A Logarithmic Extension of the Hölder Inequality

J. GINIBRE[a] and G. VELO[b]

[a] Laboratoire de Physique Théorique et Hautes Energies, *Université de Paris XI, bâtiment 211, 91405 Orsay Cedex, France; [b] Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna, Italy

We prove a logarithmic extension of the Hölder inequality, motivated by an application to the complex Ginzburg–Landau equation.

Keywords: Hölder inequality; Logarithmic

AMS classification: Primary 26D15, Secondary 26A51, 39B72

In their classic book [1] Hardy *et al.* present the Hölder inequality in a general setting which can be rephrased as follows. Let f, g and H be three functions satisfying the following assumption:

(A) f, g and H are suitably regular (for definiteness, continuous) functions from $[x_0, \infty)$, from $[y_0, \infty)$ and from $[x_0, \infty) \times [y_0, \infty)$ respectively to $\mathbb{R}^+$, such that
  (a) $f(x)g(y) \leq H(x, y)$ for all $x \geq x_0$, $y \geq y_0$,
  (b) H is jointly concave in (x, y).

The following result is then an immediate consequence of concavity:

PROPOSITION 1 *Let* f, g, H *satisfy the assumption* (A). *Then for any probability space* (S, m) *and any measurable functions* φ *and* ψ *from* S *to*

---

* Laboratoire associé au Centre National de la Recherche Scientifique – URA D0063.

$[x_0, \infty)$ *and to* $[y_0, \infty)$ *respectively, the following inequality holds*:

$$\int \mathrm{d}m(s) f(\varphi(s)) g(\psi(s)) \le H\left(\int \mathrm{d}m(s) \varphi(s), \int \mathrm{d}m(s) \psi(s)\right). \quad (1)$$

The Hölder inequality is obtained by taking $x_0 = y_0 = 0$, $f(x) = x^a$, $g(y) = y^b$ with $0 \le a, b \le 1$, and $H = f \otimes g$, namely $H(x, y) = f(x)g(y)$. The concavity condition on H reduces to $a + b \le 1$, and the Hölder inequality follows as the limiting case $a + b = 1$. In that case, both members of (1) are homogeneous of degree 1 in m, and the condition that m has mass 1 or has finite mass can be eliminated by a limiting procedure.

In order to generalize the Hölder inequality, it would be interesting to analyse the assumption (A) in a systematic way. Meanwhile, one may try to look for useful examples, namely to look for functions f, g and H satisfying (A) that are reasonably simple or at least easily computable. When looking for such examples, inspiration may be gained from the problem of convergence of the integral $\int^\infty \mathrm{d}x f(x)^{-1}$ for large x. If $f(x) = f_0(x) = x^\alpha$, the convergence condition is $\alpha > 1$. In order to get closer to the limiting case, one may consider successively $f_1(x) = x(\log x)^\alpha$, $f_2(x) = x(\log x)(\log \log x)^\alpha$, etc., still with the convergence condition $\alpha > 1$. Coming back to the assumption (A), the analogue of f_1 consists of functions behaving at infinity like

$$f(x) = x^a(\log x)^\alpha, \qquad g(y) = y^b(\log y)^{-\beta} \quad (2)$$

in the limiting power case $a + b = 1$. We assume furthermore $\alpha > 0$ in order to get interesting examples. If the function $f(x)g(y)$ is to admit a concave majorant, then the same must hold for f, g and for the restriction to rays $h_\lambda(x) = f(x)g(\lambda x)$ $(\lambda \in \mathbb{R}^+)$, and we must therefore assume $a < 1$ and $\beta \ge \alpha$ (actually we shall need $\beta > \alpha$). The purpose of this note is to exhibit a simple explicit function H such that the assumption (A) holds with the previous choice of f and g. The main result is as follows:

