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A Maximal Inequality of Non-negative Submartingale

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In this paper, we prove the maximal inequality $\lambda P(\sup_{n\geq 0}(f_n + |g_n|) \geq \lambda) \leq (Q(1)+2)\|f\|_1$, $\lambda > 0$, between a non-negative submartingale f, g is strongly subordinate to f and $1 - 2f_{n-1} - Q(1) \leq 0$,where Q is real valued function such that $0 < Q(s) \leq s$ for each $s > 0$, $Q(0) = 0$. This inequality improves Burkholder's inequality in which $Q(1) = 1$.

Keywords: Maximal inequality; Submartingale; Strongly subordinate; Inner product

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1. A MAXIMAL INEQUALITY

Suppose that $f = (f_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$ are adapted to a filtration $(\mathcal{F}_n)_{n\geq 0}$ of a probability space $(\Omega, \mathcal{F}, P)$. Here f is a non-negative submartingale and g is $\mathbb{R}^\nu$-valued, where ν is positive integer. With $f_n = d_0 + \cdots + d_n$ and $g_n = e_0 + \cdots + e_n$ $(n \geq 0)$. Consider the two conditions;

$$|e_n| \leq |d_n| \quad \text{for } n \geq 0, \tag{1.1}$$

$$|\mathbb{E}(e_n \mid \mathcal{F}_{n-1})| \leq |\mathbb{E}(d_n \mid \mathcal{F}_{n-1})| \quad \text{for } n \geq 1. \tag{1.2}$$

The process g *is called differentially subordinate to* f if (1.1) holds. If (1.2) is satisfied, then g *is conditionally differentially subordinate to* f.

If both of the conditions (1.1) and (1.2) are satisfied, then g *is strongly differentially subordinate to* f, or more simply, g *is strongly subordinate to* f. Of course, if f and g are martingales, then both sides of (1.2) vanish and (1.2) is trivially satisfied. It will be convenient to allow g to have its values in a space of possibly more than one dimension. So we assume throughout this paper that the Euclidean norm of $y \in \mathbb{R}^\nu$ is denoted by $|y|$ and the inner product of y and $k \in \mathbb{R}^\nu$ by $y \cdot k$. We set $\|f\|_1 = \sup_{n \geq 0}\|f_n\|_1$, the maximal function of g is defined by $g^* = \sup_{n\geq 0}|g_n|$ and the function Q is real valued function such that $0 < Q(s) \leq s$ for each $s > 0$, $Q(0) = 0$.

THEOREM 1.1 *If* f *is a non-negative submartingale,* g *is strongly subordinate to* f *and* $1 - 2f_{n-1} - Q(1) \leq 0$, *then, for all* $\lambda > 0$,

$$\lambda P\left(\sup_{n\geq 0}(f_n + |g_n|) \geq \lambda\right) \leq (Q(1) + 2)\|f\|_1.$$

COROLLARY 1.2 *If* f *is a non-negative submartingale,* g *is strongly subordinate to* f *and* $1 - 2f_{n-1} - Q(1) \leq 0$, *then, for all* $\lambda > 0$,

$$\lambda P\left(\sup_{n\geq 0} g^* \geq \lambda\right) \leq (Q(1) + 2)\|f\|_1.$$

COROLLARY 1.3 (D.L. Burkholder [2]) *If* f *is a non-negative submartingale and* g *is strongly subordinate to* f, *then, for all* $\lambda > 0$,

$$\lambda P\left(\sup_{n\geq 0}(f_n + |g_n|) \geq \lambda\right) \leq 3\|f\|_1.$$

Remark 1.1 In [2] Burkholder proved the inequality in Theorem when $Q(1) = 1$.

