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Research Article

Some Shannon-McMillan Approximation Theorems for Markov Chain Field on the Generalized Bethe Tree

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A class of small-deviation theorems for the relative entropy densities of arbitrary random field on the generalized Bethe tree are discussed by comparing the arbitrary measure μ with the Markov measure μ_Q on the generalized Bethe tree. As corollaries, some Shannon-Mcmillan theorems for the arbitrary random field on the generalized Bethe tree, Markov chain field on the generalized Bethe tree are obtained.

1. Introduction and Lemma

Let T be a tree which is infinite, connected and contains no circuits. Given any two vertices $x \neq y \in T$, there exists a unique path $x = x_1, x_2, \ldots, x_m = y$ from x to y with $x_1, x_2, \ldots, x_m$ distinct. The distance between x and y is defined to $m-1$, the number of edges in the path connecting x and y. To index the vertices on T, we first assign a vertex as the "root" and label it as O. A vertex is said to be on the nth level if the path linking it to the root has n edges. The root O is also said to be on the 0th level.

Definition 1.1. Let T be a tree with root O, and let $\{N_n, n \geq 1\}$ be a sequence of positive integers. T is said to be a generalized Bethe tree or a generalized Cayley tree if each vertex on the nth level has N_{n+1} branches to the $n+1$th level. For example, when $N_1 = N + 1 \geq 2$ and $N_n = N$ $(n \geq 2)$, T is rooted Bethe tree $T_{B,N}$ on which each vertex has $N + 1$ neighboring

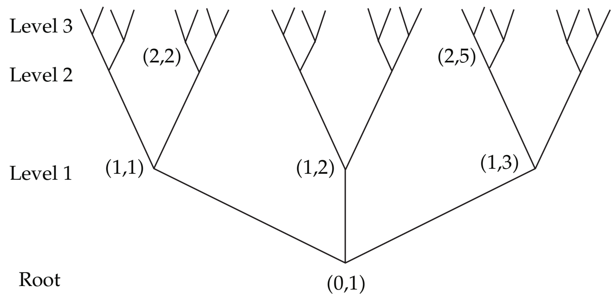

Figure 1: Bethe tree $T_{B,2}$.

vertices (see Figure 1, $T_{B,2}$), and when $N_n = N \geq 1$ $(n \geq 1)$, T is rooted Cayley tree $T_{C,N}$ on which each vertex has N branches to the next level.

In the following, we always assume that T is a generalized Bethe tree and denote by $T^{(n)}$ the subgraph of T containing the vertices from level 0 (the root) to level n. We use (n, j) $(1 \leq j \leq N_1 \cdots N_n, n \geq 1)$ to denote the jth vertex at the nth level and denote by $|B|$ the number of vertices in the subgraph B. It is easy to see that, for $n \geq 1$,

$$\left|T^{(n)}\right| = \sum_{m=0}^{n} N_0 \cdots N_m = 1 + \sum_{m=1}^{n} N_1 \cdots N_m. \tag{1.1}$$

Let $S = \{s_0, s_1, s_2, \ldots\}$, $\Omega = S^T$, $\omega = \omega(\cdot) \in \Omega$, where $\omega(\cdot)$ is a function defined on T and taking values in S, and let F be the smallest Borel field containing all cylinder sets in Ω. Let $X = \{X_t, t \in T\}$ be the coordinate stochastic process defined on the measurable space (Ω, F); that is, for any $\omega = \{\omega(t), t \in T\}$, define

$$X_t(\omega) = \omega(t), \quad t \in T. \tag{1.2}$$

$$X^{T^{(n)}} \triangleq \left\{X_t, t \in T^{(n)}\right\}, \qquad \mu\left(X^{T^{(n)}} = x^{T^{(n)}}\right) = \mu\left(x^{T^{(n)}}\right). \tag{1.3}$$

Now we give a definition of Markov chain fields on the tree T by using the cylinder distribution directly, which is a natural extension of the classical definition of Markov chains (see [1]).

Definition 1.2. Let $Q = Q(j \mid i)$. One has a strictly positive stochastic matrix on S, $q = (q(s_0), q(s_1), q(s_2) \ldots)$ a strictly positive distribution on S, and μ_Q a measure on (Ω, F). If

$$\begin{gathered} \mu_Q(x_{0,1}) = q(x_{0,1}), \\ \mu_Q\left(x^{T^{(n)}}\right) = q(x_{0,1}) \prod_{m=0}^{n-1} \prod_{i=1}^{N_0 \cdots N_m} \prod_{j=N_{m+1}(i-1)+1}^{N_{m+1} i} q\left(x_{m+1,j} \mid x_{m,i}\right), \quad n \geq 1. \end{gathered} \tag{1.4}$$

Then μ_Q will be called a Markov chain field on the tree T determined by the stochastic matrix Q and the distribution q.

