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An Inequality for the Product of Two Integrals Relating to the Incomplete Gamma Function

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The inequality $\int_0^x e^{t^p}\,dt \cdot \int_x^\infty e^{-t^p}\,dt < \frac{1}{4}$ is proved for $p > 1.87705\ldots$ and $x \geq 0$, and new inequalities are established for the integrals $\int_0^x e^{t^p}\,dt$ and $\int_0^x e^{-t^p}\,dt$, $p > 1$.

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1. INTRODUCTION

Gautschi [3] proved the following inequalities

$$\frac{1}{2}\left[(x^p+2)^{1/p}-x\right] < e^{x^p}\int_x^\infty e^{-t^p}\,dt \leq a_p\left[\sqrt{x^2+\frac{1}{a_p}}-x\right],\quad p>1,\ x\geq 0, \tag{1.1}$$

where

$$a_p = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{p/(p-1)}.$$

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The integral in (1.1) can be expressed in terms of the incomplete Gamma function

$$\int_x^\infty e^{-t^p}\, dt = \frac{1}{p}\Gamma\left(\frac{1}{p}, x^p\right),$$

where

$$\Gamma(\alpha, z) = \int_z^\infty e^{-t} t^{\alpha-1}\, dt, \quad \alpha > 0,\ z > 0.$$

Alzer [2] refined these bounds showing new inequalities for $0 < p < 1$ and for $p > 1$. In the case $p > 1$ and $x > 0$ he found

$$\Gamma\left(1+\frac{1}{p}\right)(1-e^{-x^p})^{1/p} < \int_0^x e^{-t^p}\, dt < \Gamma\left(1+\frac{1}{p}\right)(1-e^{-\alpha x^p})^{1/p}, \tag{1.2}$$

where

$$\alpha = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{-p}.$$

Recently Feng Qi and Sen-lin Guo [5] established, among others, the inequality

$$\int_0^x e^{t^p}\, dt < \frac{e^{x^p}-1}{x^{p-1}}, \quad x > 0,\ p > 1. \tag{1.3}$$

In Section 3 we shall give lower and upper bounds for this integral and as a particular case we also recover this bound.

In this paper we are essentially interested in the product of the two integrals

$$F(x,p) = \int_0^x e^{t^p}\, dt \cdot \int_x^\infty e^{-t^p}\, dt \quad x \geq 0,\ p \geq 1$$

and our main purpose is to establish an upper bound for $F(x,p)$.

First of all, we observe that by the inequalities (1.1) and (1.3), the following relations

$$F(0,p) = 0, \quad \lim_{x\to+\infty} F(x,p) = 0 \qquad \text{for } p > 1 \tag{1.4}$$

hold. However, using twice the l'Hôspital rule, we can obtain the more informative result

$$\lim_{x\to\infty} F(x,p) = \lim_{x\to\infty} \frac{\int_x^\infty e^{-t^p}\,dt}{1/(\int_0^x e^{t^p}\,dt)} = \lim_{x\to\infty} \frac{1}{px^{p-1}} = \begin{cases} 0 & \text{if } p > 1, \\ 1 & \text{if } p = 1, \\ \infty & \text{if } 0 < p < 1. \end{cases}$$

By relations (1.4) it follows for any fixed $p > 1$ that the function $F(x,p)$ must have an absolute maximum. Let the function $\varphi(p)$ be defined by

$$\varphi(p) = \sup_{x>0} F(x,p), \quad p \geq 1.$$

In the special case $p = 1$, a direct calculation gives $F(x,1) = 1 - e^{-x}$ hence $\varphi(1) = 1$.

Concerning the function $F(x,p)$ we have the following two results which will be proved in the Section 2.