2. TECHNICAL LEMMAS

Put $S = \{(x, y)\colon x > 0 \text{ and } y \in \mathbb{R}^\nu \text{ with } |y| > 0\}$. Define two functions U and V on S by

$$U(x,y) = \begin{cases} [|y| - (Q(1)+1)x](x+|y|)^{1/(Q(1)+1)} & \text{if } x + |y| < 1, \\ 1 - (Q(1)+2)x & \text{if } x + |y| \geq 1 \end{cases}$$

and

$$V(x,y) = \begin{cases} -(Q(1)+2)x & \text{if } x+|y|<1, \\ 1-(Q(1)+2)x & \text{if } x+|y|\geq 1. \end{cases}$$

Observe that U is continuous on S.

LEMMA 2.1 (a) *$V \leq U$ on S.* (b) *$U(x,y) \leq 0$ if $x \geq |y|$.*

Proof For (a) we may assume $x+|y|<1$. Write $x+|y| = R^{Q(1)+1}$. Since $0<R<1$, we have

$$V(x,y) - U(x,y) = -R^{Q(1)+2} - (1-R)(Q(1)+2)x < 0.$$

In order to prove (b) assume $x \geq |y|$. If $x+|y|<1$, then $|y| - (Q(1)+1)x \leq |y| - x \leq 0$. Hence $U(x,y) \leq 0$. If $x+|y| \geq 1$, then $U(x,y) = 1-(Q(1)+2)x \leq x+|y|-(Q(1)+2)x = |y|-(Q(1)+1)x \leq 0$.

LEMMA 2.2 *If $x+|y|<1$ and $1-2x-Q(1) \leq 0$, then $U_x(x,y) + |U_y(x,y)| < 0$.*

Proof If $x+|y|<1$, then differentiation gives

$$U_x(x,y) = -\frac{(Q(1)+1)(Q(1)+2)x + Q(1)(Q(1)+2)|y|}{(Q(1)+1)(x+|y|)^{Q(1)/(Q(1)+1)}},$$
$$U_y(x,y) = \frac{(Q(1)+2)y}{(Q(1)+1)(x+|y|)^{Q(1)/(Q(1)+1)}}.$$

On the other hand, since

$$\begin{aligned} &-(Q(1)+1)(Q(1)+2)x - Q(1)(Q(1)+2)|y| + (Q(1)+2)|y| \\ &\quad = [Q(1)+2][-(Q(1)+1)x + (1-Q(1))|y|] \\ &\quad < [Q(1)+2][1-2x-Q(1)], \end{aligned}$$

hence $U_x(x,y) + |U_y(x,y)| < 0$ by the assumption and above inequality.

LEMMA 2.3 *If $(x,y) \in S$, $h \in \mathbb{R}$, $x+h>0$, $k \in \mathbb{R}^\nu$, $|h| \geq |k|$ and $|y+kt|>0$ for all $t \in \mathbb{R}$, then*

$$U(x+h, y+k) \leq U(x,y) + U_x(x,y)h + U_y(x,y)\cdot k. \tag{2.1}$$

Proof Put $I=\{t\in\mathbb{R}\colon x+th>0,\ |y+tk|>0\}$ and observe that $0\in I$ and I is an open set. Define a function G on I by

$$G(t) = U(x+ht, y+kt).$$

From the chain rule we have

$$G'(t) = U_x(x+th, y+tk)h + U_y(x+th, y+tk)\cdot k.$$

Thus it suffices to show $G(1)\le G(0)+G'(0)$. For this we define more functions r, N and m on I by $r(t)=m(t)+N(t)$, $m(t)=x+ht$ and $N(t)=|y+kt|$ we will write m for $m(t)$, etc. Therefore, put $I_1=\{t\in\mathbb{R}\colon m(t)>0,\ N(t)>0$ and $r(t)<1\}$ and $I_2=\{t\in\mathbb{R}\colon m(t)>0,\ N(t)>0$ and $r(t)>1\}$. On I_1, we have

$$G(t) = r^{(Q(1)+2)/(Q(1)+1)} - (Q(1)+2)mr^{1/(Q(1)+1)}.$$

Differentiating G, we get

$$\alpha G''(t) = r''r^2 + \frac{1}{Q(1)+1}(r')^2 r - 2hrr' - mrr'' + \frac{Q(1)}{Q(1)+1}m(r')^2,$$

