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Research Article

Nonsquareness and Locally Uniform Nonsquareness in Orlicz-Bochner Function Spaces Endowed with Luxemburg Norm

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Criteria for nonsquareness and locally uniform nonsquareness of Orlicz-Bochner function spaces equipped with Luxemburg norm are given. We also prove that, in Orlicz-Bochner function spaces generated by locally uniform nonsquare Banach space, nonsquareness and locally uniform nonsquareness are equivalent.

1. Introduction

A lot of nonsquareness concepts in Banach spaces are known (see [1]). Nonsquareness are important notions in geometry of Banach space. One of reasons is that these properties are strongly related to the fixed point property (see [2]). The criteria for nonsquareness and locally uniform nonsquareness in the classical Orlicz function spaces have been given in [3, 4] already. However, because of the complicated structure of Orlicz-Bochner function spaces equipped with the Luxemburg norm, the criteria for nonsquareness and locally uniform nonsquareness of them have not been found yet. The aim of this paper is to give criteria for nonsquareness and locally uniform nonsquareness of Orlicz-Bochner function spaces equipped with Luxemburg norm.

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and unit ball, respectively. Let us recall some geometrical notions concerning nonsquareness. A Banach space X is said to be nonsquare if for any $x, y \in S(X)$ we have $\min\{\|(x+y)/2\|, \|(x-y)/2\|\} < 1$. A Banach space X is said to be uniformly nonsquare if there exists $\delta > 0$ such that for any $x, y \in S(X)$, $\min\{\|(x+y)/2\|, \|(x-y)/2\|\} < 1-\delta$. A Banach space X is said to be locally uniformly nonsquare if for any $x \in S(X)$, there exists $\delta_x > 0$ such that $\min\{\|(x+y)/2\|, \|(x-y)/2\|\} < 1-\delta_x$, where $y \in S(X)$.

Let R be set of real numbers. A function $M : R \to R^+$ is called an N-function if M is convex, even, $M(0) = 0$, $M(u) > 0$ $(u \neq 0)$ and $\lim_{u\to 0}(M(u)/u) = 0$, and $\lim_{u\to 0}(M(u)/u) = \infty$.

Let (T, Σ, μ) be a nonatomic measurable space. p denotes right derivative of M. Moreover, for a given Banach space $(X, \|\cdot\|)$, we denote by X_T the set of all strongly μ-measurable function from T to X, and for each $u \in X_T$, we define the modular of u by

$$\rho_M(u) = \int_G M(\|u(t)\|)dt. \tag{1.1}$$

Put

$$L_M = \left\{u(t) \in X_T : \int_G M(\|\lambda u(t)\|)dt < \infty \text{ for some } \lambda > 0\right\}. \tag{1.2}$$

The linear set L_M endowed with the Luxemburg norm

$$\|u\| = \inf\left\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) \le 1\right\} \tag{1.3}$$

is a Banach space. We say that an Orlicz function M satisfies condition Δ_2 $(M \in \Delta_2)$ if there exist $K > 2$ and $u_0 \ge 0$ such that

$$M(2u) \le KM(u) \quad (u \ge u_0). \tag{1.4}$$

First let us recall a known result that will be used in the further part of the paper.

Lemma 1.1 (see [3]). *Suppose $M \in \Delta_2$. Then*

$$\rho_M(u_n) \longrightarrow 0 \Longleftrightarrow \|u_n\| \longrightarrow 0, \qquad \rho_M(u_n) \longrightarrow 1 \Longleftrightarrow \|u_n\| \longrightarrow 1(n \longrightarrow \infty). \tag{1.5}$$

2. Main Results

Theorem 2.1. *L_M is nonsquare if and only if*

(a) *$M \in \Delta_2$;*

(b) *X is nonsquare.*

In order to prove the theorem, we give a lemma.

Lemma 2.2. *If X is nonsquare, then for any $x, y \neq 0$, we have*

$$\|x\| + \|y\| - \min\{\|x + y\|, \|x - y\|\} > 0. \tag{2.1}$$

Proof.

Case 1. If $\|x\| < \|y\|$, then

$$\begin{aligned}\|x+y\| &\le \left\|x+\frac{\|x\|}{\|y\|}\cdot y\right\| + \left(1-\frac{\|x\|}{\|y\|}\right)\cdot\|y\| \\ &< \|x\|+\|x\|+\|y\|-\|x\| \\ &= \|x\|+\|y\|\end{aligned} \tag{2.2}$$

or

$$\|x-y\| < \|x\|+\|y\|. \tag{2.3}$$

Case 2. If $\|x\| \ge \|y\|$, then

$$\begin{aligned}\|x+y\| &\le \left\|\frac{\|y\|}{\|x\|}\cdot x+y\right\| + \left(\frac{\|x\|}{\|y\|}-1\right)\cdot\|y\| \\ &< \|y\|+\|y\|+\|x\|-\|y\| \\ &= \|x\|+\|y\|\end{aligned} \tag{2.4}$$

or

$$\|x-y\| < \|x\|+\|y\|. \tag{2.5}$$

This implies $\|x\|+\|y\|-\min\{\|x+y\|,\|x-y\|\} > 0$. This completes the proof. □