THEOREM 1 *Let $p > 1$. Then the function $F(x,p)$ has the property of unimodality, i.e. there exists a unique point $x_p > 0$ such that the function $F(x,p)$ increases for $0 < x < x_p$ and decreases for $x_p < x < \infty$.*

THEOREM 2 *Let $p^* = 1.87705\ldots$ Then the following inequality*

$$F(x,p) = \int_0^x e^{t^p}\,dt \cdot \int_x^\infty e^{-t^p}\,dt < \frac{1}{4}, \quad x \geq 0,\ p > p^* \tag{1.5}$$

holds. The value p^* is the unique solution of the equation $\varphi(p) = \frac{1}{4}$.

The case $p = 2$ is of special interest in view of the connection between the integral $\int_x^\infty e^{-t^2}\,dt$ and the complementary error function $\operatorname{erfc}(x)$:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt.$$

Moreover, by means of the Gamma function, $F(x,p)$ can be written as

$$F(x,p) = \int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t \cdot \left[\Gamma\left(1+\frac{1}{p}\right) - \int_0^x \mathrm{e}^{-t^p}\,\mathrm{d}t\right]. \tag{1.6}$$

Remark By (1.6), for a fixed $x \in (0,1)$, the function $F(x,p)$ tends to $x(1-x)$ when $p \to \infty$ and, particularly, $\lim_{p\to\infty} F(\frac{1}{2},p) = \frac{1}{4}$. Hence $\limsup_{p\to\infty} \varphi(p) \geq \frac{1}{4}$. Then by Theorem 2 it follows $\lim_{p\to\infty} \varphi(p) = \frac{1}{4}$. This justifies the choice of the constant $\frac{1}{4}$ in Theorem 2.

2. PROOF OF THE THEOREMS

Proof of Theorem 1 Fix the value $p > 1$. Then by the relations (1.4), $F(0,p) = F(\infty,p) = 0$ and by Rolle's theorem, the function $\mathrm{d}F(x,p)/\mathrm{d}x$ has at least one positive zero in $(0,\infty)$. Differentiation of $F(x,p)$ with respect to x gives

$$\begin{aligned} F'(x,p) &= \frac{\mathrm{d}}{\mathrm{d}x}F(x,p) = \mathrm{e}^{x^p}\int_x^\infty \mathrm{e}^{-t^p}\,\mathrm{d}t - \mathrm{e}^{-x^p}\int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t, \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{x^p}F'(x,p)\right] &= 2x^{p-1}\mathrm{e}^{2x^p}\left[p\int_x^\infty \mathrm{e}^{-t^p}\,\mathrm{d}t - x^{1-p}\mathrm{e}^{-x^p}\right] \\ &= 2x^{p-1}\mathrm{e}^{2x^p} f(x), \end{aligned}$$

where

$$\lim_{x\to\infty} f(x) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) = (p-1)x^{-p}\mathrm{e}^{-x^p} > 0.$$

Therefore $f(x) < 0$ and consequently the function $\mathrm{e}^{x^p}F'(x,p)$ strictly decreases. Thus $F'(x,p)$ can have at most one zero. This and the existence of a zero of $F'(x,p)$ show that $F'(x,p)$ has exactly one zero $x_p > 0$. Clearly, the function $F(x,p)$ increases on $(0,x_p)$ and decreases on (x_p,∞). This completes the proof of Theorem 1.

Proof of Theorem 2 First we show that the point x_p where the function $F(x,p)$ takes on its maximum value belongs to the interval $(0,1)$. To this end, by Theorem 1, it is sufficient to show that $F'(1,p) < 0$.