where

$$\alpha = \frac{Q(1)+1}{Q(1)+2}r^{(2Q(1)+1)/(Q(1)+1)}.$$

Rearranging terms and inserting $(r')^2\, r - r(r')^2$, we have

$$\begin{aligned}\alpha G''(t) &= \left(r''r - mr'' - 2hr' + (r')^2\right)r \\ &\quad + \left(-r + \frac{1}{Q(1)+1}r + \frac{1}{Q(1)+1}m\right)(r')^2 \\ &= (|k|^2 - h^2)r - \frac{Q(1)}{Q(1)+1}N(r')^2 \le (|k|^2 - h^2)r.\end{aligned}$$

Here we used the observation that $m'=h$, $N'=r'-h$, $NN'=k\cdot(y+tk)$ and $Nr''=NN''=|k|^2-(N')^2$. On I_2 we have

$$G(t) = 1 - (Q(1)+2)(x+ht).$$

Differentiating G, we get $G''(t) = 0$. Therefore, on each component of $I_1 \cup I_2$, the derivative G'' is non-positive and G' is non-increasing. So, by Mean Value Theorem we have

$$G(1) - G(0) = G'(\tau) \quad (0 < \tau < 1).$$

Hence $G(1) - G(0) \leq G'(0)$. The case $x + |y| = 1$ follows by replacing x by $x + 2^j$ in the inequality (2.1) and then taking the limit of both sides as $j \to \infty$. This proves the lemma.

3. PROOF OF THE INEQUALITY IN THEOREM 1.1

We can assume that $\|f\|_1$ is finite. A stopping time argument leads to a further reduction: it is enough to prove that if $n \geq 0$, then

$$P(f_n + |g_n| \geq 1) \leq (Q(1) + 2)\mathbb{E}f_n. \tag{3.1}$$

We may further assume that

$$f_{n-1} > 0 \quad \text{and} \quad |g_{n-1} + te_n| > 0 \qquad \text{for all } t \in \mathbb{R} \text{ and } n \geq 1. \tag{3.2}$$

Indeed, for each $0 < \epsilon < 1$, the processes f^ϵ and g^ϵ, where $f_n^\epsilon = f_n + \epsilon$ and $g_n^\epsilon = (g_n, \epsilon)$, satisfy (3.2) and all the assumptions in Section 1. Here g^ϵ is a process in $\mathbb{R}^{\nu+1}$. Now, the inequality

$$P(f_n^\epsilon + |g_n^\epsilon| \geq 1) \leq (Q(1) + 2)\mathbb{E}f_n^\epsilon$$

yields, as $\epsilon \to 0$, the inequality (3.1) because $P(f_n + |g_n| \geq 1) \leq P(f_n + \epsilon + |(g_n, \epsilon)| \geq 1)$ and $\mathbb{E}f_n \leq \mathbb{E}f_n + \epsilon$.

Let the functions U and V be as in the previous section. Observe, from the assumption (3.2), that $(f_{n-1}, g_{n-1}) \in S$. The inequality (3.1) is equivalent to

$$\mathbb{E}V(f_n, g_n) \leq 0.$$

According to (a) of Lemma 2.1 it suffices to prove

$$\mathbb{E}U(f_n, g_n) \leq 0.$$

Also, (b) of Lemma 2.1 and the Assumption (1.1) imply $U(f_0, g_0) \leq 0$. Hence the proof is complete if we can show

$$\mathbb{E}U(f_n, g_n) \leq \mathbb{E}U(f_{n-1}, g_{n-1}),$$

which holds for all $n \geq 1$. By Lemma 2.3 we have, with $x = f_{n-1}$, $h = d_n$, $y = g_{n-1}$ and $k = e_n$,