Let μ be an arbitrary probability measure defined as (1.3), denote

$$f_n(\omega) = -\frac{1}{|T^{(n)}|}\log \mu\left(X^{T^{(n)}}\right). \tag{1.5}$$

$f_n(\omega)$ is called the entropy density on subgraph $T^{(n)}$ with respect to μ. If $\mu = \mu_Q$, then by (1.4), (1.5) we have

$$f_n(\omega) = -\frac{1}{|T^{(n)}|}\left[\log q(X_{0,1}) + \sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\log q\left(X_{m+1,j} \mid X_{m,i}\right)\right]. \tag{1.6}$$

The convergence of $f_n(\omega)$ in a sense (L_1 convergence, convergence in probability, or almost sure convergence) is called the Shannon-McMillan theorem or the entropy theorem or the asymptotic equipartition property (AEP) in information theory. The Shannon-McMillan theorem on the Markov chain has been studied extensively (see [2, 3]). In the recent years, with the development of the information theory scholars get to study the Shannon-McMillan theorems for the random field on the tree graph (see [4]). The tree models have recently drawn increasing interest from specialists in physics, probability and information theory. Berger and Ye (see [5]) have studied the existence of entropy rate for G-invariant random fields. Recently, Ye and Berger (see [6]) have also studied the ergodic property and Shannon-McMillan theorem for PPG-invariant random fields on trees. But their results only relate to the convergence in probability. Yang et al. [7–9] have recently studied a.s. convergence of Shannon-McMillan theorems, the limit properties and the asymptotic equipartition property for Markov chains indexed by a homogeneous tree and the Cayley tree, respectively. Shi and Yang (see [10]) have investigated some limit properties of random transition probability for second-order Markov chains indexed by a tree.

In this paper, we study a class of Shannon-McMillan random approximation theorems for arbitrary random fields on the generalized Bethe tree by comparison between the arbitrary measure and Markov measure on the generalized Bethe tree. As corollaries, a class of Shannon-McMillan theorems for arbitrary random fields and the Markov chains field on the generalized Bethe tree are obtained. Finally, some limit properties for the expectation of the random conditional entropy are discussed.

Lemma 1.3. *Let μ_1 and μ_2 be two probability measures on $(\Omega, \mathcal{F})$, $D \in \mathcal{F}$, and let $\{\tau_n, n \geq 0\}$ be a positive-valued stochastic sequence such that*

$$\liminf_{n} \frac{\tau_n}{|T^{(n)}|} > 0, \quad \mu_1\text{-a.s. } \omega \in D, \tag{1.7}$$

then

$$\limsup_{n\to\infty} \frac{1}{\tau_n}\log\frac{\mu_2\left(X^{T^{(n)}}\right)}{\mu_1\left(X^{T^{(n)}}\right)} \leq 0, \quad \mu_1\text{-a.s. } \omega \in D. \tag{1.8}$$

In particular, let $\tau_n = |T^{(n)}|$*, then*

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu_2\left(X^{T^{(n)}}\right)}{\mu_1\left(X^{T^{(n)}}\right)} \leq 0, \quad \mu_1\text{-a.s. } \omega \in D. \tag{1.9}$$

Proof (see [11]). Let

$$\varphi(\mu \mid \mu_Q) = \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu\left(X^{T^{(n)}}\right)}{\mu_Q\left(X^{T^{(n)}}\right)}. \tag{1.10}$$

$\varphi(\mu \mid \mu_Q)$ is called the sample relative entropy rate of μ relative to μ_Q. $\varphi(\mu \mid \mu_Q)$ is also called the asymptotic logarithmic likelihood ratio. By (1.9)

$$\varphi(\mu \mid \mu_Q) \geq \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu\left(X^{T^{(n)}}\right)}{\mu_Q\left(X^{T^{(n)}}\right)} \geq 0, \quad \mu\text{-a.s.} \tag{1.11}$$

Hence $\varphi(\mu \mid \mu_Q)$ can be look on as a type of measures of the deviation between the arbitrary random fields and the Markov chain fields on the generalized Bethe tree. □

2. Main Results

Theorem 2.1. *Let* $X = \{X_t, t \in T\}$ *be an arbitrary random field on the generalized Bethe tree.* $f_n(\omega)$ *and* $\varphi(\mu \mid \mu_Q)$ *are, respectively, defined as* (1.5) *and* (1.10)*. Denote* $\alpha \geq 0$, $H_m^Q(X_{m+1,j} \mid X_{m,i})$ *the random conditional entropy of* $X_{m+1,j}$ *relative to* $X_{m,i}$ *on the measure* μ_Q*, that is,*

$$H_m^Q(X_{m+1,j} \mid X_{m,i}) = -\sum_{x_{m+1,j}\in S} q(x_{m+1,j} \mid X_{m,i}) \log q(x_{m+1,j} \mid X_{m,i}) . \tag{2.1}$$

Let

$$D(c) = \{\omega : \varphi(\mu \mid \mu_Q) \leq c\}, \tag{2.2}$$

$$b_\alpha = \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\cdots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} E_Q\left[\log^2 q(X_{m+1,j} \mid X_{m,i}) \cdot q(X_{m+1,j} \mid X_{m,i})^{-\alpha} \mid X_{m,i}\right] < \infty, \tag{2.3}$$

when $0 \le c \le \alpha^2 b_\alpha/2$,

$$\limsup_{n\to\infty}\left\{ f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} H_m^Q(X_{m+1,j}\mid X_{m,i})\right\} \le \sqrt{2cb_\alpha},\quad \mu\text{-a.s. } \omega\in D(c). \tag{2.4}$$

$$\liminf_{n\to\infty}\left\{ f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} H_m^Q(X_{m+1,j}\mid X_{m,i})\right\} \ge -\sqrt{2cb_\alpha}-c,\quad \mu\text{-a.s. } \omega\in D(c). \tag{2.5}$$

In particular,

$$\lim_{n\to\infty}\left[f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} H_m^Q(X_{m+1,j}\mid X_{m,i})\right] = 0,\quad \mu\text{-a.s. } \omega\in D(0), \tag{2.6}$$

where log *is the natural logarithmic,* E_Q *is expectation with respect to the measure* μ_Q*.*