PROPOSITION 2   *Let* $0 \le a < 1$, $0 < \alpha < \beta \equiv \alpha + \gamma$, *and define* δ_0, δ_1 *and* δ *by*

$$\delta_0 = [2a(1 - a)]^{-1}\left[\beta(2a - 1) + \beta(1 + 4a(1 - a)\gamma^{-1})^{1/2}\right], \quad (3)$$

$$\delta_1 = 2\beta\alpha^{-1} + \beta(1-a)^{-1} - [2a(1-a)\alpha]^{-1}$$
$$\times \left\{ \alpha\beta + \left[16a^2(1-a)^2\beta\gamma - 4a(1-a)\alpha\beta(\beta+\gamma) + \alpha^2\beta^2 \right]^{1/2} \right\}, \quad (4)$$

which is well defined for $4a(1-a)\gamma \geq \alpha\beta,$

$$
\begin{cases}
\delta = \delta_0 & \text{for } 4a(1-a)\gamma \leq \alpha(\beta+\gamma), \quad\quad (5) \\[2mm]
\delta = \delta_1 & \text{for } 4a(1-a)\gamma \geq \alpha(\beta+\gamma). \quad\quad (6)
\end{cases}
$$

Let $x_0 = 1$ and $y_0 = e^\delta$ and define f, g and H by

$$f(x) = x^a(\log x)^\alpha, \qquad g(y) = y^{1-a}(\log y)^{-\beta}, \quad (7)$$

$$H(x,y) = f(x)g(y) \ \text{for } x \geq 1, \quad y_0 \leq y \leq x^{\beta/\alpha}, \quad (8)$$

$$H(x,y) = \alpha^\alpha \beta^{-\beta}(\beta-\alpha)^{\beta-\alpha} x^a y^{1-a}(\log y/x)^{\alpha-\beta}$$
$$\text{for } x \geq 1, \ y \geq \text{Max}(y_0, x^{\beta/\alpha}). \quad (9)$$

Then the assumption (A) is satisfied (and therefore Proposition 1 applies).

Remark 1   Define

$$\delta_2 = \beta\left[(1-a)^{-1} + 2(\beta+\gamma)^{-1}\right]. \quad (10)$$

One can check directly on (3)–(6) but it follows more simply from the proof of Proposition 2 given below that

$$
\begin{aligned}
\delta_0 < \delta_1 < \delta_2 &\quad \text{for } 4a(1-a)\gamma > \alpha(\beta+\gamma), \\
\delta_0 = \delta_1 = \delta_2 &\quad \text{for } 4a(1-a)\gamma = \alpha(\beta+\gamma), \\
\delta_1 > \delta_0 > \delta_2 &\quad \text{for } \alpha\beta \leq 4a(1-a)\gamma < \alpha(\beta+\gamma), \\
\delta_0 > \delta_2 &\quad \text{for } 4a(1-a)\gamma < \alpha(\beta+\gamma),
\end{aligned}
$$

while δ_1 is not defined for $4a(1-a)\gamma < \alpha\beta$. On the other hand, it follows from (5) that $\delta = \delta_0$ for a sufficiently small, for a sufficiently close to 1, and for all a if $\gamma \leq \alpha(\beta+\gamma)$ or equivalently $\beta^2 \geq \gamma(\gamma+1)$.

Remark 2   Clearly Proposition 2 remains true if one replaces δ by a larger quantity in the definition of y_0. It follows from the ordering given in Remark 1 that

$$\delta \leq \text{Max}(\delta_0, \delta_2)$$

and one can therefore replace δ by $\text{Max}(\delta_0, \delta_2)$ in the definition of y_0, thereby avoiding the cumbersome δ_1. In the same spirit, let

$$\delta_3 = \beta\left[(1-a)^{-1} + \gamma^{-1}\right]. \tag{11}$$

One can check that $\delta_3 \geq \delta_0$ (more precisely $\delta_3 = \delta_0$ for $a = 0$ and $\delta_3 > \delta_0$ for $a > 0$) and therefore obviously $\delta_3 \geq \text{Max}(\delta_0, \delta_2)$, so that one can *a fortiori* replace δ by the very simple δ_3 in the definition of y_0. The quantity δ_3 is optimal for $a = 0$, but it overestimates the singularity at $\gamma = 0$ for $a > 0$.

Remark 3   The function H can be defined and shown to be concave in a larger region than indicated in (8), (9). In (8), (9) we have restricted it to the largest product region where it can be used to implement the assumption (A).

