

Research Article

Strong Converse Inequality for a Spherical Operator

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In the paper titled as “Jackson-type inequality on the sphere” (2004), Ditzian introduced a spherical nonconvolution operator $O_{t,r}$, which played an important role in the proof of the well-known Jackson inequality for spherical harmonics. In this paper, we give the lower bound of approximation by this operator. Namely, we prove that there are constants C_1 and C_2 such that $C_1\omega_{2r}(f,t)_p \leq \|O_{t,r}f - f\|_p \leq C_2\omega_{2r}(f,t)_p$ for any p th Lebesgue integrable or continuous function f defined on the sphere, where $\omega_{2r}(f,t)_p$ is the $2r$ th modulus of smoothness of f .

1. Introduction

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ ($d \geq 3$) be the unit sphere of \mathbb{R}^d endowed with the usual rotation invariant measure $d\omega(x)$. We denote by \mathcal{H}_k^d the space of all spherical harmonics of degree k on \mathbb{S}^{d-1} and Π_n^d the space of all spherical harmonics of degree at most n . The spaces \mathcal{H}_k^d ($k = 0, 1, \dots$) are mutually orthogonal with respect to the inner product

$$\langle f, g \rangle := \int_{\mathbb{S}^{d-1}} f(x)g(x)d\omega(x), \quad (1.1)$$

so there holds

$$\Pi_n^d = \mathcal{H}_0^d \oplus \mathcal{H}_1^d \oplus \dots \oplus \mathcal{H}_n^d. \quad (1.2)$$

By $C(\mathbb{S}^{d-1})$ and $L^p(\mathbb{S}^{d-1})$, $1 \leq p < +\infty$, we denote the space of continuous, real-value functions and the space of (the equivalence classes of) p -integrable functions defined on \mathbb{S}^{d-1} endowed with the respective norms

$$\|f\|_{C(\mathbb{S}^{d-1})} := \max_{\mu \in \mathbb{S}^{d-1}} |f(\mu)|, \quad \|f\|_p := \left(\int_{\mathbb{S}^{d-1}} |f(\mu)|^p d\omega(\mu) \right)^{1/p}, \quad 1 \leq p < \infty. \quad (1.3)$$

In the following, $L^p(\mathbb{S}^{d-1})$ will always be one of the spaces $L^p(\mathbb{S}^{d-1})$ for $1 \leq p < \infty$, or $C(\mathbb{S}^{d-1})$ for $p = \infty$.

For an arbitrary number θ , $0 < \theta < \pi$, we define the spherical translation operator with step θ as (see [1, 2])

$$S_\theta(f) := S_\theta(f; \mu) = \frac{1}{|\mathbb{S}^{d-2}| \sin^{d-2} \theta} \int_{\mu \cdot \nu = \cos \theta} f(\nu) d\omega_{d-2}(\nu), \quad (1.4)$$

where ω_{d-2} means the $(d-2)$ -dimensional surface area of sphere embedded into \mathbb{R}^{d-1} . Here we integrate over the family of points $\nu \in \mathbb{S}^{d-2}$ whose spherical distance from the given point $\mu \in \mathbb{S}^{d-1}$ (i.e., the length of minor arc between μ and ν on the great circle passing through them) is equal to θ . Thus $S_\theta(f; \mu)$ can be interpreted as the mean value of the function f on the surface of a $(d-2)$ -dimensional sphere with radius $\sin \theta$.

By the help of translation operator, we can define the modulus of smoothness of $f \in L_p(\mathbb{S}^{d-1})$ as (see [3, Chapter 10] or [4])

$$\omega_r(f, t)_p := \sup_{0 < \theta_i \leq t} \|(S_{\theta_1} - I)(S_{\theta_2} - I) \cdots (S_{\theta_r} - I)f\|_p. \quad (1.5)$$

Clearly, the modulus is meaningful to describe the approximation degree and the smoothness of f , which has been widely used in the study of approximation on sphere.