Proof of Theorem 2.1. (a) *Necessity*. Suppose that $M \notin \Delta_2$, then there exist $u \in S(L_M)$ and $\delta > 0$ such that $\rho_M(u) = 1-\delta < 1$. Pick $c > 0$ such that $E = \{t \in T : \|u(t)\| \le c\}$ is not a null set. Since $M \notin \Delta_2$, there exist sequence $\{r_n\}_{n=1}^{\infty}$ and disjont subsets $\{E_n\}_{n=1}^{\infty}$ of E such that

$$r_n > 2nc, \quad M\left(\left(1+\frac{1}{n}\right)r_n\right) > 2^n M\left(\left(1+\frac{1}{2n}\right)r_n\right), \quad 2^n M\left(\left(1+\frac{1}{2n}\right)r_n\right)\mu E_n = 2^{-n}\delta. \tag{2.6}$$

Therefore, if we define $v = \sum_{n=1}^{\infty} r_n \chi_{E_n}$, then for any $l > 1$, we have

$$\begin{aligned}\rho_M(lv) &= \sum_{n=1}^{\infty}\rho_M(lr_n\chi_{E_n}) \ge \sum_{n=m}^{\infty}\rho_M\left(\left(1+\frac{1}{n}\right)r_n\chi_{E_n}\right) > \sum_{n=m}^{\infty}2^n\rho_M\left(\left(1+\frac{1}{2n}\right)r_n\chi_{E_n}\right) \\ &\ge \sum_{n=m}^{\infty}2^n M\left(\left(1+\frac{1}{2n}\right)r_n\right)\mu E_n = \sum_{n=m}^{\infty}2^n\cdot 2^{-n}\delta = \infty, \\ \rho_M(v) &= \sum_{n=1}^{\infty}M(r_n)\mu E_n < \sum_{n=1}^{\infty}M(r_n+c)\mu E_n < \sum_{n=1}^{\infty}M\left(\left(1+\frac{1}{2n}\right)r_n\right)\mu E_n = \delta.\end{aligned} \tag{2.7}$$

This yields

$$\|v\| = 1, \qquad \rho_M(u \pm v) \leq \rho_M(u) + \sum_{n=1}^{\infty} M(r_n + c)\mu E_n = 1 - \delta + \delta = 1. \tag{2.8}$$

Hence, $\|u \pm v\| \leq 1$. But $\|u + v\| + \|u - v\| \leq \|2u\| = 2$, and we deduce that $\|u + v\| = \|u - v\| = 1$. Moreover, we have $\|(1/2)(u + v) + (1/2)(u - v)\| = 1$ and $\|(1/2)(u + v) - (1/2)(u - v)\|\} = 1$, a contradiction with nonsquareness of L_M.

If (b) is not true, then there exist $x, y \in S(X)$ such that $\|x\| = \|y\| = \|(1/2)(x + y)\| = \|(1/2)(x - y)\|$. Pick $\alpha > 0$ such that $\int_T M(\alpha)dt = 1$. Put

$$u(t) = \alpha \cdot x \cdot \chi_T(t), \qquad v(t) = \alpha \cdot y \cdot \chi_T(t). \tag{2.9}$$

Then we have

$$\begin{aligned} \rho_M(u) &= \int_T M(\|\alpha x\|)dt = \int_T M(\alpha)dt = 1, \\ \rho_M(v) &= \int_T M(\|\alpha y\|)dt = \int_T M(\alpha)dt = 1. \end{aligned} \tag{2.10}$$

It is easy to see $u, v \in S(L_M)$. We know that

$$\frac{u(t) + v(t)}{2} = \alpha \cdot \frac{x + y}{2} \cdot \chi_T(t), \qquad \frac{u(t) - v(t)}{2} = \alpha \cdot \frac{x - y}{2} \cdot \chi_T(t). \tag{2.11}$$

Hence, we have

$$\begin{aligned} \rho_M\left(\frac{u + v}{2}\right) &= \int_T M\left(\left\|\alpha \cdot \frac{x + y}{2}\right\|\right)dt = \int_T M(\alpha)dt = 1, \\ \rho_M\left(\frac{u - v}{2}\right) &= \int_T M\left(\left\|\alpha \cdot \frac{x - y}{2}\right\|\right)dt = \int_T M(\alpha)dt = 1. \end{aligned} \tag{2.12}$$

It is easy to see $(1/2)(u + v), (1/2)(u - v) \in S(L_M)$, a contradiction!

Sufficiency. Suppose that there exists $u, v \in S(L_M)$ such that

$$\|u\| = \|v\| = \left\|\frac{1}{2}(u + v)\right\| = \left\|\frac{1}{2}(u - v)\right\| = 1. \tag{2.13}$$

We will derive a contradiction for each of the following two cases.