Remark 4   With m a probability measure, the main issue in Proposition 1 lies in the behaviour of f, g and H for large values of their arguments. In particular one could replace f and g in (7) by various equivalent forms at infinity in order to ensure preferred behaviour of these functions for small values of (x, y). It turns out however that the explicit forms (7) or rescaled versions thereof as used in Proposition 2′ below are especially suitable to make subsequent computations simple and lead to simple formulas for H.

   For the applications, it may be convenient to rescale the variable y in order to get a fixed range for it and to reformulate Proposition 2 in the following equivalent form:

PROPOSITION 2′   *Let $0 \leq a < 1$, $0 < \alpha < \beta \equiv \alpha + \gamma$; define δ by (3)–(6); let $x_0 = y_0 = 1$; and define f, g and H by*

$$f(x) = x^a(\log x)^\alpha, \qquad g(y) = y^{1-a}(\delta + \log y)^{-\beta}, \tag{7′}$$

$$H(x, y) = f(x)g(y) \text{ for } x \geq 1, \quad 1 \leq y \leq e^{-\delta}x^{\beta/\alpha}, \tag{8′}$$

$$H(x,y) = \alpha^\alpha \, \beta^{-\beta} (\beta - \alpha)^{\beta - \alpha} \, x^a \, y^{1-a} (\delta + \log \, y/x)^{\alpha - \beta}$$
$$\text{for } x \geq 1, \;\; y \geq \text{Max}(1, e^{-\delta} \, x^{\beta/\alpha}). \tag{9'}$$

Then the assumption (A) *is satisfied (and therefore Proposition 1 applies).*

Proof of Proposition 2   Clearly one should try to define $H = f \otimes g$ insofar as this is a concave function of (x, y), and only large values of (x, y) matter since smaller values can be eliminated by suitably choosing x_0 and y_0. It will turn out (see below) that $f \otimes g$ is concave for (x, y) not too small and $y \leq C x^{\beta/\alpha}$ for some constant C. Therefore the product form $H = f \otimes g$ is suitable except in the region $y \geq C x^{\beta/\alpha}$ where it has to be modified. In that region, we modify $f \otimes g$ by replacing its graph by the concave envelope of that graph and of the origin of coordinates, namely by the cone with apex at the origin and tangent to that graph. For that purpose we consider for $\lambda > 1$ the restriction to rays

$$h_\lambda(x) = f(x) g(\lambda x) = \lambda^{1-a} \, x (\log \, x)^\alpha (\log \, \lambda x)^{-\beta} \tag{12}$$

which is well defined for $x \geq 1$.
  Now

$$h'_\lambda / h_\lambda = x^{-1} (1 + \alpha / \log \, x - \beta / \log \, \lambda x). \tag{13}$$

The tangent from the origin to the graph of h_λ touches the latter for $h'_\lambda / h_\lambda = x^{-1}$, namely $\alpha \log \lambda x = \beta \log x$, or equivalently at the intersection of the ray $y = \lambda x$ with the curve S (hereafter called separatrix) defined by $y = x^{\beta/\alpha}$ ($\geq x \geq 1$). For $y \geq \text{Max}(x, x^{\beta/\alpha})$ we replace $f \otimes g$ by the conic function

$$\tilde{H}(x, y) = \theta f(x/\theta) \, g(y/\theta) = x^a \, y^{1-a} (\log \, x/\theta)^\alpha (\log \, y/\theta)^{-\beta} \tag{14}$$

where $\theta \leq 1$ is defined by the condition that $(x/\theta, y/\theta) \in S$, namely

$$\alpha \log \, y/\theta = \beta \log \, x/\theta \tag{15}$$

so that (14) reduces to

$$\tilde{H}(x, y) = \alpha^\alpha \, \beta^{-\beta} (\beta - \alpha)^{\beta - \alpha} \, x^a \, y^{1-a} (\log \, y/x)^{\alpha - \beta} \tag{16}$$

by an elementary computation. That function is well defined for $0 \le x < y$. Furthermore it follows from (13) that $\tilde{H}(x,y) \ge f(x)g(y)$ for $1 \le x \le x^{\beta/\alpha} \le y$. In fact let $\tilde{h}_\lambda(x) = \tilde{H}(x, \lambda x)$. Then by (13)

$$\frac{d}{dx} \log \tilde{h}_\lambda = \frac{\tilde{h}_\lambda'}{\tilde{h}_\lambda} = \frac{1}{x} \le \frac{h_\lambda'}{h_\lambda} = \frac{d}{dx} \log h_\lambda$$

for $1 \le x \le x^{\beta/\alpha} \le \lambda x$ while $\tilde{h}_\lambda(x) = h_\lambda(x)$ for $x^{\beta/\alpha} = \lambda x$, thereby yielding the required inequality by integration.