Making use of the substitution

$$s = t^p, \qquad s = t^{-p}$$

in the first and in the second integral of $F(x,p)$, respectively, we obtain

$$F'(1,p) = \frac{1}{p}\int_1^\infty \left[e^{1-s} s^{(1/p)-1} - e^{-1+(1/s)} s^{-1-(1/p)} \right] ds.$$

Introducing the notations

$$\sigma = \frac{1}{p},$$

$$\Phi(\sigma) = pF'(1,p) = \int_1^\infty \left[e^{1-s} s^{\sigma-1} - e^{-1+(1/s)} s^{-1-\sigma} \right] ds,$$

we see that the function $\Phi(\sigma)$ has the same sign as $F'(1,p)$. A differentiation gives

$$\Phi'(\sigma) = \frac{d}{d\sigma}\Phi(\sigma) = \int_1^\infty \left[e^{1-s} s^{\sigma-1} + e^{-1+(1/s)} s^{-1-\sigma} \right] \log s \, ds > 0.$$

Therefore $\Phi(\sigma)$ increases, hence $\Phi(\sigma) \leq \Phi(1/p^*)$ for $\sigma \leq 1/p^*$, or, equivalently for $p \geq p^*$. A direct calculation shows that $F'(1,p^*) = -0.134 < 0$. This implies that $F'(x,p) < 0$ for $x \geq 1$ and $p \geq p^*$, and that

$$\varphi(p) = \max_{0 \leq x \leq 1} F(x,p), \quad \text{for } p \geq p^*.$$

Now we need only to prove that $\varphi(p) < \frac{1}{4}$ for $p > p^*$. We use two different approaches to attack this problem: one for large values of p, say $p > p_0 = 2.099376\ldots > 2$, and another for moderate values of p, $p^* \leq p < p_0$. The value of p_0 will be specified later.

First we consider the case $p > p_0$. By the series expansion of the exponential function e^{t^p}, we have

$$\int_0^x e^{t^p} \, dt = \sum_{n=0}^\infty \frac{x^{np+1}}{(np+1)n!}. \tag{2.1}$$

Concerning the second integral $\int_x^\infty e^{-t^p}\,dt$, the series expansion of e^{-t^p} yields

$$\int_0^x e^{-t^p}\,dt = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{np}}{(np+1)n!}. \tag{2.2}$$

Moreover

$$\int_0^\infty e^{-t^p}\,dt = \frac{1}{p}\int_0^\infty e^{-s}s^{(1/p)-1}\,ds = \Gamma\left(1+\frac{1}{p}\right).$$

This and (2.2) give

$$\int_x^\infty e^{-t^p}\,dt = \Gamma\left(1+\frac{1}{p}\right) - x\sum_{n=0}^{\infty}(-1)^n\frac{x^{np}}{(np+1)n!}.$$

Introducing the functions

$$A(z) = \sum_{n=0}^{\infty}\frac{z^n}{(np+1)n!}, \qquad B(z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(np+1)n!},$$

we have to show that

$$x\left[\Gamma\left(1+\frac{1}{p}\right) - xB(z)\right]A(z) < \frac{1}{4}, \quad z = x^p. \tag{2.3}$$

The left-hand side can be considered as the quadratic polynomial $-u^2AB + A\Gamma u$ taken at $u = x$. Since this polynomial has its maximum $\frac{1}{4}(A\Gamma^2)/B$ at $u = \Gamma/(2B)$, inequality (2.3) will be satisfied if we show that

$$A(z)\Gamma^2\left(1+\frac{1}{p}\right) < B(z). \tag{2.4}$$

Now we introduce the notation

$$c_p = (p+1)\frac{1-\Gamma^2(1+1/p)}{1+\Gamma^2(1+1/p)}, \quad p \geq 2,$$

and we are going to show that (2.4) holds for $0<z<z_0$, where $z_0=0.358035\ldots$ is the solution of the equation

$$c_2 = 0.36059\ldots = z + \frac{z^3}{3\cdot 3!} + \frac{z^5}{5\cdot 5!} + \cdots = \int_0^z \frac{\sinh t}{t}\,\mathrm{d}t.$$

To prove inequality (2.4) we replace $A(z)$ and $B(z)$ by their series expansions given above. In this way (2.4) becomes equivalent to

$$c_p \geq z - c_p\frac{1}{(2p+1)2!}z^2 + \frac{p+1}{(3p+1)3!}z^3 - c_p\frac{1}{(4p+1)4!}z^4 + \frac{p+1}{(5p+1)5!}z^5 - \cdots$$