Proof. Let $(\Omega, \mathcal{F}, \mu)$ be the probability space we consider, λ an arbitrary constant. Define

$$E_Q\left[q(X_{m+1,j}\mid X_{m,i})^{-\lambda}\mid X_{m,i}=x_{m,i}\right] = \sum_{x_{m+1,j}\in S} q(x_{m+1,j}\mid x_{m,i})^{1-\lambda}; \tag{2.7}$$

denote

$$\mu_Q\left(\lambda, x^{T^{(n)}}\right) = q(x_{0,1})\prod_{m=0}^{n-1}\prod_{i=1}^{N_0\cdots N_m}\prod_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\frac{q(x_{m+1,j}\mid x_{m,i})^{1-\lambda}}{E_Q\left[q(X_{m+1,j}\mid X_{m,i})^{-\lambda}\mid X_{m,i}=x_{m,i}\right]}. \tag{2.8}$$

We can obtain by (2.7), (2.8) that in the case $n \geq 1$,

$$\begin{aligned}
&\sum_{x^{L_n}\in S} \mu_Q\left(\lambda; x^{T^{(n)}}\right) \\
&\quad = \sum_{x^{L_n}\in S} q(x_{0,1}) \prod_{m=0}^{n-1} \prod_{i=1}^{N_0\cdots N_m} \prod_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \frac{q\left(x_{m+1,j} \mid x_{m,i}\right)^{1-\lambda}}{E_Q\left[q\left(X_{m+1,j} \mid X_{m,i}\right)^{-\lambda} \mid X_{m,i} = x_{m,i}\right]} \\
&\quad = \mu_Q\left(\lambda; x^{T^{(n-1)}}\right) \sum_{x^{L_n}\in S} \prod_{i=1}^{N_0\cdots N_{n-1}} \prod_{j=N_n(i-1)+1}^{N_n i} \frac{q\left(x_{n,j} \mid x_{n-1,i}\right)^{1-\lambda}}{E_Q\left[q\left(X_{n,j} \mid X_{n-1,i}\right)^{-\lambda} \mid X_{n-1,i} = x_{n-1,i}\right]} \\
&\quad = \mu_Q\left(\lambda; x^{T^{(n-1)}}\right) \prod_{i=1}^{N_0\cdots N_{n-1}} \prod_{j=N_n(i-1)+1}^{N_n i} \sum_{x_{n,j}\in S} \frac{q\left(x_{n,j} \mid x_{n-1,i}\right)^{1-\lambda}}{E_Q\left[q\left(x_{n,j} \mid x_{n-1,i}\right)^{-\lambda} \mid X_{n-1,i} = x_{n-1,i}\right]} \\
&\quad = \mu_Q\left(\lambda; x^{T^{(n-1)}}\right) \prod_{i=1}^{N_0\cdots N_{n-1}} \prod_{j=N_n(i-1)+1}^{N_n i} \frac{E_Q\left[q\left(X_{n,j} \mid X_{n-1,i}\right)^{-\lambda} \mid X_{n-1,i} = x_{n-1,i}\right]}{E_Q\left[q\left(x_{n,j} \mid x_{n-1,i}\right)^{-\lambda} \mid X_{n-1,i} = x_{n-1,i}\right]} \\
&\quad = \mu_Q\left(\lambda; x^{T^{(n-1)}}\right),
\end{aligned} \tag{2.9}$$

$$\sum_{x^{L_0}\in S} \mu_Q\left(\lambda; x^{T^{(0)}}\right) = \sum_{x_{0,1}\in S} q(x_{0,1}) = 1. \tag{2.10}$$

Therefore, $\mu_Q(\lambda, x^{T^{(n)}})$, $n = 0, 1, 2, \ldots$ are a class of consistent distributions on $S^{T^{(n)}}$. Let

$$U_n(\lambda, \omega) = \frac{\mu_Q\left(\lambda, X^{T^{(n)}}\right)}{\mu\left(X^{T^{(n)}}\right)}, \tag{2.11}$$

then $\{U_n(\lambda, \omega), \mathcal{F}_n,\ n \geq 1\}$ is a nonnegative supermartingale which converges almost surely (see [12]). By Doob's martingale convergence theorem we have

$$\lim_{n\to\infty} U_n(\lambda, \omega) = U_\infty(\lambda, \omega) < \infty. \quad \mu\text{-a.s.} \tag{2.12}$$

Hence by (1.3), (1.9), (2.9), and (2.11) we get

$$\limsup_{n\to\infty} \frac{1}{\left|T^{(n)}\right|} \log U_n(\lambda, \omega) \leq 0. \quad \mu\text{-a.s.} \tag{2.13}$$

By (1.4), (2.8), and (2.11), we have

$$\frac{1}{|T^{(n)}|}\log U_n(\lambda,\omega)$$
$$= \frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\left[-\lambda\log q(X_{m+1,j}\mid X_{m,i}) - \log E_Q\left(q(X_{m+1,j}\mid X_{m,i})^{-\lambda}\mid X_{m,i}\right)\right]$$
$$+ \frac{1}{|T^{(n)}|}\log\frac{\mu_Q\left(X^{T^{(n)}}\right)}{\mu(X^{T^{(n)}})}. \tag{2.14}$$

By (1.10), (2.2), (2.13), and (2.14) we have

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\left[-\lambda\log q(X_{m+1,j}\mid X_{m,i}) - \log E_Q\left(q(X_{m+1,j}\mid X_{m,i})^{-\lambda}\mid X_{m,i}\right)\right]$$
$$\le \varphi(\mu\mid\mu_Q)\le c,\quad \mu\text{-a.s. } \omega\in D(c). \tag{2.15}$$