Case 1. $\mu(\{t \in T : \|u(t)\| \neq 0\} \cap \{t \in T : \|v(t)\| \neq 0\}) = 0$. Let $G = \{t \in T : \|u(t)\| \neq 0\}$. Hence, we have

$$\begin{aligned}
\frac{1}{2}\rho_M(u) + \frac{1}{2}\rho_M(v) &= \frac{1}{2}\int_G M(\|u(t)\|)dt + \frac{1}{2}\int_{T\setminus G} M(\|v(t)\|)dt \\
&= \frac{1}{2}\int_G M(\|u(t) + v(t)\|)dt + \frac{1}{2}\int_{T\setminus G} M(\|u(t) + v(t)\|)dt \\
&> \int_G M\left(\frac{1}{2}\|u(t) + v(t)\|\right)dt + \int_{T\setminus G} M\left(\frac{1}{2}\|u(t) + v(t)\|\right)dt \\
&= \int_T M\left(\frac{1}{2}\|u(t) + v(t)\|\right)dt \\
&= \rho_M\left(\frac{1}{2}(u + v)\right).
\end{aligned} \tag{2.14}$$

Since $M \in \Delta_2$, we have $\rho_M(u) = \rho_M(v) = 1$. Hence, $\rho_M((u + v)/2) < 1$. This implies $\|(u + v)/2\| < 1$, a contradiction!

Case 2. $\mu(\{t \in T : \|u(t)\| \neq 0\} \cap \{t \in T : \|v(t)\| \neq 0\}) > 0$. By Lemma 2.2, without loss of generality, we may assume that there exists $T_1 \subset \{t \in T : \|u(t)\| \neq 0\} \cap \{t \in T : \|v(t)\| \neq 0\}$ such that $\|u(t)\| + \|v(t)\| > \|u(t) + v(t)\|$, $t \in T_1$ and $\mu T_1 > 0$. Therefore,

$$\begin{aligned}
\frac{1}{2}\rho_M(u) + \frac{1}{2}\rho_M(v) &= \frac{1}{2}\int_T M(\|u(t)\|)dt + \frac{1}{2}\int_T M(\|v(t)\|)dt \\
&= \int_T \frac{1}{2}M(\|u(t)\|) + \frac{1}{2}M(\|v(t)\|)dt \\
&\geq \int_{T_1} M\left(\frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right)dt + \int_{T\setminus T_1} M\left(\frac{1}{2}\|u(t)\| + \frac{1}{2}\|v(t)\|\right)dt \\
&> \int_{T_1} M\left(\frac{1}{2}\|u(t) + v(t)\|\right)dt + \int_{T\setminus T_1} M\left(\frac{1}{2}\|u(t) + v(t)\|\right)dt \\
&= \rho_M\left(\frac{u + v}{2}\right).
\end{aligned} \tag{2.15}$$

Since $M \in \Delta_2$, we have $\rho_M(u) = \rho_M(v) = 1$. Hence, $\rho_M((u + v)/2) < 1$. This implies $\|(u + v)/2\| < 1$, a contradiction! □

Theorem 2.3. *L_M is locally uniformly nonsquare if and only if*

(a) $M \in \Delta_2$;

(b) X *is locally uniformly nonsquare.*

In order to prove the theorem, we give a lemma.

Lemma 2.4. *If X is locally uniformly nonsquare, then*

(a) *For any* $x \neq 0$, $r_1 \geq r_2 > 0$, *we have*

$$\inf_{y\neq 0}\{\|x\| + \|y\| - \min\{\|x+y\|, \|x-y\|\} : x \in X, r_2 \leq \|y\| \leq r_1\} > 0 \tag{2.16}$$

(b) *If* $x_n \to x$, *then* $\lim_{n\to\infty}\delta(x_n) = \delta(x)$, *where*

$$\delta(x) = \inf_{y\neq 0}\{\|x\| + \|y\| - \min\{\|x+y\|, \|x-y\|\} : x \in X, r_2 \leq \|y\| \leq r_1\}. \tag{2.17}$$

Proof. (a) Since X is locally uniformly nonsquare, we have $\eta_x > 0$ and $\eta_{\lambda x} = \lambda\eta_x$, where $\lambda > 0$ and

$$\eta_x = \inf_y\{\|x\| + \|y\| - \min\{\|x+y\|, \|x-y\|\} : \|x\| = \|y\| > 0\}. \tag{2.18}$$

In fact, since X is locally uniformly nonsquare, we have

$$\begin{aligned}
\eta_x &= \inf_y\{\|x\| + \|y\| - \min\{\|x+y\|, \|x-y\|\} : \|x\| = \|y\| > 0\} \\
&= \|x\| \cdot \inf_y\left\{2 - \min\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right\} : \|x\| = \|y\| > 0\right\} > 0, \\
\eta_{\lambda x} &= \inf_y\{\|\lambda x\| + \|y\| - \min\{\|\lambda x+y\|, \|\lambda x-y\|\} : \|\lambda x\| = \|y\| > 0\} \\
&= \lambda \cdot \inf_y\left\{\|x\| + \left\|\frac{1}{\lambda}y\right\| - \min\left\{\left\|x + \frac{1}{\lambda}y\right\|, \left\|x - \frac{1}{\lambda}y\right\|\right\} : \|x\| = \left\|\frac{1}{\lambda}y\right\| > 0\right\} \\
&= \lambda \cdot \inf_y\{\|x\| + \|y\| - \min\{\|x+y\|, \|x-y\|\} : \|x\| = \|y\| > 0\} \\
&= \lambda \cdot \eta_x.
\end{aligned} \tag{2.19}$$

