delta\}\}. \tag{3.2}$$

Lemma 3.2. *Let $\alpha \ge 0$ and $0 \le \beta < 1$*

$$R_a = \begin{cases} a-\alpha\beta, & \text{for } a \le 2\alpha+\alpha\beta, \\ 2\sqrt{\alpha(a-\alpha-\alpha\beta)}, & \text{for } a \ge 2\alpha+\alpha\beta. \end{cases} \tag{3.3}$$

Then

$$\{w : |w-a| \le R_a\} \subseteq \{w : |w-(\alpha+\alpha\beta)| \le \operatorname{Re}\{w+\alpha-\alpha\beta\}\}. \tag{3.4}$$

Proof. Since $\alpha \ge 0$ and $0 \le \beta < 1$, then $\alpha > \alpha\beta$. Then in Lemma 3.1, put $\mu = \alpha$ and $\delta = \alpha\beta$. □

Theorem 3.3. *Let $f \in \Sigma_p$. Then f is in the class $\Gamma_\lambda^*(n,\alpha,\beta)$ if and only if*

$$\sum_{k=p+1}^{\infty} (k+p\alpha\beta)(k\lambda+p\lambda+1)^n a_k \le p(1-\alpha\beta) \quad \left(\alpha \ge \frac{1}{2+\beta};\ 0 \le \beta < 1;\ p \in N;\ n \in N_0\right). \tag{3.5}$$

Proof. Suppose that $f \in \Gamma_\lambda^*(n,\alpha,\beta)$. Then by the inequality (1.9), we get that

$$\left|\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))} + \alpha + \alpha\beta\right| \le \operatorname{Re}\left\{-\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}\right\} + \alpha - \alpha\beta. \tag{3.6}$$

That is,

$$\begin{aligned} \operatorname{Re}\left\{\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))} + \alpha + \alpha\beta\right\} &\le \left|\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))} + \alpha + \alpha\beta\right| \\ &\le \operatorname{Re}\left\{-\frac{z(D_\lambda^n f(z))'}{p(D_\lambda^n f(z))}\right\} + \alpha - \alpha\beta, \end{aligned} \tag{3.7}$$

that is,

$$\operatorname{Re}\left\{\frac{2z\left(D_{\lambda}^{n}f(z)\right)'}{p\left(D_{\lambda}^{n}f(z)\right)}+2\alpha\beta\right\}\le 0. \tag{3.8}$$

Hence by the inequality (1.3),

$$\operatorname{Re}\left\{\frac{-2p(1-\alpha\beta)+\sum_{k=p+1}^{\infty}2(k+p\alpha\beta)(k\lambda+p\lambda+1)^{n}a_{k}z^{k+p}}{p+\sum_{k=p+1}^{\infty}p(k\lambda+p\lambda+1)^{n}a_{k}z^{k+p}}\right\}\le 0. \tag{3.9}$$

Taking z to be real and putting $z\to 1^{-}$ through real values, then the inequality (3.9) yields

$$\frac{-2p(1-\alpha\beta)+\sum_{k=p+1}^{\infty}2(k+p\alpha\beta)(k\lambda+p\lambda+1)^{n}a_{k}}{p+\sum_{k=p+1}^{\infty}p(k\lambda+p\lambda+1)^{n}a_{k}}\le 0, \tag{3.10}$$

which leads us at once to (3.5).

In order to prove the converse, consider that the inequality (3.5) holds true. In Lemma 3.2 above, since $\alpha>\alpha\beta$ and $\alpha\ge 1/(2+\beta)$, that is, $1\le 2\alpha+\alpha\beta$, we can put $a=1$. Then for $p\in N$ and $n\in N_0$, let $w_{np}=-z(D_{\lambda}^{n}f(z))'/p(D_{\lambda}^{n}f(z))$. Now, if we let $z\in\partial U^{*}=\{z\in C:|z|=1\}$, we get from the inequalities (1.3) and (3.5) that $|w_{np}-1|\le R_1$. Thus by Lemma 3.2 above, we ge that