The Laplace-Beltrami operator Δ is defined by (see [5, 6])

$$\Delta f := \sum_{i=1}^d \frac{\partial^2 g(x)}{\partial x_i^2} \Big|_{|x|=1}, \quad g(x) = f\left(\frac{x}{|x|}\right), \quad (1.6)$$

where $|x| = (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2}$, $x = (x_1, x_2, \dots, x_d)$. We also need a K -functional on sphere \mathbb{S}^{d-1} defined by (see [3])

$$K_{2r}(f; t)_p := \inf_{g \in C^{2r}(\mathbb{S}^{d-1})} \left\{ \|f - g\|_p + t^{2r} \|\Delta^r g\|_p \right\}, \quad (1.7)$$

where $\Delta^k := \Delta^{k-1} \Delta$. For the modulus of smoothness and K -functional, the following equivalent relationship has been proved (see [3, Section 10.6])

$$\omega_{2r}(f, t)_p \sim K_{2r}(f, t)_p. \quad (1.8)$$

Throughout this paper, we denote by C_i ($i = 1, 2, \dots$) the positive constants independent of f and n and by $C(a)$ the positive constants depending only on a . Their value will be different at different occurrences, even within the same formula. By $A \sim B$ we denote that there are positive constants C_1 and C_2 such that $C_1 B \leq A \leq C_2 B$.

In [3], Ditzian introduced a spherical operator $O_{t,r}$ and used it to prove the well-known Jackson type inequality for spherical harmonics. Before giving the definition of $O_{t,r}$, we need to introduce some preliminaries. Denote

$$T(\rho)f(x) := f(\rho x) \quad \text{for } \rho \in \text{SO}(d), \quad x \in \mathbb{S}^{d-1}, \tag{1.9}$$

where $\text{SO}(d)$ denotes the group of orthogonal matrices on \mathbb{R}^d with determinants 1. We denote further

$$\overline{\Delta}_\rho^{2r} f(x) := \left(T(\rho) - 2I + T(\rho^{-1}) \right)^r f(x). \tag{1.10}$$

For an orthogonal matrix Q with determinant 1, we define

$$M(t, Q) := Q^{-1} \begin{pmatrix} \cos t & \sin t & 0 & 0 & \dots & 0 \\ -\sin t & \cos t & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{1.11}$$

Now we are in the position to define the operator $O_{t,r}$. At first we define the averaging operator $A_{t,r}f$ by (see [3])

$$f - A_{t,r}f := \frac{1}{\binom{2r}{r}} \int_{Q \in \text{SO}(d)} \overline{\Delta}_{M(t,Q)}^{2r} f dQ, \tag{1.12}$$

where dQ represents the Haar measure on $\text{SO}(d)$ normalized so that

$$\int_{Q \in \text{SO}(d)} dQ = 1, \tag{1.13}$$

where the definition of the Haar measure can be found in [7]. Furthermore, for a measure $\mu_t(u)$ supported in $[0, t]$ (t being fixed and u is the variable) such that $\int d\mu_t(u) = 1$ ($d\mu_t(u) = 0$ for $u > |t|$), the operator $O_{t,r}$ is defined by

$$O_{t,r}f := \int A_{u,r}f(x) d\mu_t(u). \tag{1.14}$$

In [3], Ditzian gave a converse inequality for $O_{t,r}$ as follows.

Theorem A. For any $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p \leq \infty$, and some fixed $\eta > 1$, there holds

$$C^{-1} \|f - O_{t,r}f\|_p \leq K_{2r}(f;t)_p \leq C \left(\|f - O_{t,r}f\|_p + \|f - O_{\eta t,r}f\|_p \right). \quad (1.15)$$

In this paper, we improve this result. Motivated by [8, 9], we obtain the following Theorem 1.1.