We now define the function $H_0(x,y)$, in the domain $(x \ge 1, y > 1)$ where $f(x) g(y)$ is naturally defined, by

$$\begin{cases} H_0(x,y) = \tilde{H}(x,y) & \text{for } 1 \le x \le x^{\beta/\alpha} \le y, \\ \\ H_0(x,y) = f(x)g(y) & \text{for } 1 < y \le x^{\beta/\alpha}. \end{cases} \tag{17}$$

We have seen that $H_0(x,y) \ge f(x)g(y)$ for $x \ge 1$, $y > 1$, and it remains only to be shown that H_0 is concave in $[1,\infty) \times [y_0,\infty)$ so that its restriction H to that domain fulfils the condition (b) of the assumption (A).

We recall a few elementary properties of concave functions. A real valued function F defined in a convex subset X of $\mathbb{R}^n$ is concave by definition if the hypograph $\mathcal{G}_-(F) = \{(x,y) \in X \times \mathbb{R}: y \le F(x)\}$ is a convex subset of $X \times \mathbb{R}$. If X is open with closure $\bar{X}$, if $F \in \mathcal{C}(X)$ is concave in X and if F extends by continuity to a function $\bar{F} \in \mathcal{C}(\bar{X})$, then $\bar{F}$ is concave in $\bar{X}$. If X is open and $F \in \mathcal{C}^1(X)$, a necessary and sufficient condition for F to be concave is that the graph $\mathcal{G}(F) = \{(x,y) \in X \times \mathbb{R}: y = F(x)\}$ lies below its tangent plane locally at every point x of (a dense subset of) X. If X is open and $F \in \mathcal{C}^2(X)$ it is sufficent for F to be concave that the Hessian matrix of F be non-positive (as a matrix) at every point of (a dense subset of) X.

The function H_0 defined above lies in $\mathcal{C}(\bar{Q}) \cap \mathcal{C}^1(Q) \cap \mathcal{C}^2(Q \backslash S)$, where $Q = (1,\infty) \times (y_0,\infty)$. From the previous properties, it follows that for H_0 to be concave in $\bar{Q}$, it suffices that the Hessian matrix of H_0 be non-positive in $Q \backslash S$.

We first consider the conic form $\tilde{H}$. Omitting the prefactor, we take

$$\tilde{H}(x,y) = x^a y^{1-a} (\log y/x)^{-\gamma}$$

so that

$$\tilde{H}'_x = (y/x)^{1-a}\left\{a(\log y/x)^{-\gamma} + \gamma(\log y/x)^{-(\gamma+1)}\right\} \equiv k(y/x),$$

$$\tilde{H}'_y = (x/y)^{a}\left\{(1-a)(\log y/x)^{-\gamma} - \gamma(\log y/x)^{-(\gamma+1)}\right\} \equiv \ell(y/x),$$

$$\tilde{H}''_{x^2} = -(y/x^2)\,k'(y/x), \qquad \tilde{H}''_{y^2} = (1/x)\,\ell'(y/x),$$

$$\tilde{H}''_{xy} = (1/x)\,k'(y/x) = -(y/x^2)\,\ell'(y/x).$$

An elementary computation shows that

$$k'(\lambda) = -\lambda \ell'(\lambda)$$

$$= \lambda^{-a}\left\{a(1-a)(\log \lambda)^{-\gamma} + \gamma(1-2a)(\log \lambda)^{-(\gamma+1)}\right.$$

$$\left. - \gamma(\gamma+1)(\log \lambda)^{-(\gamma+2)}\right\}.$$

One checks that the Hessian of $\tilde{H}$ vanishes, and the concavity condition reduces to $k'(\lambda) \geq 0$, namely