Case 1. If $\|x\| \geq \|y\|$, then

$$\begin{aligned}
\|x+y\| &\leq \left\| \left(1-\frac{\|y\|}{\|x\|}\right)x + y + \frac{\|y\|}{\|x\|}x \right\| \\
&\leq \left\| \left(1-\frac{\|y\|}{\|x\|}\right)x \right\| + \left\| y + \frac{\|y\|}{\|x\|}x \right\| \\
&\leq \|x\| - \|y\| + \|y\| + \|y\| - \eta_{(\|y\|/\|x\|)x} \\
&\leq \|x\| + \|y\| - \eta_{(r_2/\|x\|)x}
\end{aligned} \tag{2.20}$$

or

$$\|x-y\| \leq \|x\| + \|y\| - \eta_{(r_2/\|x\|)x}. \tag{2.21}$$

Case 2. If $\|x\| < \|y\|$, then

$$\begin{aligned}
\|x+y\| &\leq \left\| \frac{\|x\|}{\|y\|}\cdot y + x \right\| + \left(1-\frac{\|x\|}{\|y\|}\right)\cdot\|y\| \\
&\leq \|x\| + \|x\| - \eta_x + \|y\| - \|x\| \\
&= \|x\| + \|y\| - \eta_x \\
&\leq \|x\| + \|y\| - \eta_x
\end{aligned} \tag{2.22}$$

or

$$\|x-y\| \leq \|x\| + \|y\| - \eta_x. \tag{2.23}$$

Therefore, we get, the following inequality

$$\inf_y\{\|x\| + \|y\| - \min\{\|x+y\|, \|x-y\|\} : x \in X\} \geq \min\{\eta_{(r_2/\|x\|)x}, \eta_x\} > 0 \tag{2.24}$$

holds.

(b1) Suppose that $\limsup_{n\to\infty}\delta(x_n) > \delta(x)$, where $x_n \to x$ $(n \to \infty)$. Then there exist $a > 0$ and subsequence $\{n\}$ of $\{n\}$, such that $\delta(x_n) - \delta(x) \geq a$. By definition of $\delta(x)$, there exist $y_0 \in X$ such that

$$\|x\| + \|y_0\| - \min\{\|x+y_0\|, \|x-y_0\|\} < \delta(x) + \frac{a}{8}, \quad r_1 \leq \|y_0\| \leq r_2. \tag{2.25}$$

We will derive a contradiction for each of the following two cases.

Case 1. $\|x+y_0\| = \|x-y_0\|$. Since $x_n \to x$ $(n \to \infty)$, there exists n_0 such that $\|x_{n_0} - x\| < a/8$. Therefore,

$$\begin{aligned}
\|x_{n_0}\| + \|y_0\| - \|x_{n_0} + y_0\| &\le \|x\| + \|x_{n_0} - x\| + \|y_0\| - \|x_{n_0} + y_0\| \\
&\le \|x\| + \|x_{n_0} - x\| + \|y_0\| - \left(\|x + y_0\| - \|x_{n_0} - x\|\right) \\
&= \|x\| + \|y_0\| - \|x + y_0\| + 2\|x_{n_0} - x\| \\
&\le \delta(x) + \frac{a}{8} + 2\|x_{n_0} - x\| \\
&< \delta(x) + \frac{a}{8} + 2 \cdot \frac{a}{8} \\
&= \delta(x) + \frac{3}{8}a.
\end{aligned} \tag{2.26}$$

This implies $\delta(x_{n_0}) \le \delta(x) + (3/8)a$, a contradiction!

Case 2. $\|x - y_0\| \ne \|x + y_0\|$. Without loss of generality, we may assume $\|x - y_0\| > \|x + y_0\| + r$, where $r > 0$. Since $x_n \to x$ $(n \to \infty)$, there exists n_0 such that $\|x_{n_0} - x\| < \min\{(1/8)a, (1/8)r\}$. Therefore, we have

$$\begin{aligned}
\|x_{n_0} - y_0\| &= \|x - y_0 + x_{n_0} - x\| \\
&\ge \|x - y_0\| - \|x_{n_0} - x\| \\
&\ge \|x - y_0\| - \frac{1}{8}r, \\
\|x_{n_0} + y_0\| &= \|x + y_0 + x_{n_0} - x\| \\
&\le \|x + y_0\| + \|x_{n_0} - x\| \\
&\le \|x + y_0\| + \frac{1}{8}r.
\end{aligned} \tag{2.27}$$

This implies

$$\|x_{n_0} - y_0\| \ge \|x - y_0\| - \frac{1}{8}r \ge \|x + y_0\| + r - \frac{1}{8}r \ge \|x + y_0\| + \frac{1}{8}r \ge \|x_{n_0} + y_0\|. \tag{2.28}$$

Similarly, we have

$$\|x_{n_0}\| + \|y_0\| - \|x_{n_0} + y_0\| \le \delta(x) + \frac{3}{8}a. \tag{2.29}$$

Therefore, we have

$$\|x_{n_0}\| + \|y_0\| - \min\{\|x_{n_0} + y_0\|, \|x_{n_0} - y_0\|\} \le \delta(x) + \frac{3}{8}a. \tag{2.30}$$

This implies $\delta(x_{n_0}) \le \delta(x) + (3/8)a$, a contradiction! Hence, $\limsup_{n\to\infty}\delta(x_n) \le \delta(x)$.

