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Equivalence of Markov's and Schur's Inequalities on Compact Subsets of the Complex Plane

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We prove that, on an arbitrary compact subset of the complex plane, Markov's and Schur's inequalities are equivalent.

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We recall first the two classical inequalities of the title.

Markov's inequality ([5], 1889): for any polynomial P

$$\max\{|P'(x)|:\ x \in [-1,1]\} \leq (\deg P)^2 \max\{|P(x)|:\ x \in [-1,1]\}.$$

Schur's inequality ([8], 1919): for any polynomial P

$$\max\{|P(x)|:\ x \in [-1,1]\} \leq (1+\deg P) \max\{|xP(x)|:\ x \in [-1,1]\}.$$

These inequalities are extensively used in approximation theory and have been widely generalized in many ways (see e.g. [1,4,6,7]).

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For example in the one-dimensional case, it has been proved for some compact subset E of $\mathbf{C}$ that

$$\|P'\|_{L^q(E)} \leq C_1(\deg P)^{m_1}\|P\|_{L^q(E)}, \tag{M}$$

$$\|P\|_{L^q(E)} \leq C_2(\deg P)^{m_2}\|(x-x_0)P(x)\|_{L^q(E)}, \tag{S}$$

where $x_0 \in \mathbf{C}$ and C_1, m_1, C_2, m_2 are positive constants depending only on E and q,

$$\|f\|_{L^q(E)} = \left[\int_E |f(x)|^q \,\mathrm{d}x\right]^{1/q}; \quad q \in [1, +\infty),$$
$$\|f\|_{L^\infty(E)} = \max\{|f(x)|:\ x \in E\}.$$

If $E=[-1,1]$, it is not difficult to show that (S) can be established using (M) (see [2, Lemma 2]); moreover (S) implies (M).

In this note we show that for an arbitrary compact set of $\mathbf{C}$, (M) and (S) are equivalent:

PROPOSITION *Let E be a compact subset of* $\mathbf{R}$ (*or* $\mathbf{C}$) *and* $q \in [1, +\infty]$. *Then the following conditions are equivalent*:

(i) *There exist two positive constants C_1, m_1, depending only on E and q such that for any polynomial $P \in \mathcal{P}(\mathbf{R})$ (resp. $\mathcal{P}(\mathbf{C})$),*

$$\|P'\|_{L^q(E)} \leq C_1(\deg P)^{m_1}\|P\|_{L^q(E)},$$

(ii) *There exist two positive constants C_2, m_2, depending only on E and q such that for any polynomial $P \in \mathcal{P}(\mathbf{R})$ (resp. $\mathcal{P}(\mathbf{C})$) and any $x_0 \in \mathbf{R}$ (resp. $\mathbf{C}$),*

$$\|P\|_{L^q(E)} \leq C_2(\deg P)^{m_2}\|(x-x_0)P(x)\|_{L^q(E)},$$

(iii) *There exist two positive constants C_3, m_3, depending only on E* and *q such that for any polynomial $P \in \mathcal{P}(\mathbf{R})$ (resp. $\mathcal{P}(\mathbf{C})$) and any $a, b, c \in \mathbf{R}$ (resp. $\mathbf{C}$),*

$$\|(ax^2+bx+c)P'(x)\|_{L^q(E)} \leq C_3(\deg P)^{m_3}\|(ax^2+bx+c)P(x)\|_{L^q(E)},$$

(iv) *There exist two positive constants* C_4, m_4*, depending only on E and q such that for any polynomials P and any polynomial* $R \in \mathcal{P}(\mathbf{R})$ *(resp.* $\mathcal{P}(\mathbf{C})$*),*

$$\|RP'\|_{L^q(E)} \leq C_4(\deg P + degR)^{m_4} \|RP\|_{L^q(E)}.$$

Inequalities of types (iii) and (iv) were investigated by many authors (see e.g. Nevai [6, Chapter 9] and Goetgheluck [1]).

Proof We will give the proof only for the real case; the complex case is similar.