We claim that

$$z + \frac{z^3}{3\cdot 3!} + \frac{z^5}{5\cdot 5!} + \cdots > z - c_p\frac{z^2}{(2p+1)2!} + \frac{p+1}{(3p+1)3!}z^3 - c_p\frac{z^4}{(4p+1)4!} + \frac{p+1}{(5p+1)5!}z^5 - \cdots$$

for $0<z\leq 1$. For $n=1,2,\ldots$ we have to show

$$\frac{z^{2n+1}}{(2n+1)(2n+1)!} > -c_p\frac{z^{2n}}{(2np+1)(2n)!} + \frac{p+1}{[(2n+1)p+1](2n+1)!}z^{2n+1}$$

or

$$c_p > \frac{2n}{(2n+1)^2}\frac{2np+1}{[(2n+1)p+1]}z. \tag{2.5}$$

Since the right-hand side is less than $\frac{2}{9}$ for $n\geq 1$, $p>0$ and $0<z\leq 1$, hence it is sufficient to show that

$$c_p > c_2 = 0.36059\ldots, \quad \text{for } p \geq p_0 > 2.$$

This inequality has the form

$$(p+1)\frac{1-\Gamma^2(1+1/p)}{1+\Gamma^2(1+1/p)} \ge c_2, \quad p>2,$$

and using the notation $1/p = t$, this inequality is equivalent to

$$g(t) = 2\log\Gamma(t+1) - \log\frac{1+(1-c_2)t}{1+(1+c_2)t} \le 0, \quad 0 \le t \le \tfrac{1}{2}. \tag{2.6}$$

Clearly we have $g(0) = g(\frac{1}{2}) = 0$. Inequality (2.6) will be proved if we show that the function $g(t)$ is convex. To this end we are going to show that

$$g''(t) = 2\psi'(1+t) - 4c_2\frac{1+(1-c_2^2)t}{[1+(1-c_2)t]^2[1+(1+c_2)t]^2} > 0, \tag{2.7}$$

where $\psi(x)$ denotes the logarithmic derivative of $\Gamma(x)$. Using the inequality [4, p. 288]

$$\psi'(t) > \frac{1}{t}, \quad t>0,$$

we shall prove inequality (2.7) if we show that

$$\frac{2}{1+t} - 4c_2\frac{1+t}{[1+(1-c_2)t]^2[1+(1+c_2)t]^2} > 0$$

which is equivalent to

$$[1+(1-c_2)t][1+(1+c_2)t] - \sqrt{2c_2}(1+t) > 0$$

or

$$(1-c_2^2)t^2 + (2-2\sqrt{c_2})t + 1 - \sqrt{2c_2} > 0.$$

But this is true because the polynomial on the left-hand side has no positive zeros. Thus the function $g(t)$ defined above is convex and this proves the inequality (2.6). So inequality (2.4) is proved for $0 < z \leq z_0$ and $p \geq 2$.

Now let us define $x_0(p) = z_0^{1/p}$. We have proved that the function $F(x,p)$ is less than $\frac{1}{4}$ for $0 < x \leq x_0(p)$. If we prove that the function $F(x,p)$ has its derivative negative at $x = x_0(p)$, then by Theorem 1, we can conclude that $F(x,p) < \frac{1}{4}$, for every $x > 0$. Now we prove that this is true for $p \geq p_0$. We have to show that

$$\begin{aligned} e^{-x^p} F'(x,p) &= \int_x^\infty e^{-t^p}\, dt - e^{-2x^p} \int_0^x e^{t^p}\, dt \\ &= \Gamma\left(1 + \frac{1}{p}\right) - xB(x^p) - e^{-2x^p} xA(x^p) < 0 \end{aligned} \tag{2.8}$$