$$\begin{aligned}
&\left|\frac{z\left(D_{\lambda}^{n}f(z)\right)'}{p\left(D_{\lambda}^{n}f(z)\right)}+\alpha+\alpha\beta\right|\\
&\quad=\left|-\frac{z\left(D_{\lambda}^{n}f(z)\right)'}{p\left(D_{\lambda}^{n}f(z)\right)}-(\alpha+\alpha\beta)\right|\\
&\quad=\left|w-(\alpha+\alpha\beta)\right|\\
&\quad\le\operatorname{Re}\{w+\alpha-\alpha\beta\}=\operatorname{Re}\{w\}+\alpha-\alpha\beta\\
&\quad=\left\{-\frac{z\left(D_{\lambda}^{n}f(z)\right)'}{p\left(D_{\lambda}^{n}f(z)\right)}\right\}+\alpha-\alpha\beta,\quad\left(\alpha\ge\frac{1}{2+\beta};\ 0\le\beta<1;\ p\in N;\ n\in N_0\right).
\end{aligned} \tag{3.11}$$

Therefore by the maximum modulus theorem, we obtain $f\in\Gamma_{\lambda}^{*}(n,\alpha,\beta)$. □

Corollary 3.4. *If $f\in\Gamma_{\lambda}^{*}(n,\alpha,\beta)$, then*

$$a_k\le\frac{p(1-\alpha\beta)}{(k+p\alpha\beta)(k\lambda+p\lambda+1)^{n}}\quad\left(\alpha\ge\frac{1}{2+\beta};\ 0\le\beta<1;\ p\in N;\ n\in N_0\right). \tag{3.12}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \frac{p(1-\alpha\beta)}{(k+p\alpha\beta)(k\lambda+p\lambda+1)^n} z^k \quad \left(\alpha \geq \frac{1}{2+\beta};\ 0 \leq \beta < 1;\ p \in N;\ n \in N_0\right). \tag{3.13}$$

Theorem 3.5. *The class $\Gamma_\lambda^*(n,\alpha,\beta)$ is closed under convex linear combinations.*

Proof. This proof is similar as the proof of Theorem 2.4. □

The following are the growth and distortion theorems for the class $\Gamma_\lambda^*(n,\alpha,\beta)$.

Theorem 3.6. *If $f \in \Gamma_\lambda^*(n,\alpha,\beta)$, then*

$$\begin{aligned} &\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(1-\alpha\beta)}{(1+\alpha\beta)(p+1)^{n-1}} \cdot \frac{p!2^{-n}}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \leq \left|f^{(m)}(z)\right| \\ &\quad \leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(1-\alpha\beta)}{(1+\alpha\beta)(p+1)^{n-1}} \cdot \frac{p!2^{-n}}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \\ &\quad \left(0 < |z| = r < 1;\ \alpha \geq \frac{1}{2+\beta};\ 0 \leq \beta < 1;\ p \in N;\ n, m \in N_0;\ p > m\right). \end{aligned} \tag{3.14}$$

The result is sharp for the function f given by

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \frac{(1-\alpha\beta)}{(1+\alpha\beta)(2p+2)^n} z^p, \quad (n \in N_0;\ p \in N). \tag{3.15}$$

Next we determine the radii of meromorphically p-valent starlikeness of order μ $(0 \leq \mu < p)$ and meromorphically p-valent convexity of order μ $(0 \leq \mu < p)$ for the class $\Gamma_\lambda^*(n,\alpha,\beta)$.

Theorem 3.7. *If $f \in \Gamma_\lambda^*(n,\alpha,\beta)$, then f is meromorphically p-valent starlike of order μ $(0 \leq \mu < 1)$ in the disk $|z| < r_1$, that is,*

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \mu \quad (0 \leq \mu < p;\ |z| < r_1;\ p \in N), \tag{3.16}$$

where

$$r_1 = \inf_{k \geq p+1} \left\{\frac{(p-\mu)(k+p\alpha\beta)(k\lambda+p\lambda+1)^n}{p(k+\mu)(1-\alpha\beta)}\right\}^{1/(k+p)}. \tag{3.17}$$

Proof. This proof is similar to the proof of Theorem 2.6. □

Theorem 3.8. *If $f \in \Gamma_\lambda^*(n, \alpha, \beta)$, then f is meromorphically p-valent convex of order μ $(0 \le \mu < 1)$ in the disk $|z| < r_2$, that is,*

$$\operatorname{Re}\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \mu \quad (0 \le \mu < p;\ |z| < r_2;\ p \in N), \tag{3.18}$$

where

$$r_2 = \inf_{k \ge p+1} \left\{ \frac{(p-\mu)(k + p\alpha\beta)(k\lambda + p\lambda + 1)^n}{k(k+\mu)(1-\alpha\beta)} \right\}^{1/(k+p)}. \tag{3.19}$$