(1) Inequalities (i) and (ii) are equivalent.
The implication (i) $\Rightarrow$ (ii) is due to Goetgheluck [3].
(ii) $\Rightarrow$ (i). We can write $P(x) = (x - x_1)(x - x_2) \ldots (x - x_k)(x^2 + b_1 x + c_1)\ (x^2 + b_2 x + c_2) \ldots (x^2 + b_l x + c_l)$, where $x_1, x_2, \ldots, x_k, b_1, b_2, \ldots, b_l,\ c_1, c_2, \ldots, c_l \in \mathbf{R}$ and $b_j^2 < 4c_j$ for every $j \in \{1, 2, \ldots, l\}$. Then we have

$$\begin{aligned}
&\|P'\|_{L^q(E)} \\
&\quad \leq \sum_{i=1}^{k} \left\| \left[\prod_{j=1, j\neq i}^{k} (x - x_j) \right] \left[\prod_{n=1}^{l} (x^2 + b_n x + c_n) \right] \right\|_{L^q(E)} \\
&\qquad + \sum_{i=1}^{l} \left\| (2x + b_i) \left[\prod_{j=1}^{k} (x - x_j) \right] \left[\prod_{n=1, n\neq i}^{l} (x^2 + b_n x + c_n) \right] \right\|_{L^q(E)} \\
&\quad \leq C_2 k (\deg P)^{m_2} \|P\|_{L^q(E)} \\
&\qquad + 2C_2 (\deg P)^{m_2} \sum_{i=1}^{l} \left\| \left(x + \frac{b_i}{2} \right)^2 \left[\prod_{j=1}^{k} (x - x_j) \right] \right. \\
&\qquad \times \left. \left[\prod_{n=1, n\neq i}^{l} (x^2 + b_n x + c_n) \right] \right\|_{L^q(E)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|P'\|_{L^q(E)} &\leq C_2 k (\deg P)^{m_2} \|P\|_{L^q(E)} + 2lC_2 (\deg P)^{m_2} \|P\|_{L^q(E)} \\
&= C_2 (\deg P)^{m_2 + 1} \|P\|_{L^q(E)}.
\end{aligned}$$

(2) Inequalities (i), (iii) and (iv) are equivalent.

(i) $\Rightarrow$ (iv). Fix an arbitrary unitary polynomial R. We have $R(x) = (x-x_1)(x-x_2)\dots(x-x_k)(x^2+b_1x+c_1)(x^2+b_2x+c_2)\dots(x^2+b_lx+c_l)$, for some $x_1, x_2, \dots, x_k, b_1, b_2, \dots, b_l, c_1, c_2, \dots, c_l, \in \mathbf{R}$ with $b_j^2 < 4c_j$ for every $j \in \{1, 2, \dots, l\}$. Then

$$\begin{aligned}
&\|RP'\|_{L^q(E)} \\
&\quad \le \|(RP)'\|_{L^q(E)} \\
&\qquad + \sum_{i=1}^{k} \left\| P(x) \left[\prod_{j=1, j\ne i}^{k} (x-x_j) \right] \left[\prod_{n=1}^{l} (x^2+b_nx+c_n) \right] \right\|_{L^q(E)} \\
&\qquad + \sum_{i=1}^{l} \left\| P(x)(2x+b_i) \left[\prod_{j=1}^{k} (x-x_j) \right] \left[\prod_{n=1, n\ne i}^{l} (x^2+b_nx+c_n) \right] \right\|_{L^q(E)}.
\end{aligned}$$

By (i) which is equivalent to (ii), we have

$$\begin{aligned}
&\|RP'\|_{L^q(E)} \\
&\quad \le C_1(\deg P + \deg R)^{m_1} \|RP\|_{L^q(E)} \\
&\qquad + C_2k(\deg P + \deg R)^{m_2} \|RP\|_{L^q(E)} \\
&\qquad + 2C_2l(\deg P + \deg R)^{m_2} \|RP\|_{L^q(E)} \\
&\quad \le 2\max\{C_1, C_2\}(\deg P + \deg R)^{\max\{m_1, m_2+1\}} \|RP\|_{L^q(E)}.
\end{aligned}$$

The obvious implications (iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) complete the proof.

Remark For the complex case it is easily seen that $m_1 = m_2 = m_3 = m_4$.

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