$$a(1-a)(\log \lambda)^2 + \gamma(1-2a)\log \lambda - \gamma(\gamma+1) \geq 0$$

or equivalently $\lambda \geq \lambda_0 = \exp(\gamma\delta_0/\beta)$ with δ_0 defined by (3). Therefore the function $\tilde{H}$ is concave for $0 \leq \lambda_0 x \leq y$, and by the appropriate restriction, the function H_0 is concave for $x \geq 1$, $y \geq \text{Max}(\lambda_0 x, x^{\beta/\alpha})$. Furthermore, the ray $y = \lambda_0 x$ intersects the separatrix S at the point $(\tilde{x}_0, \tilde{y}_0)$ with $\tilde{x}_0 = \exp(\alpha\delta_0/\beta)$, $\tilde{y}_0 = \exp(\delta_0)$ so that by further restriction, the function H is concave for $x \geq 1$, $y \geq \text{Max}(\tilde{y}_0, x^{\beta/\alpha})$.

We next consider the product form $f \otimes g$. The concavity of that function in some domain is equivalent to the non-positivity of the Hessian matrix at each point of that domain. With f and g non-negative functions, the latter at (x, y) reduces to the conditions $f''(x) \leq 0$, $g''(y) \leq 0$, and

$$f(x)f''(x)\,g(y)\,g''(y) - (f'(x)\,g'(y))^2 \geq 0. \tag{18}$$

Using the redundant parameter $b = 1 - a$ for convenience and the variables $u = \log x$, $v = \log y$, we compute

$$f'(x) = x^{a-1} u^{\alpha-1}(au + \alpha),$$

$$f''(x) = x^{a-2} u^{\alpha-2}\left(-abu^2 + \alpha(a - b)u + \alpha(\alpha - 1)\right),$$

$$g'(y) = y^{b-1} v^{-(\beta+1)}(bv - \beta),$$

$$g''(y) = y^{b-2} v^{-(\beta+2)}\left(-abv^2 + \beta(a - b)v + \beta(\beta + 1)\right),$$

so that the concavity conditions reduce to

$$abu^2 - \alpha(a - b)u - \alpha(\alpha - 1) \geq 0, \tag{19}$$

$$abv^2 - \beta(a - b)v - \beta(\beta + 1) \geq 0, \tag{20}$$

$$F(u, v) \equiv \left[abu^2 - \alpha(a - b)u - \alpha(\alpha - 1)\right]$$
$$\times \left[abv^2 - \beta(a - b)v - \beta(\beta + 1)\right] - (au + \alpha)^2(bv - \beta)^2 \geq 0. \tag{21}$$

The conditions (19), (20) are satisfied for u and v large and therefore by continuity remain satisfied as long as one does not cross the curve $F(u, v) = 0$. It is therefore sufficient to consider the condition (21) starting from large u and v. By an elementary computation one obtains

$$F(u, v) = ab\, uv(\beta u - \alpha v) - a\beta(\beta + b)u^2 - b\alpha(\alpha - a)v^2$$
$$+ \alpha\beta uv - \alpha\beta(\beta + b - a)u + \alpha\beta(\alpha + b - a)v$$
$$- \alpha\beta(1 + \beta - \alpha). \tag{22}$$

It is convenient to introduce the variable $w = \beta u - \alpha v$ and the new function $G(w, v) = \beta F(u, v)$. The separatrix is now the line $w = 0$ and an elementary computation yields

$$G(w, v) = ab\, vw^2 + ab\, \alpha v^2 w - a(b + \beta)w^2 + ab\, \alpha\gamma v^2$$
$$- ((a - b)\alpha\beta + 2ab\alpha)vw - \alpha\beta(b - a + \beta)w$$
$$- (a - b)\alpha\beta\,\gamma v - \alpha\beta^2(1 + \gamma) \tag{23}$$