Proof. This proof is similar to the proof of Theorem 2.7. □

4. Subordination Properties

If f and g are analytic functions in U, we say that f is *subordinate* to g, written symbolically as follows:

$$f \prec g \quad \text{in } U \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U) \tag{4.1}$$

if there exists a function w which is analytic in U with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in U), \tag{4.2}$$

such that

$$f(z) = g(w(z)) \quad (z \in U). \tag{4.3}$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in U) \Longrightarrow f(0) = g(0), \quad f(U) \subset g(U). \tag{4.4}$$

In particular, if the function g is univalent in U we have the following equivalence (see [18]):

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0), \quad f(U) \subset g(U). \tag{4.5}$$

Let $\phi : C^2 \to C$ be a function and let h be univalent in U. If J is analytic function in U and satisfied the differential subordination $\phi(J(z), J'(z)) \prec h(z)$ then J is called a *solution of the differential subordination* $\phi(J(z), J'(z)) \prec h(z)$. The univalent function q is called a *dominant* of the solution of the differential subordination, $J \prec q$.

Lemma 4.1 (see [19]). *Let $q(z)\neq 0$ be univalent in U. Let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set*

$$Q(z)=zq'(z)\phi(q(z)),\qquad h(z)=\theta(q(z))+Q(z). \tag{4.6}$$

Suppose that

(i) *$Q(z)$ is starlike univalent in U,*

(ii) *$\mathrm{Re}\{zh'(z)/Q(z)\}>0$ for $z\in U$.*

If J is analytic function in U and

$$\theta(J(z))+zJ'(z)\phi(J(z))\prec\theta(q(z))+zq'(z)\phi(q(z)), \tag{4.7}$$

then $J(z)\prec q(z)$ and q is the best dominant.

Lemma 4.2 (see [20]). *Let $w,\gamma\in C$ and ϕ is convex and univalent in U with $\phi(0)=1$ and $\mathrm{Re}\{w\phi(z)+\gamma\}>0$ for all $z\in U$. If q is analytic in U with $q(0)=1$ and*

$$q(z)+\frac{zq'(z)}{wq(z)+\gamma}\prec\phi(z)\quad(z\in U), \tag{4.8}$$

then $q(z)\prec\phi(z)$ and ϕ is the best dominant.

Theorem 4.3. *Let $q(z)\neq 0$ be univalent in U such that $zq'(z)/q(z)$ is starlike univalent in U and*

$$\mathrm{Re}\left\{1+\frac{\epsilon}{\gamma}q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0,\quad(\epsilon,\gamma\in C,\ \gamma\neq 0). \tag{4.9}$$

If $f\in\Sigma_p$ satisfies the subordination

$$\epsilon\frac{z\left[D_\lambda^n f(z)\right]'}{\left[D_\lambda^n f(z)\right]}+\gamma\left[1+\frac{z\left[D_\lambda^n f(z)\right]''}{\left[D_\lambda^n f(z)\right]'}-\frac{z\left[D_\lambda^n f(z)\right]'}{\left[D_\lambda^n f(z)\right]}\right]\prec\epsilon q(z)+\frac{\gamma zq'(z)}{q(z)}, \tag{4.10}$$

then $z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)]\prec q(z)$ and q is the best dominant.

Proof. Our aim is to apply Lemma 4.1. Setting

$$J(z)=\frac{z\left[D_\lambda^n f(z)\right]'}{\left[D_\lambda^n f(z)\right]}=\frac{-p+\sum_{k=p+1}^{\infty}k\left(k\lambda+p\lambda+1\right)^n a_k z^{k+p}}{1+\sum_{k=p+1}^{\infty}\left(k\lambda+p\lambda+1\right)^n a_k z^{k+p}},\quad(n\in N_0;\ p\in N), \tag{4.11}$$

$\theta(w)=w$ and $\phi(w)=\gamma/w$, $\gamma\neq 0$. It can be easily observed that J is analytic in U, θ is analytic in C, ϕ is analytic in $C/\{0\}$ and $\phi(w)\neq 0$. By computation shows that

$$\frac{zJ'(z)}{J(z)}=1+\frac{z\left[D_\lambda^n f(z)\right]''}{\left[D_\lambda^n f(z)\right]'}-\frac{z\left[D_\lambda^n f(z)\right]'}{\left[D_\lambda^n f(z)\right]} \tag{4.12}$$

which yields, by (4.10), the following subordination:

$$\epsilon J(z) + \gamma \frac{zJ'(z)}{J(z)} \prec \epsilon q(z) + \frac{\gamma z q'(z)}{q(z)}, \tag{4.13}$$

that is,

$$\theta(J(z)) + zJ'(z)\phi(J(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)). \tag{4.14}$$