at $x = x_0(p)$ where the functions $A(z)$ and $B(z)$ have been introduced above. We prove (2.8) by using the following lower bound for $A(z)$ and $B(z)$:

$$A(z) = \sum_{n=0}^{\infty} \frac{z^n}{(np+1)n!} > 1 + \frac{u(z)}{p+1}, \tag{2.9a}$$

$$B(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(np+1)n!} > 1 - \frac{v(z)}{p+1} \tag{2.9b}$$

where

$$u(z) = \sum_{n=1}^{\infty} \frac{z^n}{n \cdot n!}, \qquad v(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n \cdot n!}. \tag{2.10}$$

Clearly (2.9a) is valid for $z > 0$ and $p > 1$.

The proof of (2.9b) is not so immediate. To show this inequality we start from

$$B(z) - 1 + \frac{v(z)}{p+1} = \frac{1}{p+1} \sum_{n=2}^{\infty} (-1)^n \frac{(n-1)z^n}{n(np+1)n!}. \tag{2.11}$$

We wish to prove that the function on the right-hand side is positive, at least for $0 < z < 1$. The series in (2.11) is of the Leibniz type and

$$\frac{(n-1)z^n}{n(np+1)n!} > \frac{nz^{n+1}}{[(n+1)p+1](n+1)(n+1)!}, \quad n = 2, 3, \ldots$$

So $B(z) - 1 + v(z)/(p+1) > 0$ for $0 < z < 1$ and the proof of the inequality (2.8) is reduced to the proof of the one where $A(z)$ and $B(z)$ are replaced by their lower bounds (2.9a) and (2.9b), respectively. Thus we need to show that

$$\Gamma\left(1 + \frac{1}{p}\right) < z_0^{1/p}\left[1 + \mathrm{e}^{-2z_0} - \frac{v(z_0) - \mathrm{e}^{-2z_0}u(z_0)}{p+1}\right]$$

which, with $1/p = t$ is equivalent to

$$\Gamma(1+t) < z_0^t\left[1 + \mathrm{e}^{-2z_0} - \frac{v(z_0) - \mathrm{e}^{-2z_0}u(z_0)}{1/t + 1}\right].$$

Taking the logarithms, this inequality can be written as

$$h(t) = \log\Gamma(2+t) - t\log z_0 - \log(\alpha + \beta t) < 0,$$

where

$$\alpha = 1 + \mathrm{e}^{-2z_0}, \qquad \beta = \alpha + \mathrm{e}^{-2z_0}u(z_0) - v(z_0).$$

The function $h(t)$ has derivative

$$h'(t) = \psi(2+t) - \log z_0 - \frac{\beta}{\alpha + \beta t}.$$

Moreover $h(0) < 0$, $h'(0) > 0$ and $h''(t) > 0$. This shows that $h(t)$ increases for $t > 0$. Numerically we find that $h(\frac{1}{2}) > 0$, therefore $h(t)$ has exactly one zero at $t_0 = 0.476331\ldots \in (0, \frac{1}{2})$. This zero give $p_0 = 1/t_0 = 2.099376\ldots$ and, consequently, Theorem 2 is true for $p \geq p_0$.

Now we consider the case $p^* \leq p < p_0$. First we observe that by (1.6) $F(x,p) < F(x,p^*)$ for $0 < x < 1$. Indeed, the function $\Gamma(z)$ has its minimum at $\tilde{z} = 1.46163\ldots$ [1, p. 253], and $\Gamma(z)$ is increasing for $z > \tilde{z}$, i.e.

$$\Gamma\left(1+\frac{1}{p}\right) < \Gamma\left(1+\frac{1}{p^*}\right), \quad \text{for } p^* < p < p_0,$$

because $1 + 1/p \geq 1 + 1/p_0 = 1 + t_0 = 1.476331\ldots > \tilde{z}$. Moreover for $0 < x < 1$ we have

$$\int_0^x e^{t^p}\, dt < \int_0^x e^{t^{p^*}}\, dt, \qquad \int_0^x e^{-t^p}\, dt > \int_0^x e^{-t^{p^*}}\, dt.$$