(with $\gamma = \beta - \alpha$). The equation $G(w, v) = 0$ defines a cubic curve Γ in the (w, v) plane, with asymptotes $w = -\gamma$, $v = 1 + \beta/b$ and $\alpha v + w(\equiv \beta u) = \beta(1 - \alpha/a)$, except in the special case $a = 0$ where that curve reduces to the hyperbola with equation

$$(w + \gamma)(v - 1 - \beta) - \alpha = 0. \tag{24}$$

In that case $G(w, v)$ is non-negative in $[0, \infty) \times [v_0, \infty)$ where $v_0 = 1 + \beta + \alpha/\gamma = \beta(1 + 1/\gamma)$, which is the common value of δ_0 and $\tilde{\delta}$ given by (3), (11) for $a = 0$. We next consider the general case and we look for the largest region $(w, v) \in [0, \infty) \times [v_0, \infty)$ where $G(w, v) \geq 0$. For that purpose it is convenient to change variables and use the first two asymptotes of Γ as coordinate axes. Accordingly we define $w' = w + \gamma$, $v' = v - 1 - \beta/b$, so that the separatrix is now the line $w' = \gamma$, and we proceed in two steps by defining

$$G(w,v) = G_1(w, v') = G_2(w', v'). \tag{25}$$

We first compute

$$G_1(w, v') = ab\, v'w^2 + (ab\, \alpha v'^2 + \alpha\beta v' - ab\, \alpha)w$$
$$+ ab\, \alpha\gamma v'^2 + \alpha\beta\gamma v' + 2ab\, \alpha\gamma v' + ab\alpha\gamma - \alpha^2\beta. \tag{26}$$

It follows from (26) that $G_1(w, v')$ is quadratic convex in w for fixed $v' > 0$ and strictly increasing in v' for $v' \geq 0$ and fixed $w \geq 0$. We next compute

$$G_2(w', v') = ab\{(v'w' - \alpha)(w' + \alpha v' - 2\gamma) + \beta^2 v'\} + \alpha\beta(v'w' - \alpha) \tag{27}$$

which is of course also quadratic convex in w' for fixed $v' > 0$ and strictly increasing in v' for $v' \geq 0$ and fixed $w' \geq \gamma$. We next rewrite G_2 as follows

$$G_2(w', v') = abv'\left\{(w' - \alpha/v')^2 + 2(\alpha z - \mu)(w' - \alpha/v') + \beta^2\right\} \tag{28}$$

where

$$\mu = \gamma - \alpha\beta/2ab, \tag{29}$$

$$z \equiv z(v') = (v'^2 + 1)/2v' \geq 1. \tag{30}$$

For fixed $v' > 0$, the function $G_2(w', v')$ reaches a minimum $M(v')$ for

$$w' = w_1'(v') \equiv \alpha/v' - \alpha z + \mu = \alpha(1 - v'^2)/2v' + \mu, \qquad (31)$$

and

$$M(v') = abv'\left(\beta^2 - (\alpha z - \mu)^2\right). \qquad (32)$$

Clearly $w_1'(v')$ decreases strictly from $+\infty$ to $-\infty$ when v' increases from 0 to ∞, and there exists a unique $\tilde{v}' > 0$ such that $w_1'(\tilde{v}') = \gamma$, which is the positive root of the equation

$$ab(v'^2 - 1) + \beta v' = 0. \qquad (33)$$

On the other hand $M(v')$ is strictly increasing in v' insofar as $w_1'(v') \geq \gamma$, namely for $0 < v' \leq \tilde{v}'$, since $G_2(w', v')$ is strictly increasing in v' for fixed $w' \geq \gamma$.

We look for the largest product region $(w, v) \in [0, \infty) \times [v_0, \infty)$ where $G(w, v) \geq 0$ or equivalently for the largest product region $(w', v') \in [\gamma, \infty) \times [v_0', \infty)$ where $G_2(w', v') \geq 0$. Clearly the condition $v_0' \geq \delta_0'$ is necessary, where δ_0' is the larger root of the equation

$$G_2(\gamma, v') \equiv \alpha\gamma\{abv'^2 + (\beta + 2ab)v' + ab - \alpha\beta/\gamma\}$$
$$= 0. \qquad (34)$$

That condition reduces to $v_0 \geq \delta_0$, where δ_0 is defined by (3). Two cases can then occur.