Theorem 1.1. For any $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p \leq \infty$, there holds

$$\|O_{t,r}f - f\|_p \sim \omega_{2r}(f,t)_p \sim K_{2r}(f,t)_p. \quad (1.16)$$

2. The Proof of Main Result

Before proceeding the proof, we state some useful lemmas at first. The first one can be find in [3, page 6].

Lemma 2.1. For any $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p \leq \infty$, there exists a constant $C(r)$ depending only on r such that

$$\|O_{t,r}(f)\|_p \leq C(r) \|f\|_p. \quad (2.1)$$

The following three lemmas reveal some important properties of $O_{t,r}(f)$. Their proofs can be found in [3, Theorem 6.1], [3, Theorem 6.2], and [3, equation (4.8)], respectively.

Lemma 2.2. For $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p \leq \infty$, and $m \geq 2k$, one has

$$\left\| \Delta^k O_{t,r}^m f \right\|_p \leq \frac{C(k)}{t^{2k}} \left\| O_{t,r}^{m-2k} f \right\|_p, \quad (2.2)$$

where $O_{t,r}^m f := O_{t,r}^{m-1} O_{t,r} f$.

Lemma 2.3. For $g \in C^{2r+2}(\mathbb{S}^{d-1})$, and $1 \leq p \leq \infty$, there holds

$$\left\| O_{t,r} g - g - t^{2r} P_r(\Delta) g \right\|_p \leq C t^{2r+2} \left\| \Delta^{r+1} g \right\|_p, \quad (2.3)$$

where $P_r(\Delta) := \sum_{i=1}^r a_i \Delta^i g$ is a polynomial of degree r in Δ . Moreover, $P_r(\Delta)g = 0$ only for $g = \text{const}$.

Lemma 2.4. For any $g \in C^{2r+2}(\mathbb{S}^{d-1})$, any $k \leq r$, and $m \in \mathbb{Z}$, there holds

$$O_{t,r}^m \Delta^k f = \Delta^k O_{t,r}^m f. \quad (2.4)$$

From (1.8) and [10, Theorem 3.2] (see also [3, page 16]) we deduce the following Lemma 2.5 easily.

Lemma 2.5. *Let $P_r(\Delta)$ be defined in (2.3) and $1 \leq p \leq \infty$, then one has*

$$K_{2r}(f, t)_p \sim \omega_{2r}(f, t)_p \sim \inf_{g \in C^{2r}} (\|f - g\|_p + t^{2r} \|P_r(\Delta)g\|_p). \quad (2.5)$$

Now, we give the last lemma, which can easily be deduced from [10, Theorem 3.1].

Lemma 2.6. *Let $P_r(\Delta)$ be defined in (2.3) and $1 \leq p \leq \infty$, then one has*

$$t^{2r+2} \left\| \Delta^{r+1} O_{t,r}^m f \right\|_p \leq C t^{2r} \left\| P_r(\Delta) O_{t,r}^{m-2} f \right\|_p. \quad (2.6)$$

We now give the proof of Theorem 1.1. It has been shown in (1.15) and (1.8) that there exists a constant C_1 such that

$$\|f - O_{t,r}(f)\|_p \leq C_1 \omega_{2r}(f, t)_p, \quad (2.7)$$

hence we only need to prove that there exists a constant C_2 such that

$$\omega_{2r}(f, t)_p \leq C_2 \|f - O_{t,r}(f)\|_p. \quad (2.8)$$

From (2.5) it is sufficient to prove that, for $m \geq 2r + 1$, there holds

$$\left\| f - O_{t,r}^m(f) \right\|_p + t^{2r} \left\| P_r(\Delta) O_{t,r}^m(f) \right\|_p \leq C_3 \|f - O_{t,r}(f)\|_p. \quad (2.9)$$