By (2.15) we have

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}(-\lambda)\{\log q(X_{m+1,j}\mid X_{m,i}) - E_Q(\log q(X_{m+1,j}\mid X_{m,i})\mid X_{m,i})\}$$
$$\le \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\left[\log E_Q\left(q(X_{m+1,j}\mid X_{m,i})^{-\lambda}\mid X_{m,i}\right)\right.$$
$$\left. - E_Q(-\lambda\log q(X_{m+1,j}\mid X_{m,i})\mid X_{m,i})\right] + c,\quad \mu\text{-a.s. } \omega\in D(c). \tag{2.16}$$

By the inequality

$$x^{-\lambda} - 1 + \lambda\log x \le \left(\frac{1}{2}\right)\lambda^2(\log x)^2 x^{-|\lambda|},\quad 0\le x\le 1, \tag{2.17}$$

$\log x \le x - 1$ $(x \ge 0)$ and (2.16), (2.17), (2.3), we have in the case of $|\lambda| < \alpha$,

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}(-\lambda)\{\log q(X_{m+1,j}\mid X_{m,i})-E_Q(\log q(X_{m+1,j}\mid X_{m,i})\mid X_{m,i})\}$$
$$\le\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\Big[E_Q\Big(q(X_{m+1,j}\mid X_{m,i})^{-\lambda}\mid X_{m,i}\Big)-1$$
$$-E_Q\Big(-\lambda\log q(X_{m+1,j}\mid X_{m,i})\mid X_{m,i}\Big)\Big]+c$$
$$\le\limsup_{n\to\infty}\frac{1}{2|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}E_Q\Big[\lambda^2\log^2(q(X_{m+1,j}\mid X_{m,i}))$$
$$\cdot q(X_{m+1,j}\mid X_{m,i})^{-|\lambda|}\mid X_{m,i}\Big]+c$$
$$\le\limsup_{n\to\infty}\frac{\lambda^2}{2|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}E_Q\Big[\log^2(q(X_{m+1,j}|X_{m,i}))\cdot q(X_{m+1,j}\mid X_{m,i})^{-\alpha}\mid X_{m,i}\Big]$$
$$+c=\left(\frac{1}{2}\right)\lambda^2 b_\alpha+c.\quad \mu\text{-a.s. }\omega\in D(c).\tag{2.18}$$

When $0<\lambda<\alpha$, we get by (2.18)

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}-\{\log q(X_{m+1,j}\mid X_{m,i})-E_Q(\log q(X_{m+1,j}\mid X_{m,i})\mid X_{m,i})\}$$
$$\le\left(\frac{1}{2}\right)\lambda b_\alpha+\frac{c}{\lambda},\quad \mu\text{-a.s. }\omega\in D(c).\tag{2.19}$$

Let $g(\lambda)=(1/2)\lambda b_\alpha+c/\lambda$, in the case $0<c\le(\alpha^2 b_\alpha)/2$, then it is obvious $g(\lambda)$ attains, at $\lambda=\sqrt{(2c)/b_\alpha}$, its smallest value $g(\sqrt{(2c)/b_\alpha})=\sqrt{2cb_\alpha}$ on the interval $(0,\alpha)$. We have

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}-\{\log q(X_{m+1,j}\mid X_{m,i})-E_Q(\log q(X_{m+1,j}\mid X_{m,i})\mid X_{m,i})\}$$
$$\le\sqrt{2cb_\alpha},\quad \mu\text{-a.s. }\omega\in D(c).\tag{2.20}$$

When $c = 0$, we select $0 < \lambda_i < \alpha$ such that $\lambda_i \to 0$ $(i \to \infty)$. Hence for all i, it follows from (2.19) that

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} - \{\log q(X_{m+1,j} \mid X_{m,i}) - E_Q(\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i})\} \\ \le 0, \quad \mu\text{-a.s. } \omega \in D(0). \tag{2.21}$$

It is easy to see that (2.20) also holds if $c = 0$ from (2.21).

Analogously, when $-\alpha < \lambda < 0$, it follows from (2.18) if $0 \le c \le (\alpha^2 b_\alpha)/2$,

$$\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} - \{\log q(X_{m+1,j} \mid X_{m,i}) - E_Q(\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i})\} \\ \ge -\sqrt{2cb_\alpha}, \quad \mu\text{-a.s. } \omega \in D(c). \tag{2.22}$$

Setting $\lambda = 0$ in (2.14), by (2.14) we have

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log U_n(0,\omega) = \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log\frac{\mu_Q\left(X^{T^{(n)}}\right)}{\mu(X^{T^{(n)}})} \le 0, \quad \mu\text{-a.s.} \tag{2.23}$$

Noticing

$$H_m^Q(X_{m+1,j} \mid X_{m,i}) = E_Q[-\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i}]. \tag{2.24}$$

By (1.4), (1.5), (2.20), and (2.23), we obtain

$$\begin{aligned}\limsup_{n\to\infty}&\left[f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} H_m^Q(X_{m+1,j}, X_{m,i})\right] \\ &\le \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log\frac{\mu_Q\left(X^{T^{(n)}}\right)}{\mu(X^{T^{(n)}})} \\ &\quad + \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \\ &\quad - \{\log q(X_{m+1,j} \mid X_{m,i}) - E_Q(\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i})\} \\ &\le \sqrt{2\mathrm{cb}_\alpha}, \quad \mu\text{-a.s. } \omega \in D(c).\end{aligned} \tag{2.25}$$