Case 1   One has $\delta_0' \geq \tilde{v}'$ or equivalently $G_2(\gamma, \tilde{v}') \leq 0$ or equivalently $w_1'(\delta_0') \leq \gamma$. One checks that this is the case provided $4ab\gamma \leq \alpha(\beta + \gamma)$. In that case, the condition $v_0' \geq \delta_0'$ is also sufficient.

Case 2   One has $\delta_0' < \tilde{v}'$ or equivalently $G_2(\gamma, \tilde{v}') > 0$. This is the case provided $4ab\gamma > \alpha(\beta + \gamma)$. In that case the condition $v_0' \geq \text{Max}(\delta_0', 0)$ is necessary but not sufficient, while the condition $v_0' \geq \tilde{v}'$ is obviously sufficient but not necessary. The necessary and sufficient condition is obtained as follows. Since $M(v')$ increases from $-\infty$ to $G_2(\gamma, \tilde{v}') \geq 0$ when v' increases from 0 to $\tilde{v}'$, there is a unique δ_1' with $0 < \delta_1' \leq \tilde{v}'$ such that $M(\delta_1') = 0$. Since $z \geq 1$ and $\mu \leq \gamma$ imply $\alpha z - \mu > \alpha - \gamma > -\beta$, the

equation $M(v') = 0$ reduces to

$$\alpha z - \mu = \beta \tag{35}$$

so that by (31), $v' \le \delta_2' \equiv \alpha/(\beta + \gamma) < 1$ and δ_1' is the value of v' determined by (35) and

$$v' = z - \sqrt{z^2 - 1},$$

namely

$$\delta_1' = \alpha^{-1}\left\{\beta + \mu - \left[(\beta + \mu)^2 - \alpha^2\right]^{1/2}\right\}. \tag{36}$$

Since $G_2(w', v')$ is increasing in v' for fixed $w' \ge \gamma$, the necessary and sufficient condition for the positivity of G_2 in the present case becomes $v_0' \ge \delta_1'$, which reduces to $v_0 \ge \delta_1$ with δ_1 defined by (4) by an elementary computation, thereby completing the proof of Proposition 2. In addition, when returning to the variable v, δ_2' becomes δ_2 defined by (10), and the previous discussion essentially yields the ordering given in Remark 1.   QED

Remark 5   The fact, mentioned in Remark 2, that $\delta_0 \le \delta_3$ with δ_3 defined by (11) follows easily from (27). In fact for $v' > 0$, $v'w' = \alpha + \varepsilon \ge \alpha$, we obtain from (27)

$$G_2(w', v') \ge ab\{\varepsilon(\alpha + \varepsilon)/v' + (\varepsilon\alpha + \beta^2)v' - 2\varepsilon\gamma\} \ge 0$$

since $\varepsilon(\alpha + \varepsilon)(\varepsilon\alpha + \beta^2) - \varepsilon^2\gamma^2 \ge 0$. As a consequence we obtain $G_2(w', v') \ge 0$ for $(w', v') \in [\gamma, \infty) \times [\alpha/\gamma, \infty)$ or equivalently $G(w, v) \ge 0$ for $[w, v) \in [0, \infty) \times [\delta_3, \infty)$, so that $\delta_3 \ge \delta_0$ since $G(\gamma, \delta_0) = 0$ and $G(\gamma, v)$ is increasing in v in the relevant region.

We conclude this note by mentioning that the present work was motivated by the problem of strict localization of L^2 and related estimates for the complex Ginzburg–Landau equation

$$\partial_t u = \lambda u + (1 + i\nu)\Delta u - (1 + i\mu)u\, g_0(|u|^2)$$

where u is a complex function defined in space time $\mathbb{R}^{n+1}$, in the case where the non-linearity $g_0(\rho)$ behaves as $(\log \rho)^\beta$ for large ρ. The crux of

　　　　　J. GINIBRE AND G. VELO

the argument is an application of Propositions 1 and 2 in the special case $a = 0$, $\beta > \alpha > 2$ [2].

References

[1] G. Hardy, J. Littlewood and G. Polya. *Inequalities*, Cambridge Univ. Press, London (1951).
[2] J. Ginibre and G. Velo. Localized estimates and Cauchy problem for the logarithmic complex Ginzburg Landau equation, *J. Math. Phys.* **38** (1997), 2475–2482.