Hence by (1.6) $F(x,p) < F(x,p^*)$ and consequently, by the definition of p^*

$$\varphi(p) < \varphi(p^*) = \tfrac{1}{4},$$

i.e. $F(x^*,p^*) = \frac{1}{4}$ and $dF(x^*,p^*)/dx = 0$. The values of p^* and $x^* = 0.677050\ldots$ are calculated numerically. The proof of Theorem 2 is complete.

3. INEQUALITIES FOR $\int_0^x e^{t^p}\, dt$ AND $\int_0^x e^{-t^p}\, dt$

The estimation of the integrals $\int_0^x e^{t^p}\, dt$, $\int_0^x e^{-t^p}\, dt$ occurred in Section 2 may have an independent interest. For this reason we point out some inequalities here which can he deduced directly from the inequalities occurring between these integrals and the functions $u(z)$ and $v(z)$ introduced by (2.10)

$$u(z) = \int_0^z \frac{e^s - 1}{s}\, ds, \qquad v(z) = \int_0^z \frac{1 - e^{-s}}{s}\, ds.$$

For the first integral we have

$$1 + \frac{u(x^p)}{p+1} < \frac{1}{x}\int_0^x e^{t^p}\, dt < 1 + \frac{u(x^p)}{p}, \quad p > 1,\ x > 0. \qquad (3.1)$$

Similarly we get for the second integral

$$1-\frac{\nu(x^p)}{p+1}<\frac{1}{x}\int_0^x \mathrm{e}^{-t^p}\,\mathrm{d}t, \quad p>1,\ 0<x^p<\frac{9(3p+1)}{4(2p+1)}, \tag{3.2}$$

where the bound for x^p in this inequality comes from a closer investigation of the properties of the Leibniz type series met in the proof of Theorem 2.

We deduce other bounds for $\int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t$:

$$\frac{1}{p}\frac{\mathrm{e}^{x^p}-1}{x^{p-1}}+\frac{p-1}{p}x<\int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t<\frac{2}{p+1}\frac{\mathrm{e}^{x^p}-1}{x^{p-1}}+\frac{p-1}{p+1}x, \quad p>1. \tag{3.3}$$

Let us observe that these inequalities become equalities at $p=1$ and that they are reversed for $0<p<1$. The inequalities (3.3) are based on the following reasoning: We wish to find the values of μ and ν such that the function $\mu(\mathrm{e}^{x^p}-1)/(x^{p-1})+\nu x$ is an upper (lower bound) for the integral. We consider the function

$$\begin{aligned}\int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t-\mu\frac{\mathrm{e}^{x^p}-1}{x^{p-1}}-\nu x \\ = x\left[1-\mu-\nu+\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\left(\frac{n+1}{np+1}-\mu\right)x^{np}\right].\end{aligned}$$

If we choose $\mu\geq 2/(p+1)$, $\nu=1-\mu$, then we have $1-\mu-\nu=0$ and the coefficients of x^{np} are negative except the case $n=1$ where we have the coefficient equal to zero. The optimal choice is clearly $\mu=2/(p+1)$ and consequently $\nu=(p-1)/(p+1)$. Similarly considerations show that the choice $\mu=1/p$ and $\nu=(p-1)/p$ gives the lower bound of (3.3). When $\mu=1$ and $\nu=0$ we obtain the upper bound

$$\int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t<\frac{\mathrm{e}^{x^p}-1}{x^{p-1}}, \quad x>0,\ p>1$$

which is the inequality (1.3) mentioned by Feng Qi and Sen-lin Guo [5].

Finally we observe that using different values of the parameters μ and ν, other inequalities of the type considered here can be obtained.

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