Hence (2.4) follows from (2.25). By (1.4), (1.5), (1.10), (2.2), and (2.22), we have

$$\begin{aligned}
&\liminf_{n\to\infty}\left[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}H_m^Q(\omega)\right]\\
&\geq \liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\log\left[\frac{\mu_Q\left(X^{T^{(n)}}\right)}{\mu(X^{T^{(n)}})}\right]\\
&\quad+\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\\
&\quad-\left\{\log q(X_{m+1,j}\mid X_{m,i})-E_Q(\log q(X_{m+1,j}\mid X_{m,i})\mid X_{m,i})\right\}\\
&\geq -\varphi(\mu\mid\mu_Q)-\sqrt{2cb_\alpha}\geq -\sqrt{2cb_\alpha}-c,\quad \mu\text{-a.s. } \omega\in D(c).
\end{aligned} \tag{2.26}$$

Therefore (2.5) follows from (2.26). Set $c = 0$ in (2.4) and (2.5), (2.6) holds naturally. □

Corollary 2.2. *Let $X = \{X_t, t \in T\}$ be the Markov chains field determined by the measure μ_Q on the generalized Bethe tree $T\cdot f_n(\omega)$, b_α are, respectively, defined as* (1.6) *and* (2.3), *and $H_m^Q(X_{m+1,j} \mid X_{m,i})$ is defined by* (2.1). *Then*

$$\lim_{n\to\infty}\left\{f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}H_m^Q(X_{m+1,j}\mid X_{m,i})\right\}=0.\quad \mu_Q\text{-a.s.} \tag{2.27}$$

Proof. We take $\mu \equiv \mu_Q$, then $\varphi(\mu \mid \mu_Q) \equiv 0$. It implies that (2.2) always holds when $c = 0$. Therefore $D(0) = \Omega$ holds. Equation(2.27) follows from (2.3) and (2.6). □

3. Some Shannon-McMillan Approximation Theorems on the Finite State Space

Corollary 3.1. *Let $X = \{X_t, t \in T\}$ be an arbitrary random field which takes values in the alphabet $S = \{s_1, \ldots, s_N\}$ on the generalized Bethe tree. $f_n(\omega)$, $\varphi(\mu \mid \mu_Q)$ and $D(c)$ are defined as* (1.5), (1.10), *and* (2.2). *Denote $0 \leq \alpha < 1$, $0 \leq c \leq 2N\alpha^2/[(1-\alpha)e]^2$. $H_m^Q(X_{m+1,j} \mid X_{m,i})$ is defined as above. Then*

$$\begin{aligned}
&\limsup_{n\to\infty}\left[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}H_m^Q(X_{m+1,j}\mid X_{m,i})\right]\\
&\leq \frac{2e^{-1}}{(1-\alpha)}\sqrt{2cN},\quad \mu\text{-a.s. } \omega\in D(c),
\end{aligned} \tag{3.1}$$

$$\liminf_{n\to\infty}\left[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}H_m^Q(X_{m+1,j}\mid X_{m,i})\right]$$
$$\geq -\frac{2e^{-1}}{1-\alpha}\sqrt{2cN}-c,\quad \mu\text{-a.s. } \omega\in D(c). \tag{3.2}$$

Proof. Set $0<\alpha<1$ we consider the function

$$\phi(x)=(\log x)^2x^{1-\alpha},\quad 0<x\leq 1,\ 0<\alpha<1\ \ (\text{Set } \phi(0)=0). \tag{3.3}$$

Then

$$\phi'(x)=x^{-\alpha}\left[2(\log x)+(\log x)^2(1-\alpha)\right]. \tag{3.4}$$

Let $\phi'(x)=0$ thus $x=e^{2/(\alpha-1)}$. Accordingly it can be obtained that

$$\max\{\phi(x),0\leq x\leq 1\}=\phi\left(e^{2/(\alpha-1)}\right)=\left(\frac{2}{\alpha-1}\right)^2e^{-2}. \tag{3.5}$$

By (2.3) and (3.5) we have

$$b_\alpha=\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}E_Q\left[\log^2q(X_{m+1,j}\mid X_{m,i})\cdot q(X_{m+1,j}\mid X_{m,i})^{-\alpha}\mid X_{m,i}\right]$$
$$=\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\sum_{x_{m+1,j}\in S}\log^2q(x_{m+1,j}\mid X_{m,i})\cdot q(x_{m+1,j}\mid X_{m,i})^{1-\alpha}$$
$$\leq\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\sum_{x_{m+1,j}=s_1}^{s_N}\left(\frac{2}{\alpha-1}\right)^2e^{-2}$$
$$=N\left(\frac{2}{\alpha-1}\right)^2e^{-2}\cdot\limsup_{n\to\infty}\frac{|T^{(n)}|-1}{|T^{(n)}|}=N\left(\frac{2}{\alpha-1}\right)^2e^{-2}<\infty. \tag{3.6}$$

Therefore, (2.3) holds naturally. By (2.18) and (3.6) we have

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}(-\lambda)\{\log q(X_{m+1,j}\mid X_{m,i})-E_Q(\log q(X_{m+1,j}\mid X_{m,i})\mid X_{m,i})\}$$
$$\leq N\lambda^2\frac{2e^{-2}}{(\alpha-1)^2}+c,\quad \mu\text{-a.s. } \omega\in D(c). \tag{3.7}$$