(b2) Suppose that $\liminf_{n\to\infty}\delta(x_n) < \delta(x)$, where $x_n \to x$ $(n \to \infty)$. Then there exist $b > 0$ and subsequence $\{n\}$ of $\{n\}$, such that $\delta(x) - \delta(x_n) \geq b$. Since $x_n \to x$ $(n \to \infty)$, then there exist $n_0 \in N$ such that $\|x_{n_0} - x_n\| < (1/8)b$, whenever $n \geq n_0$. By definition of $\delta(x_{n_0})$, there exist $y_0 \in X$ such that

$$\|x_{n_0}\| + \|y_0\| - \min\{\|x_{n_0} + y_0\|, \|x_{n_0} - y_0\|\} < \delta(x_{n_0}) + \frac{b}{8}, \qquad r_2 \leq \|y_0\| \leq r_1. \tag{2.31}$$

Therefore, we have

$$\begin{aligned}
&\|x_n\| + \|y_0\| - \min\{\|x_n + y_0\|, \|x_n - y_0\|\} \\
&\quad = \|x_{n_0} - x_{n_0} + x_n\| + \|y_0\| - \min\{\|x_{n_0} - x_{n_0} + x_n + y_0\|, \|x_{n_0} - x_{n_0} + x_n - y_0\|\} \\
&\quad \leq \|x_{n_0}\| + \frac{1}{8}b + \|y_0\| - \min\{\|x_{n_0} + y_0\|, \|x_{n_0} - y_0\|\} + \frac{1}{8}b \\
&\quad = \|x_{n_0}\| + \|y_0\| - \min\{\|x_{n_0} + y_0\|, \|x_{n_0} - y_0\|\} + \frac{1}{4}b \\
&\quad < \delta(x_{n_0}) + \frac{1}{8}b + \frac{1}{4}b \\
&\quad < \delta(x_{n_0}) + \frac{3}{8}b
\end{aligned} \tag{2.32}$$

whenever $n \geq n_0$. Since $x_n \to x$ $(n \to \infty)$, there exists $n_1 > n_0$ such that $|\eta(x) - \eta(x_{n_1})| < (1/8)b$, where

$$\eta(x) = \|x\| + \|y_0\| - \min\{\|x + y_0\|, \|x - y_0\|\}. \tag{2.33}$$

Hence, we have

$$\eta(x_{n_1}) > \eta(x) - \frac{1}{8}b \geq \delta(x) - \frac{1}{8}b \geq \delta(x_{n_0}) + b - \frac{1}{8}b = \delta(x_{n_0}) + \frac{7}{8}b. \tag{2.34}$$

This implies

$$\|x_{n_1}\| + \|y_0\| - \min\{\|x_{n_1} + y_0\|, \|x_{n_1} - y_0\|\} \geq \delta(x_{n_0}) + \frac{7}{8}b, \tag{2.35}$$

which contradict (2.32). Hence, $\liminf_{n\to\infty}\delta(x_n) \geq \delta(x)$.

Combing (b1) with (b2), we get $\lim_{n\to\infty}\delta(x_n) = \delta(x)$. This completes the proof. □

Proof of Theorem 2.3. Necessity. By Theorem 2.1, $M \in \Delta_2$. If (b) is not true, then there exist $x \in S(X)$, $\{y_n\}_{n=1}^{\infty} \subset S(X)$ such that $\|(1/2)(x+y_n)\| \to 1$ and $\|(1/2)(x-y_n)\| \to 1$ as $n \to \infty$. Pick $\alpha > 0$ such that $\int_T M(\alpha)dt = 1$. Put

$$u(t) = \alpha \cdot x \cdot \chi_T(t), v_n(t) = \alpha \cdot y_n \cdot \chi_T(t). \tag{2.36}$$

Then we have

$$\begin{aligned} \rho_M(u) &= \int_T M(\|\alpha x\|)dt = \int_T M(\alpha)dt = 1, \\ \rho_M(v_n) &= \int_T M(\|\alpha y_n\|)dt = \int_T M(\alpha)dt = 1. \end{aligned} \tag{2.37}$$

It is easy to see $u, v_n \in S(L_M)$. We know that

$$\frac{u(t)+v_n(t)}{2} = \alpha \cdot \frac{x+y_n}{2} \cdot \chi_T(t), \qquad \frac{u(t)-v_n(t)}{2} = \alpha \cdot \frac{x-y_n}{2} \cdot \chi_T(t). \tag{2.38}$$

Moreover, we have $M(\alpha\cdot\|(x+y_n)/2\|) \le M(\alpha)$, $M(\alpha\cdot\|(x-y_n)/2\|) \le M(\alpha)$. By the dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \int_T M\left(\alpha \cdot \left\|\frac{x+y_n}{2}\right\|\right)dt &= \int_T \lim_{n\to\infty} M\left(\alpha \cdot \left\|\frac{x+y_n}{2}\right\|\right)dt = \int_T M(\alpha)dt = 1, \\ \lim_{n\to\infty} \int_T M\left(\alpha \cdot \left\|\frac{x-y_n}{2}\right\|\right)dt &= \int_T \lim_{n\to\infty} M\left(\alpha \cdot \left\|\frac{x-y_n}{2}\right\|\right)dt = \int_T M(\alpha)dt = 1. \end{aligned} \tag{2.39}$$

It is easy to see $\rho_M((1/2)(u+v_n)) \to 1, \rho_M((1/2)(u-v_n)) \to 1$ as $n \to \infty$. By Lemma 1.1, we have $\|(1/2)(u+v_n)\| \to 1$ and $\|(1/2)(u-v_n)\| \to 1$ as $n \to \infty$, a contradiction with locally uniform nonsquareness of L_M.