In order to prove (2.9), we first prove

$$\left\| f - O_{t,r}^m f \right\|_p \leq C(m) \|f - O_{t,r}\|_p. \quad (2.10)$$

Indeed, from (2.1), we have

$$\begin{aligned}
\|f - O_{t,r}^m f\|_p &\leq \|f - O_{t,r} f\|_p + \sum_{k=1}^{m-1} \|O_{t,r}^k f - O_{t,r}^{k+1} f\|_p \\
&\leq \|f - O_{t,r} f\|_p + \sum_{k=1}^{m-1} \|O_{t,r}^k (f - O_{t,r} f)\|_p \\
&\leq \|f - O_{t,r} f\|_p + C \sum_{k=1}^{m-1} \|O_{t,r}^{k-1} (f - O_{t,r} f)\|_p \quad (2.11) \\
&\leq \dots \leq \|f - O_{t,r} f\|_p + C \sum_{k=1}^{m-1} \|O_{t,r} (f - O_{t,r} f)\|_p \\
&\leq C(m) \|f - O_{t,r} f\|_p.
\end{aligned}$$

Now we turn to prove

$$t^{2r} \|P_r(\Delta) O_{t,r}^m(f)\|_p \leq C_4 \|f - O_{t,r}(f)\|_p. \quad (2.12)$$

In fact, from (2.3), we obtain

$$t^{2r} \|P_r(\Delta) O_{t,r}^m(f)\|_p \leq \|O_{t,r} O_{t,r}^m(f) - O_{t,r}^m(f)\|_p + C_5 t^{2r+2} \|\Delta^{r+1} O_{t,r}^m(f)\|_p. \quad (2.13)$$

In order to estimate $t^{2r+2} \|\Delta^{r+1} O_{t,r}^m(f)\|_p$, we use (2.6) and obtain that

$$\begin{aligned}
t^{2r+2} \|\Delta^{r+1} O_{t,r}^m(f)\|_p &\leq C t^{2r} \|P_r(\Delta) O_{t,r}^{m-2} f\|_p \\
&\leq C t^{2r} \|P_r(\Delta) O_{t,r}^m f\|_p + C t^{2r} \|P_r(\Delta) O_{t,r}^m f - P_r(\Delta) O_{t,r}^{m-2} f\|_p \\
&\leq C t^{2r} \|P_r(\Delta) O_{t,r}^m f\|_p + C t^{2r} \|a_r \Delta^r O_{t,r}^{m-2} (f - O_{t,r}^2 f)\|_p \quad (2.14) \\
&\quad + C t^{2r} \|a_{r-1} \Delta^{r-1} O_{t,r}^{m-4} (O_{t,r}^2 f - O_{t,r}^4 f)\|_p \\
&\quad + \dots + C t^{2r} \|a_1 \Delta O_{t,r}^{m-2r} (O_{t,r}^{2r-2} f - O_{t,r}^{2r} f)\|_p.
\end{aligned}$$

Using (2.2) again and (2.10), we have

$$\begin{aligned}
 t^{2r+2} \left\| \Delta^{r+1} O_{t,r}^m(f) \right\|_p &\leq C t^{2r} \left\| P_r(\Delta) O_{t,r}^m f \right\|_p + C_r \left\| O_{t,r}^{m-2-2r} (f - O_{t,r}^2 f) \right\|_p \\
 &\quad + C_{r-1} t^2 \left\| O_{t,r}^{m-2r-2} (O_{t,r}^2 f - O_{t,r}^4 f) \right\|_p \\
 &\quad + \cdots + C_1 t^{2r-2} \left\| O_{t,r}^{m-2r-2} (O_{t,r}^{2r-2} f - O_{t,r}^4 f) \right\|_p \\
 &\leq C t^{2r} \left\| P_r(\Delta) O_{t,r}^m f \right\|_p + C' \left\| f - O_{t,r} f \right\|_p.
 \end{aligned} \tag{2.15}$$

The above inequality together with (2.13) and (2.10) yields (2.12). Then we can deduce (2.9) from (2.12) and (2.10) easily. Therefore (2.8) holds. This completes the proof of Theorem 1.1.

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