In the case of $0 < \lambda < \alpha$, by (3.7) we have

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\cdots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} - \{\log q(X_{m+1,j} \mid X_{m,i}) - E_Q(\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i})\}$$
$$\leq N\lambda \frac{2e^{-2}}{(\alpha-1)^2} + \frac{c}{\lambda}, \quad \mu\text{-a.s. } \omega \in D(c). \tag{3.8}$$

Let $g(\lambda) = 2\lambda N e^{-2}/(\alpha-1)^2 + c/\lambda$, in the case $0 < c \leq 2N\alpha^2/[(1-\alpha)e]^2$, then it is obvious $g(\lambda)$ attains, at $\lambda = (1-\alpha)e\sqrt{c/2N}$, its smallest value $g((1-\alpha)e\sqrt{c/2N}) = 2e^{-1}\sqrt{2cN}/(1-\alpha)$ on the interval $(0,\alpha)$. That is

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\cdots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} - \{\log q(X_{m+1,j} \mid X_{m,i}) - E_Q(\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i})\}$$
$$\leq \frac{2e^{-1}}{(1-\alpha)}\sqrt{2cN}, \quad \mu\text{-a.s. } \omega \in D(c). \tag{3.9}$$

By the similar means of reasoning (2.21), it can be concluded that (3.9) also holds when $c = 0$. According to the methods of proving (2.4), (3.1) follows from (1.5), (2.23), and (3.9). Similarly, when $-\alpha < \lambda < 0$, $0 \leq c \leq 2N\alpha^2/[(1-\alpha)e]^2$, by (3.7) we have

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\cdots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} - \{\log q(X_{m+1,j} \mid X_{m,i}) - E_Q(\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i})\}$$
$$\geq -\frac{2e^{-1}}{(1-\alpha)}\sqrt{2cN}, \quad \mu\text{-a.s. } \omega \in D(c). \tag{3.10}$$

Imitating the proof of (2.5), (3.2) follows from (1.5), (1.10), (2.2), and (3.10). □

Corollary 3.2 (see [9]). *Let $X = \{X_t, t \in T\}$ be the Markov chains field determined by the measure μ_Q on the generalized Bethe tree $T \cdot f_n(\omega)$ is defined as* (1.6), *and $H_m^Q(X_{m+1,j} \mid X_{m,i})$ is defined as* (2.1). *Then*

$$\lim_{n\to\infty} \left\{ f_n(\omega) - \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\cdots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} H_m^Q(X_{m+1,j} \mid X_{m,i}) \right\} = 0. \quad \mu_Q\text{-a.s.} \tag{3.11}$$

Proof. By (3.1) and (3.2) in Corollary 3.1, we obtain that when $c = 0$,

$$\lim_{n\to\infty}\left\{ f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} H_m^Q(X_{m+1,j} \mid X_{m,i})\right\} = 0, \quad \mu\text{-a.s. } \omega \in D(0). \tag{3.12}$$

Set $\mu \equiv \mu_Q$, then $\varphi(\mu \mid \mu_Q) \equiv 0$. It implies (2.2) always holds when $c = 0$. Therefore $D(0) = \Omega$ holds. Equation (3.11) follows from (3.12). □

Corollary 3.3. *Under the assumption of Corollary 3.1, if $\mu \ll \mu_Q$, then*

$$\lim_{n\to\infty}\left\{ f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} H_m^Q(X_{m+1,j} \mid X_{m,i})\right\} = 0. \quad \mu\text{-a.s.} \tag{3.13}$$

Proof. It can be obtained that $\varphi(\mu \mid \mu_Q) \equiv 0, \mu$ a.s. holds if $\mu \ll \mu_Q$ (see Gray 1990 [13]), therefore $\mu(D(0)) = 1$. Equation (3.13) follows from (3.12).

Let $X = \{X_t,\ t \in T\}$ be a Markov chains field on the generalized Bethe tree with the initial distribution and the joint distribution with respect to the measure μ_P as follows:

$$\mu_P(x_{0,1}) = p(x_{0,1}), \tag{3.14}$$

$$\mu_P\left(x^{T^{(n)}}\right) = p(x_{0,1})\prod_{m=0}^{n-1}\prod_{i=1}^{N_0\cdots N_m}\prod_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} p\left(x_{m+1,j} \mid x_{m,i}\right), \quad n \geq 1, \tag{3.15}$$

where $P = p(j \mid i)$ is a strictly positive stochastic matrix on S, $p = (p(s_0), p(s_1), p(s_2)\ldots)$ is a strictly positive distribution. Therefore, the entropy density of $X = \{X_t, t \in T\}$ with respect to the measure μ_P is

$$f_n(\omega) = -\frac{1}{|T^{(n)}|}\left[\log p(X_{0,1}) + \sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \log p\left(X_{m+1,j} \mid X_{m,i}\right)\right]. \tag{3.16}$$

Let the initial distribution and joint distribution of $X = \{X_t, t \in T\}$ with respect to the measure μ_Q be defined as (1.4) and (1.5), respectively.

□

We have the following conclusion.