Sufficiency. Suppose that there exist $u \in S(L_M)$, $\{v_n\}_{n=1}^{\infty} \subset S(L_M)$ such that $\|(1/2)(u+v_n)\| \to 1, \|(1/2)(u-v_n)\| \to 1$ as $n \to \infty$. We will derive a contradiction for each of the following two cases.

Case 1. There exist $\varepsilon_0 > 0$, $\sigma_0 > 0$ such that $\mu G_n > \varepsilon_0$, where $G_n = \{t \in T : \|v_n(t)\| \ge \sigma_0\}$. Put

$$H_n = \left\{t \in T : \sigma_0 \le \|v_n(t)\| \le M^{-1}\left(\frac{4}{\varepsilon_0}\right)\right\}. \tag{2.40}$$

We have

$$1 = \int_T M(\|v_n(t)\|)dt \ge \int_{G_n\setminus H_n} M(\|v_n(t)\|)dt \ge \frac{4}{\varepsilon_0} \cdot \mu(G_n \setminus H_n). \tag{2.41}$$

This implies $\mu(G_n \setminus H_n) \le (1/4)\varepsilon_0$. Hence, $\mu H_n > (1/2)\varepsilon_0$. We define a function

$$\eta(t) = \inf_{y\neq 0}\left\{ \|u(t)\| + \|y\| - \min\{\|u(t)+y\|, \|u(t)-y\|\} : \sigma_0 \le \|y\| \le M^{-1}\left(\frac{4}{\varepsilon_0}\right)\right\} \tag{2.42}$$

on T_0, where $T_0 = \{t \in T : \|u(t)\| \neq 0\}$. By Lemma 2.4, we have $\eta(t) > 0$ μ-a.e on T_0. Let $h_n(t) \to u(t)$ μ-a.e on T_0, where h_n is simple function. Hence,

$$\eta_n(t) = \inf_{y\neq 0}\left\{ \|h_n(t)\| + \|y\| - \min\{\|h_n(t)+y\|, \|h_n(t)-y\|\} : \sigma_0 \le \|y\| \le M^{-1}\left(\frac{4}{\varepsilon_0}\right)\right\} \tag{2.43}$$

is μ-measurable. By Lemma 2.4, we have $\eta_n(t) \to \eta(t)$ μ-a.e on T_0. Then $\eta(t)$ is μ-measurable. Using

$$T \supset \bigcup_{i=1}^{\infty}\left\{ t \in T_0 : \frac{1}{i+1} < \eta(t) \le \frac{1}{i}\right\}, \tag{2.44}$$

we get that there exists $\eta_0 > 0$ such that $\mu H < (1/8)\varepsilon_0$, where

$$H = \{t \in T_0 : \eta(t) \le 2\eta_0\}. \tag{2.45}$$

Let $E_n = H_n \setminus H$, $E_n^1 = (H_n \cap \{t \in T : \|u(t)\| \neq 0\}) \setminus H$, $E_n^2 = (H_n \cap \{t \in T : \|u(t)\| = 0\}) \setminus H$. It is easy to see $E_n = E_n^1 \cup E_n^2$, $E_n^1 \cap E_n^2 = \phi$ and $\mu E_n \ge (3/8)\varepsilon_0$. If $t \in E_n^1$, by Lemma 2.4, we have

$$\|u(t)\| + \|v_n(t)\| - \min\{\|u(t)+v_n(t)\|, \|u(t)-v_n(t)\|\} \ge \eta(t) \ge 2\eta_0. \tag{2.46}$$

Without loss of generality, we may assume that there exists $F_n^1 \subset E_n^1$ such that

$$\mu F_n^1 \ge \frac{1}{2}\mu E_n^1, \qquad \|u(t)\| + \|v_n(t)\| - \|u(t)+v_n(t)\| \ge 2\eta_0 t \in F_n^1. \tag{2.47}$$

Moreover, for any $u \ge v > 0$, we have

$$\begin{aligned}
&\frac{1}{2}M(u) - M\left(\frac{1}{2}u\right) - \left[\frac{1}{2}M(v) - M\left(\frac{1}{2}v\right)\right] \\
&\quad = \frac{1}{2}\int_0^u p(t)dt - \int_0^{(1/2)u} p(t)dt - \frac{1}{2}\int_0^v p(t)dt + \int_0^{(1/2)v} p(t)dt \\
&\quad = \left[\frac{1}{2}\int_0^u p(t)dt - \frac{1}{2}\int_0^v p(t)dt\right] - \left[\int_0^{(1/2)u} p(t)dt - \int_0^{(1/2)v} p(t)dt\right] \\
&\quad = \frac{1}{2}\int_v^u p(t)dt - \int_{(1/2)v}^{(1/2)u} p(t)dt \\
&\quad \ge \int_v^{(1/2)u+(1/2)v} p(t)dt - \int_{(1/2)v}^{(1/2)u} p(t)dt \ge 0.
\end{aligned} \tag{2.48}$$

Hence, if $t \in E_n^2$, then

$$\frac{1}{2}M(\|v_n(t)\|) + \frac{1}{2}M(\|v_n(t)\|) \geq \frac{1}{2}M(\sigma_0) - M\left(\frac{\sigma_0}{2}\right) > 0. \tag{2.49}$$