Now by letting

$$\begin{aligned} Q(z) &= zq'(z)\phi(q(z)) = \frac{\gamma z q'(z)}{q(z)}, \\ h(z) &= \theta(q(z)) + Q(z) = \epsilon q(z) + \frac{\gamma z q'(z)}{q(z)}. \end{aligned} \tag{4.15}$$

We find Qi starlike univalent in U and that

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{\epsilon}{\gamma} q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{4.16}$$

Hence by Lemma 4.1, $z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)] \prec q(z)$ and q is the best dominant. □

Corollary 4.4. *If $f \in \Sigma_p$ and assume that* (4.9) *holds, then*

$$1 + \frac{z\left[D_\lambda^n f(z)\right]''}{\left[D_\lambda^n f(z)\right]'} \prec \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)} \tag{4.17}$$

implies that $z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)] \prec (1+Az)/(1+Bz)$, $-1 \leq B < A \leq 1$ and $(1+Az)/(1+Bz)$ is the best dominant.

Proof. By setting $\epsilon = \gamma = 1$ and $q(z) = (1+Az)/(1+Bz)$ in Theorem 4.3, then we can obtain the result. □

Corollary 4.5. *If $f \in \Sigma_p$ and assume that* (4.9) *holds, then*

$$1 + \frac{z\left[D_\lambda^n f(z)\right]''}{\left[D_\lambda^n f(z)\right]'} \prec e^{\alpha z} + \alpha z \tag{4.18}$$

implies that $z[D_\lambda^n f(z)]'/[D_\lambda^n f(z)] \prec e^{\alpha z}$, $|\alpha| < \pi$ and $e^{\alpha z}$ is the best dominant.

Proof. By setting $\epsilon = \gamma = 1$ and $q(z) = e^{\alpha z}$ in Theorem 4.3, where $|\alpha| < \pi$. □

Theorem 4.6. *Let $w, \gamma \in C$, and ϕ be convex and univalent in U with $\phi(0) = 1$ and $\mathrm{Re}\{w\phi(z)+\gamma\} > 0$ for all $z \in U$. If $f \in \Sigma_p$ satisfies the subordination*

$$\frac{1+\gamma+\left(z\left[D_\lambda^n f(z)\right]''/\left[D_\lambda^n f(z)\right]'\right)-((w/p)+1)\left(z\left[D_\lambda^n f(z)\right]'/\left[D_\lambda^n f(z)\right]\right)}{w-\gamma\left(p\left[D_\lambda^n f(z)\right]/z\left[D_\lambda^n f(z)\right]'\right)} \prec \phi(z), \qquad (4.19)$$

then $-z[D_\lambda^n f(z)]'/p[D_\lambda^n f(z)] \prec \phi(z)$ and ϕ is the best dominant.

Proof. Our aim is to apply Lemma 4.2. Setting

$$q(z) = \frac{-z\left[D_\lambda^n f(z)\right]'}{p\left[D_\lambda^n f(z)\right]} = \frac{p+\sum_{k=p+1}^{\infty} k\left(k\lambda+p\lambda+1\right)^n a_k z^{k+p}}{p+\sum_{k=p+1}^{\infty} p\left(k\lambda+p\lambda+1\right)^n a_k z^{k+p}}, \quad \left(n \in N_0;\ p \in N\right). \qquad (4.20)$$

It can be easily observed that q is analytic in U and $q(0) = 1$. Computation shows that

$$\frac{zq'(z)}{q(z)} = 1+\frac{z\left[D_\lambda^n f(z)\right]''}{\left[D_\lambda^n f(z)\right]'}-\frac{z\left[D_\lambda^n f(z)\right]'}{\left[D_\lambda^n f(z)\right]} \qquad (4.21)$$

which yields, by (4.19), the following subordination:

$$q(z)+\frac{zq'(z)}{wq(z)+\gamma} \prec \phi(z), \quad (z \in U). \qquad (4.22)$$

Hence by Lemma 4.2, $-z[D_\lambda^n f(z)]'/[pD_\lambda^n f(z)] \prec \phi(z)$ and ϕ is the best dominant. □

Acknowledgments

The authors express their sincere thanks to the referees for their very constructive comments and suggestions. The authors also acknowledge that this research was partially supported by the University Putra Malaysia under the Research University Grant Scheme 05-01-09-0720RU.