Corollary 3.4. *Let $X = \{X_t,\ t \in T\}$ be a Markov chains field on the generalized Bethe tree T whose initial distribution and joint distribution with respect to the measure μ_P and μ_Q are defined by* (3.14), (3.15) *and* (1.4), (1.5), *respectively. $f_n(\omega)$ is defined as* (3.16). *If*

$$\sum_{h\in S}\sum_{l\in S}\frac{\left[p(l \mid h) - q(l \mid h)\right]^+}{q(l \mid h)} \leq c, \tag{3.17}$$

then

$$\limsup_{n\to\infty}\left[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}H_m^Q(X_{m+1,j}\mid X_{m,i})\right]\le\frac{2e^{-1}}{1-\alpha}\sqrt{2cN},\quad \mu_P\text{-a.s.}\tag{3.18}$$

$$\liminf_{n\to\infty}\left[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}H_m^Q(X_{m+1,j}\mid X_{m,i})\right]\ge-\frac{2e^{-1}}{1-\alpha}\sqrt{2cN}-c.\quad \mu_P\text{-a.s,}\tag{3.19}$$

Proof. Let $\mu=\mu_P$ in Corollary 3.1, and by (1.5), (3.15) we get (3.16). By the inequalities $\log x\le x-1$ $(x>0)$, $a\le[a]^+$, (3.17), and (1.10), we obtain

$$\begin{aligned}
&\varphi(\mu_P\mid\mu_Q)\\
&=\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log\frac{\mu_P\left(X^{T^{(n)}}\right)}{\mu_Q\left(X^{T^{(n)}}\right)}\\
&=\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log\frac{p(X_{0,1})\prod_{m=0}^{n-1}\prod_{i=1}^{N_0\cdots N_m}\prod_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}p(X_{m+1,j}\mid X_{m,i})}{q(X_{0,1})\prod_{m=0}^{n-1}\prod_{i=1}^{N_0\cdots N_m}\prod_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}q(X_{m+1,j}\mid X_{m,i})}\\
&\le\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log\frac{p(X_{0,1})}{q(X_{0,1})}+\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\log\frac{p(X_{m+1,j}\mid X_{m,i})}{q(X_{m+1,j}\mid X_{m,i})}\\
&\le\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\sum_{h\in S}\sum_{l\in S}\delta_h(X_{m,i})\delta_l(X_{m+1,j})\log\frac{p(l\mid h)}{q(l\mid h)}\\
&\le\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\cdots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\sum_{h\in S}\sum_{l\in S}\delta_h(X_{m,i})\delta_l(X_{m+1,j})\left[\frac{p(l\mid h)}{q(l\mid h)}-1\right]\\
&\le\sum_{h\in S}\sum_{l\in S}\limsup_{n\to\infty}\frac{|T^{(n)}|-1}{|T^{(n)}|}\frac{p(l\mid h)-q(l\mid h)}{q(l\mid h)}\\
&\le\sum_{h\in S}\sum_{l\in S}\frac{[p(l\mid h)-q(l\mid h)]^+}{q(l\mid h)}.
\end{aligned}\tag{3.20}$$

By (3.17) and (3.20) we have

$$\varphi(\mu_P\mid\mu_Q)\le c,\quad \text{a.s.}\tag{3.21}$$

It follows from (2.2) and (3.21) that $D(c)=\Omega$; therefore (3.18), (3.19) follow from (3.1), (3.2). □

4. Some Limit Properties for Expectation of Random Conditional Entropy on the Finite State Space

Lemma 4.1 (see [8]). *Let* $X^{T^{(n)}} = \{X_t, t \in T^{(n)}\}$ *be a Markov chains field defined on a Bethe tree* $T_{B,N}$, $S_n(k,\omega)$ *be the number of* k *in the set of random variables* $X^{T^{(n)}} = \{X_t, t \in T^{(n)}\}$*. then for all* $k \in S$*,*

$$\lim_n \frac{S_n(k,\omega)}{|T^{(n)}|} = \pi(k) \quad \mu_Q\text{-a.s,} \tag{4.1}$$

where $\pi = (\pi(1), \ldots, \pi(N))$ *is the stationary distribution determined by* Q*.*

Theorem 4.2. *Let* $X^{T^{(n)}} = \{X_t, t \in T^{(n)}\}$ *be a Markov chains field defined on a Bethe tree* $T_{B,N}$*, and let* $H_m^Q(X_{m+1,j} \mid X_{m,i})$ *be defined as above. Then*

$$\begin{aligned}\lim_n \frac{1}{|T^{(n)}|}&\left[\sum_{i=1}^{N+1} H_0^Q(X_{1,i} \mid X_{0,1}) + \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni} H_m^Q(X_{m+1,j} \mid X_{m,i})\right] \\ &= -\sum_{k\in S}\sum_{l\in S}\pi(k)q(l \mid k)\log q(l \mid k), \quad \mu_Q\text{-a.s.}\end{aligned} \tag{4.2}$$

Proof. Noticing now $N_1 = N+1$, for all $n \geq 2$, $N_n = N$, that therefore we have

$$\begin{aligned}&\sum_{i=1}^{N+1} H_0^Q(X_{1,i} \mid X_{0,1}) + \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni} H_m^Q(X_{m+1,j} \mid X_{m,i}) \\ &\quad= \sum_{i=1}^{N+1} - E_Q\left[\log q(X_{1,i} \mid X_{0,1}) \mid X_{0,1}\right] \\ &\qquad+ \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni} - E_Q\left[\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i}\right] \\ &\quad= -\sum_{i=1}^{N+1}\sum_{x_{1,i}\in S} q(x_{1,i} \mid X_{0,1})\log q(x_{1,i} \mid X_{0,1}) \\ &\qquad- \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni}\sum_{x_{m+1,j}\in S} q(x_{m+1,j} \mid X_{m,i})\log q(x_{m+1,j} \mid X_{m,i})\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{N+1}\sum_{k\in S}\sum_{l\in S}\delta_k(X_{0,1})q(l\mid k)\log q(l\mid k) \\
&\quad - \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni}\sum_{k\in S}\sum_{l\in S}\delta_k(X_{m,i})q(l\mid k)\log q(l\mid k) \\
&= -\sum_{k\in S}\sum_{l\in S}q(l\mid k)\log q(l\mid k)\left[\sum_{i=1}^{N+1}\delta_k(X_{0,1}) + \sum_{m=0}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni}\delta_k(X_{m,i})\right] \\
&= -\sum_{k\in S}\sum_{l\in S}q(l\mid k)\log q(l\mid k)\left[NS_{n-1}(k,\omega) + \delta_k(X_{0,1})\right].
\end{aligned}
\tag{4.3}
$$