Let $F_n = F_n^1 \cup E_n^2$. Then $\mu F_n \geq (1/8)\varepsilon_0$. Therefore,

$$\begin{aligned}
&\frac{1}{2}\rho_M(u) + \frac{1}{2}\rho_M(v_n) - \rho_M\left(\frac{u+v_n}{2}\right) \\
&\quad = \frac{1}{2}\int_T M(\|u(t)\|)dt + \frac{1}{2}\int_T M(\|v_n(t)\|)dt - \int_T M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)dt \\
&\quad = \int_T \left[\frac{1}{2}M(\|u(t)\|) + \frac{1}{2}M(\|v_n(t)\|) - M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)\right]dt \\
&\quad \geq \int_{F_n^1} \left[\frac{1}{2}M(\|u(t)\|) + \frac{1}{2}M(\|v_n(t)\|) - M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)\right]dt \\
&\qquad + \int_{E_n^2} \left[\frac{1}{2}M(\|u(t)\|) + \frac{1}{2}M(\|v_n(t)\|) - M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)\right]dt \\
&\quad \geq \int_{F_n^1} \left[M\left(\frac{1}{2}\|u(t)\| + \frac{1}{2}\|v_n(t)\|\right) - M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)\right]dt \\
&\qquad + \int_{E_n^2} \left[\frac{1}{2}M(\|u(t)\|) + \frac{1}{2}M(\|v_n(t)\|) - M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)\right]dt \\
&\quad \geq \int_{F_n^1} \left[M\left(\frac{\|u(t)+v_n(t)\|}{2} + \eta_0\right) - M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)\right]dt \\
&\qquad + \int_{E_n^2} \left[\frac{1}{2}M(\|v_n(t)\|) - M\left(\frac{\|v_n(t)\|}{2}\right)\right]dt \\
&\quad \geq \int_{F_n^1} M(\eta_0)dt + \int_{E_n^2} \left[\frac{1}{2}M(\sigma_0) - M\left(\frac{\sigma_0}{2}\right)\right]dt \\
&\quad \geq \int_{F_n} \min\left\{M(\eta_0), \frac{1}{2}M(\sigma_0) - M\left(\frac{\sigma_0}{2}\right)\right\}dt \\
&\quad = \min\left\{M(\eta_0), \frac{1}{2}M(\sigma_0) - M\left(\frac{\sigma_0}{2}\right)\right\} \cdot \mu F_n \\
&\quad \geq \min\left\{M(\eta_0), \frac{1}{2}M(\sigma_0) - M\left(\frac{\sigma_0}{2}\right)\right\} \cdot \frac{1}{8}\varepsilon_0.
\end{aligned} \tag{2.50}$$

By Lemma 1.1, we have $\rho_M(u) = \rho_M(v_n) = 1$, $\rho_M((u+v_n)/2) \to 1$ as $n \to \infty$. This is in contradiction with $(1/2)\rho_M(u) + (1/2)\rho_M(v_n) - \rho_M((u+v_n)/2) \geq \min\{M(\eta_0), (1/2)M(\sigma_0) - M(\sigma_0/2)\} \cdot (1/8)\varepsilon_0$.

Case 2. For any $\varepsilon > 0$, $\sigma > 0$, there exists N such that $\mu\{t \in T : \|v_n(t)\| \geq \sigma\} < \varepsilon$ whenever $n > N$. By the Riesz theorem, without loss of generality, we may assume that $v_n(t) \to 0$ $\mu-$a.e on T. Using

$$\{t \in T : \|u(t)\| \neq 0\} \supset \bigcup_{n=1}^{\infty} \left\{t \in T : \frac{1}{n+1} < \|u(t)\| \leq \frac{1}{n}\right\}, \tag{2.51}$$

we get that there exist $r > 0$ such that $\mu T_1 < (1/8)\mu T_0$, where

$$T_1 = \{t \in T : 0 < \|u(t)\| < d\}, \qquad T_0 = \{t \in T : \|u(t)\| \neq 0\}. \tag{2.52}$$

Since M is N-function, we can choose $0 < h < d$ such that $(1/2)M(d) + (1/2)M(h) - M((d+h)/2) > 0$. Since $v_n(t) \to 0\mu-$ a.e on T, by the Egorov theorem, there exists N_1 such that $\|v_n(t)\| < h, t \in F$ whenever $n > N_1$, where $F \subset T$, $\mu(T \setminus F) < (1/8)\mu T_0$. Next, we will prove that if $u_1 \geq u_2 \geq v_2 \geq v_1 > 0$, then

$$\frac{1}{2}M(u_1) + \frac{1}{2}M(v_1) - M\left(\frac{u_1+v_1}{2}\right) \geq \frac{1}{2}M(u_2) + \frac{1}{2}M(v_2) - M\left(\frac{u_2+v_2}{2}\right). \tag{2.53}$$