References

[1] M. K. Aouf and H. M. Hossen, "New criteria for meromorphic p-valent starlike functions," *Tsukuba Journal of Mathematics*, vol. 17, no. 2, pp. 481–486, 1993.

[2] S. S. Kumar, V. Ravichandran, and G. Murugusundaramoorthy, "Classes of meromorphic p-valent parabolic starlike functions with positive coefficients," *The Australian Journal of Mathematical Analysis and Applications*, vol. 2, no. 2, pp. 1–9, 2005.

[3] M. Nunokawa and O. P. Ahuja, "On meromorphic starlike and convex functions," *Indian Journal of Pure and Applied Mathematics*, vol. 32, no. 7, pp. 1027–1032, 2001.

[4] M. K. Aouf, "Certain subclasses of meromorphically p-valent functions with positive or negative coefficients," *Mathematical and Computer Modelling*, vol. 47, no. 9-10, pp. 997–1008, 2008.

[5] M. K. Aouf, "Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function," *Computers & Mathematics with Applications*, vol. 55, no. 3, pp. 494–509, 2008.

[6] M. K. Aouf, "On a certain class of meromorphic univalent functions with positive coefficients," *Rendiconti di Matematica e delle sue Applicazioni. Serie VII*, vol. 11, no. 2, pp. 209–219, 1991.
[7] S. B. Joshi and H. M. Srivastava, "A certain family of meromorphically multivalent functions," *Computers & Mathematics with Applications*, vol. 38, no. 3-4, pp. 201–211, 1999.
[8] M. L. Mogra, "Meromorphic multivalent functions with positive coefficients. I," *Mathematica Japonica*, vol. 35, no. 1, pp. 1–11, 1990.
[9] M. L. Mogra, "Meromorphic multivalent functions with positive coefficients. II," *Mathematica Japonica*, vol. 35, no. 6, pp. 1089–1098, 1990.
[10] S. Owa, H. E. Darwish, and M. K. Aouf, "Meromorphic multivalent functions with positive and fixed second coefficients," *Mathematica Japonica*, vol. 46, no. 2, pp. 231–236, 1997.
[11] H. M. Srivastava, H. M. Hossen, and M. K. Aouf, "A unified presentation of some classes of meromorphically multivalent functions," *Computers & Mathematics with Applications*, vol. 38, no. 11-12, pp. 63–70, 1999.
[12] R. K. Raina and H. M. Srivastava, "A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions," *Mathematical and Computer Modelling*, vol. 43, no. 3-4, pp. 350–356, 2006.
[13] B. A. Uralegaddi and M. D. Ganigi, "Meromorphic multivalent functions with positive coefficients," *The Nepali Mathematical Sciences Report*, vol. 11, no. 2, pp. 95–102, 1986.
[14] B. A. Uralegaddi and C. Somanatha, "New criteria for meromorphic starlike univalent functions," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 137–140, 1991.
[15] D. G. Yang, "On new subclasses of meromorphic p-valent functions," *Journal of Mathematical Research and Exposition*, vol. 15, no. 1, pp. 7–13, 1995.
[16] I. Faisal, M. Darus, and A. Kılıçman, "New subclasses of meromorphic functions associated with hadamard product," in *Proceedings of the International Conference on Mathematical Sciences (ICMS '10)*, vol. 1309 of *AIP Conference Proceedings*, pp. 272–279, 2010.
[17] A. Tehranchi and A. Kılıçman, "On certain classes of p-valent functions by using complex-order and differential subordination," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 275935, 12 pages, 2010.
[18] H. M. Srivastava and S. S. Eker, "Some applications of a subordination theorem for a class of analytic functions," *Applied Mathematics Letters*, vol. 21, no. 4, pp. 394–399, 2008.
[19] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables*, vol. 48, no. 10, pp. 815–826, 2003.
[20] Z.-G. Wang, Y.-P. Jiang, and H. M. Srivastava, "Some subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function," *Computers & Mathematics with Applications*, vol. 57, no. 4, pp. 571–586, 2008.