Noticing that $\lim_{n\to\infty}(|T^{(n)}|/|T^{(n-1)}|) = N$, by (4.3) we have

$$
\begin{aligned}
&\lim_n \frac{1}{|T^{(n)}|}\left[\sum_{i=1}^{N+1}H_0^Q(X_{1,i}\mid X_{0,1}) + \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni}H_m^Q(X_{m+1,j}\mid X_{m,i})\right] \\
&\quad = -\lim_n \frac{1}{|T^{(n)}|}\sum_{k\in S}\sum_{l\in S}q(l\mid k)\log q(l\mid k)\left[NS_{n-1}(k,\omega) + \delta_k(X_{0,1})\right] \\
&\quad = -\sum_{k\in S}\sum_{l\in S}q(l\mid k)\log q(l\mid k)\lim_n \frac{1}{|T^{(n-1)}|}S_{n-1}(k,\omega) \\
&\quad = -\sum_{k\in S}\sum_{l\in S}\pi(k)q(l\mid k)\log q(l\mid k).
\end{aligned}
\tag{4.4}
$$

Equation(4.2) follows from (4.4). □

Theorem 4.3. *Let $X^{T^{(n)}} = \{X_t, t \in T^{(n)}\}$ be a Markov chains field defined on a Bethe tree $T_{B,N}$, $H_m^Q(X_{m+1,j}\mid X_{m,i})$ defined as above. Then*

$$
\begin{aligned}
&\lim_n \frac{1}{|T^{(n)}|}\left\{\sum_{i=1}^{N+1}E_Q\left[H_0^Q(X_{1,i}\mid X_{0,1})\right] + \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni}E_Q\left[H_m^Q(X_{m+1,j}\mid X_{m,i})\right]\right\} \\
&= -\sum_{k\in S}\sum_{l\in S}q(k)q(l\mid k)\log q(l\mid k). \quad \mu_Q\text{-a.s.}
\end{aligned}
\tag{4.5}
$$

Proof. By the definition of $H_m^Q(X_{m+1,j} \mid X_{m,i})$ and properties of conditional expectation, we have

$$\begin{aligned}
&\sum_{i=1}^{N+1} E_Q\left[H_0^Q(X_{1,i} \mid X_{0,1})\right] + \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni} E_Q\left[H_m^Q(X_{m+1,j} \mid X_{m,i})\right] \\
&\quad= \sum_{i=1}^{N+1} - E_Q\{E_Q[\log q(X_{1,i} \mid X_{0,1}) \mid X_{0,1}]\} \\
&\qquad+ \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni} - E_Q\{E_Q[\log q(X_{m+1,j} \mid X_{m,i}) \mid X_{m,i}]\} \\
&\quad= \sum_{i=1}^{N+1} - E_Q[\log q(X_{1,i} \mid X_{0,1})] \\
&\qquad+ \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni} - E_Q[\log q(X_{m+1,j} \mid X_{m,i})] \\
&\quad= -\sum_{i=1}^{N+1}\sum_{x_{0,1}\in S}\sum_{x_{1,i}\in S} q(x_{0,1}, x_{1,i}) \log q(x_{1,i} \mid x_{0,1}) \\
&\qquad- \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni}\sum_{x_{m,i}\in S}\sum_{x_{m+1,j}\in S} q(x_{m,i}, x_{m+1,j}) \log q(x_{m+1,j} \mid x_{\mathrm{m},i}) \\
&\quad= -\sum_{i=1}^{N+1}\sum_{k\in S}\sum_{l\in S} q(k)q(l \mid k) \log q(l \mid k) \\
&\qquad- \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni}\sum_{k\in S}\sum_{l\in S} q(k)q(l \mid k) \log q(l \mid k) \\
&\quad= -\sum_{k\in S}\sum_{l\in S} q(k)q(l \mid k) \log q(l \mid k)\left(\left|T^{(n)}\right| - 1\right).
\end{aligned} \tag{4.6}$$

Accordingly we have by (4.6)

$$\begin{aligned}
&\lim_n \frac{1}{\left|T^{(n)}\right|}\left\{\sum_{i=1}^{N+1} E_Q\left[H_0^Q(X_{1,i} \mid X_{0,1})\right] + \sum_{m=1}^{n-1}\sum_{i=1}^{(N+1)N^{m-1}}\sum_{j=N(i-1)+1}^{Ni} E_Q\left[H_m^Q(X_{m+1,j} \mid X_{m,i})\right]\right\} \\
&\quad= -\sum_{k\in S}\sum_{l\in S} q(k)q(l \mid k) \log q(l \mid k) \lim_n \frac{\left|T^{(n)}\right| - 1}{\left|T^{(n)}\right|} \\
&\quad= -\sum_{k\in S}\sum_{l\in S} q(k)q(l \mid k) \log q(l \mid k), \quad \mu_Q\text{-a.s.}
\end{aligned} \tag{4.7}$$

Therefore (4.5) also holds. □

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