In fact, we have

$$\begin{aligned}
&\frac{1}{2}M(u_1) + \frac{1}{2}M(v_1) - M\left(\frac{u_1+v_1}{2}\right) - \left[\frac{1}{2}M(u_1) + \frac{1}{2}M(v_2) - M\left(\frac{u_1+v_2}{2}\right)\right] \\
&\quad = \frac{1}{2}M(v_1) - M\left(\frac{u_1+v_1}{2}\right) - \left[\frac{1}{2}M(v_2) - M\left(\frac{u_1+v_2}{2}\right)\right] \\
&\quad = \frac{1}{2}\int_0^{v_1} p(t)dt - \int_0^{(u_1+v_1)/2} p(t)dt - \left[\frac{1}{2}\int_0^{v_2} p(t)dt - \int_0^{(u_1+v_2)/2} p(t)dt\right] \\
&\quad = \left[\frac{1}{2}\int_0^{v_1} p(t)dt - \frac{1}{2}\int_0^{v_2} p(t)dt\right] + \left[\int_0^{(u_1+v_2)/2} p(t)dt - \int_0^{(u_1+v_1)/2} p(t)dt\right] \\
&\quad = -\frac{1}{2}\int_{v_1}^{v_2} p(t)dt + \int_{(u_1+v_1)/2}^{(u_1+v_2)/2} p(t)dt \\
&\quad \geq \int_{(u_1+v_1)/2}^{(u_1+v_2)/2} p(t)dt - \int_{(v_1+v_2)/2}^{v_2} p(t)dt \geq 0.
\end{aligned} \tag{2.54}$$

Moreover, we have

$$\begin{aligned}
&\frac{1}{2}M(u_1) - M\Big(\frac{u_1+v_2}{2}\Big) - \Big[\frac{1}{2}M(u_2) - M\Big(\frac{u_2+v_2}{2}\Big)\Big] \\
&\quad = \Big[\frac{1}{2}\int_0^{u_1} p(t)dt - \frac{1}{2}\int_0^{u_2} p(t)dt\Big] - \Big[\int_0^{(u_1+v_2)/2} p(t)dt - \int_0^{(u_2+v_2)/2} p(t)dt\Big] \\
&\quad = \frac{1}{2}\int_{u_2}^{u_1} p(t)dt - \int_{(u_2+v_2)/2}^{(u_1+v_2)/2} p(t)dt \\
&\quad \geq \int_{u_2}^{(u_1+u_2)/2} p(t)dt - \int_{(u_2+v_2)/2}^{(u_1+v_2)/2} p(t)dt \geq 0.
\end{aligned} \tag{2.55}$$

By (2.54) and (2.55), we have

$$\frac{1}{2}M(u_1) + \frac{1}{2}M(v_1) - M\Big(\frac{u_1+v_1}{2}\Big) \geq \frac{1}{2}M(u_2) + \frac{1}{2}M(v_2) - M\Big(\frac{u_2+v_2}{2}\Big). \tag{2.56}$$

This shows that if $t \in T_2 = T_0 \setminus (T_1 \cup (T \setminus F))$, then

$$M(\|u(t)\|) + M(\|v_n(t)\|) - 2M\left(\frac{\|u(t)\| + \|v_n(t)\|}{2}\right) \geq M(d) + M(h) - 2M\left(\frac{d+h}{2}\right). \tag{2.57}$$

It is easy to see $\mu T_2 \geq (1/4)\mu T_0$. Therefore,

$$\begin{aligned}
&\frac{1}{2}\rho_M(u) + \frac{1}{2}\rho_M(v_n) - \rho_M\Big(\frac{u+v_n}{2}\Big) \\
&\quad = \frac{1}{2}\int_T M(\|u(t)\|)dt + \frac{1}{2}\int_T M(\|v_n(t)\|)dt - \int_T M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)dt \\
&\quad = \int_T \left[\frac{1}{2}M(\|u(t)\|) + \frac{1}{2}M(\|v_n(t)\|) - M\left(\frac{\|u(t)+v_n(t)\|}{2}\right)\right]dt \\
&\quad \geq \int_T \left[\frac{1}{2}M(\|u(t)\|) + \frac{1}{2}M(\|v_n(t)\|) - M\left(\frac{\|u(t)\|+\|v_n(t)\|}{2}\right)\right]dt \\
&\quad \geq \int_{T_2} \left[\frac{1}{2}M(\|u(t)\|) + \frac{1}{2}M(\|v_n(t)\|) - M\left(\frac{\|u(t)\|+\|v_n(t)\|}{2}\right)\right]dt \\
&\quad \geq \int_{T_2} \left[\frac{1}{2}M(d) + \frac{1}{2}M(d) - M\left(\frac{d+h}{2}\right)\right]dt \\
&\quad \geq \left[\frac{1}{2}M(d) + \frac{1}{2}M(d) - M\left(\frac{d+h}{2}\right)\right] \cdot \frac{1}{4}\mu T_0,
\end{aligned} \tag{2.58}$$

for n large enough. By Lemma 1.1, we have $\rho_M(u) = \rho_M(v_n) = 1$, $\rho_M((u+v_n)/2) \to 1$ as $n \to \infty$, which contradicts $(1/2)\rho_M(u) + (1/2)\rho_M(v_n) - \rho_M((u+v_n)/2) \geq [(1/2)M(d) + (1/2)M(d) - M((d+h)/2)] \cdot (1/4)\mu T_0$, for n large enough. This completes the proof. □

Corollary 2.5. *The following statements are equivalent:*

(a) L_M *is locally uniformly nonsquare if and only if* L_M *is nonsquare;*

(b) X *is locally uniformly nonsquare.*

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