$$U(f_n, g_n) - U(f_{n-1}, g_{n-1}) \leq U_x(f_{n-1}, g_{n-1})d_n + U_y(f_{n-1}, g_{n-1}) \cdot e_n \tag{3.3}$$

where all the random variables are integrable. The integrability follows from the boundedness of f_{n-1} and g_{n-1} although, of course, g need not be uniformly bounded. Also observe that $U(f_{n-1}, g_{n-1})$, $U_x(f_{n-1}, g_{n-1})$ and $U_y(f_{n-1}, g_{n-1})$ are $\mathcal{F}_{n-1}$ measurable. Thus conditioning on $\mathcal{F}_{n-1}$ we get

$$\mathbb{E}\big(U(f_n, g_n) - U(f_{n-1}, g_{n-1}) \,\big|\, \mathcal{F}_{n-1}\big) = \mathbb{E}\big(U(f_n, g_n) \,\big|\, \mathcal{F}_{n-1}\big) - U(f_{n-1}, g_{n-1}),$$
$$\mathbb{E}\big(U_x(f_{n-1}, g_{n-1})d_n \,\big|\, \mathcal{F}_{n-1}\big) = U_x(f_{n-1}, g_{n-1})\mathbb{E}\big(d_n \,\big|\, \mathcal{F}_{n-1}\big)$$

and

$$\mathbb{E}\big(U_y(f_{n-1}, g_{n-1}) \cdot e_n \,\big|\, \mathcal{F}_{n-1}\big) = U_y(f_{n-1}, g_{n-1}) \cdot \mathbb{E}\big(e_n \,\big|\, \mathcal{F}_{n-1}\big).$$

From (3.3) we get

$$\begin{aligned}&\mathbb{E}\big(U(f_n, g_n) \,\big|\, \mathcal{F}_{n-1}\big) - U(f_{n-1}, g_{n-1}) \\ &\quad \leq U_x(f_{n-1}, g_{n-1})\mathbb{E}\big(d_n \,\big|\, \mathcal{F}_{n-1}\big) + U_y(f_{n-1}, g_{n-1}) \cdot \mathbb{E}\big(e_n \,\big|\, \mathcal{F}_{n-1}\big).\end{aligned}$$

Since f is a submartingale,

$$\mathbb{E}(d_n \,|\, \mathcal{F}_{n-1}) \geq 0.$$

Using the Cauchy–Schwarz inequality and the assumption (1.2) we have

$$\begin{aligned}U_y(f_{n-1}, g_{n-1}) \cdot \mathbb{E}(e_n \,|\, \mathcal{F}_{n-1}) &\leq |U_y(f_{n-1}, g_{n-1})|\,|\mathbb{E}(e_n \,|\, \mathcal{F}_{n-1})| \\ &\leq |U_y(f_{n-1}, g_{n-1})|\,\mathbb{E}(d_n \,|\, \mathcal{F}_{n-1}).\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\big(U(f_n, g_n)\,\big|\,\mathcal{F}_{n-1}\big) - U(f_{n-1}, g_{n-1}) \\ \le \big(U_x(f_{n-1}, g_{n-1}) + \big|U_y(f_{n-1}, g_{n-1})\big|\big)\big(\mathbb{E}\big(d_n\,\big|\,\mathcal{F}_{n-1}\big)\big) \le 0\end{aligned}$$

or

$$\mathbb{E}\big(U(f_n, g_n)\,\big|\,\mathcal{F}_{n-1}\big) \le U(f_{n-1}, g_{n-1}). \tag{3.4}$$

In the above we used Lemma 2.2. But from the definition of conditional expectation we have

$$\mathbb{E}\big(\mathbb{E}\big(U(f_n, g_n)\,\big|\,\mathcal{F}_{n-1}\big)\big) = \mathbb{E}U(f_n, g_n).$$

Thus taking expectation in (3.4), we get

$$\mathbb{E}U(f_n, g_n) \le \mathbb{E}U(f_{n-1}, g_{n-1}).$$

This completes the proof of the inequality in Theorem 1.1.

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