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An Optimal Relative Isoperimetric Inequality in Concave Cylindrical Domains in $\mathbb{R}^n$

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We prove an optimal relative isoperimetric inequality in concave cylindrical domains in $\mathbb{R}^n$, which generalizes the well-known two-dimensional relative isoperimetric inequality $L^2 \geq 2\pi A$ in a planar sector with angle greater than or equal to π.

Keywords: Isoperimetric inequality; Sobolev inequality; Geodesic metric space

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1. INTRODUCTION

The aim of this paper is to point out an optimal relative isoperimetric inequality in concave cylindrical domains in $\mathbb{R}^n$ which plays an important role in the Sobolev inequalities and the mixed boundary value problems of partial differential equations.

Let $\mathcal{S}$ be a sector in $\mathbb{R}^2$ with the sector angle $\theta < 2\pi$. It is well-known that a domain $\Omega \subset \mathcal{S}$ satisfies the optimal relative isoperimetric inequality

$$\text{Length}(\partial\Omega - \partial\mathcal{S})^2 \geq 2\theta\text{Area}(\Omega) \quad \text{if } \theta \leq \pi, \tag{1}$$

$$\text{Length}(\partial\Omega - \partial\mathcal{S})^2 \geq 2\pi\text{Area}(\Omega) \quad \text{if } \theta \geq \pi. \tag{2}$$

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In [3] Lions and Pacella have obtained a generalization of (1) to subdomains of convex cones in $\mathbb{R}^n$, $n \geq 2$ (see also [4]). In the same paper they have also pointed out some applications to symmetrization problems and Sobolev inequalities.

In this paper we will give a higher dimensional generalization of the inequality (2). To state the result we first fix notations. Let $f: \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous convex function i.e. for any $\lambda \in [0, 1]$ and any $t_1, t_2 \in \mathbb{R}$

$$f[(1-\lambda)t_1 + \lambda t_2] \leq (1-\lambda)f(t_1) + \lambda f(t_2).$$

Let $\mathcal{W}$ be the closed cylindrical concave (the complement of a convex) domain in $\mathbb{R}^n$, $n \geq 2$, defined by

$$\mathcal{W} = \{(x_1, x_2, \ldots, x_n) \mid x_2 \leq f(x_1)\}.$$

For $n = 2$, $\mathcal{W}$ is just the set $\mathcal{V}$ of points on and under the graph of f in $\mathbb{R}^2$ and, for $n > 2$, $\mathcal{W}$ is the product of $\mathcal{V}$ and $\mathbb{R}^{n-2}$. With this notation we state

MAIN THEOREM *Let Ω be a compact domain in $\mathcal{W}$ with rectifiable boundary $\partial\Omega$. Then we have the relative isoperimetric inequality*

$$\mathrm{Vol}_{n-1}(\partial\Omega - \partial\mathcal{W}) \geq n\left(\frac{\omega_n}{2}\right)^{1/n} \mathrm{Vol}_n(\Omega)^{(n-1)/n},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. The equality case occurs if and only if Ω is a Euclidean half ball with $\partial\Omega - \partial\mathcal{W}$ an Euclidean half sphere.

Remark For $n = 2$, we recover the well-known relative isoperimetric inequality i.e. for $\Omega \subset \mathcal{V}$

$$\mathrm{Length}(\partial\Omega - \partial\mathcal{V})^2 \geq 2\pi \mathrm{Area}(\Omega).$$

By the well-known arguments we can deduce optimal Sobolev type inequality.

COROLLARY *Suppose the convex function f is continuously differentiable. Let ϕ be a smooth function with compact support in $\mathcal{W}$ (note that ϕ may not vanish on $\partial\mathcal{W}$). Then we have*

$$\int_{\mathcal{W}} |\nabla\phi| \geq n\left(\frac{\omega_n}{2}\right)^{1/n} \left(\int_{\mathcal{W}} |\phi|^{n/(n-1)}\right)^{(n-1)/n}.$$

We now sketch a brief outline of the proof of the Main Theorem. Precise statements will be given in the next two sections. We consider the abstract geodesic metric space (in the sense of Alexandrov and Gromov) $\mathcal{M}$ defined by the disjoint union of $\mathcal{W}$ with itself under the identification along $\partial\mathcal{W}$. We will show $\mathcal{M}$ is nonpositively curved in the sense of Gromov, i.e. CAT(0)-inequality holds for any geodesic triangle in $\mathcal{M}$. Since $\mathcal{M}$ is piecewise-linear we may apply the optimal isoperimetric inequality in [2] recently proved by Cao and Escobar. Then the required relative isoperimetric inequality in $\mathcal{W}$ follows from the existence of canonical isometric reflection in $\mathcal{M}$ i.e. the reflection with respect to $\partial\mathcal{W}$.

2. PRELIMINARIES

We list some aspects of the theory of geodesic metric space in the sense of Alexandrov and Gromov which will be used in the proof. For details we refer the reader to [1] and [2].

Let $(\mathcal{X}, d)$ be a locally-compact, geodesic metric space. Then any two points p and q of $\mathcal{X}$ can be connected by a geodesic (i.e. distance-realizing curve). We denote this geodesic by $\overline{pq}$. Let $\Delta = \Delta(a_0, a_1, a_2)$ be a geodesic triangle with vertices a_0, a_1 and a_2. The corresponding comparison triangle $\Delta' = \Delta'(a_0', a_1', a_2')$ is a triangle in $\mathbb{R}^2$ with the same side lengths as Δ. Δ is said to satisfy CAT(0)-inequality (comparison inequality of Alexandrov and Toponogov) if, for any $p \in \overline{a_1a_2}$ and the corresponding point $p' \in \overline{a_1'a_2'}$ such that $d(a_1, p) = d_{\mathbb{R}^2}(a_1', p')$, we have the comparison inequality

$$d(a_0, p) \leq d_{\mathbb{R}^2}(a_0', p').$$

Δ is said to satify CAT*(0)-inequality if, for any $p \in \overline{a_0a_1}$ and $q \in \overline{a_0a_2}$ and the corresponding points $p' \in \overline{a_0'a_1'}$ and $q' \in \overline{a_0'a_2'}$, we have

$$d(p, q) \leq d_{\mathbb{R}^2}(p', q').$$

We now recall three Lemmas from [1].

LEMMA 1 [1, Lemma 3] *$CAT(0)$-inequality and $CAT^*(0)$-inequality are equivalent on any convex subset of $\mathcal{X}$.*

LEMMA 2 [1, Lemma 4] *Let $\Delta(a_0, b, c)$ and $\Delta(a_1, b, c)$ be geodesic triangles satisfying $CAT^*(0)$-inequality. Suppose $\overline{a_0 b} \cup \overline{b a_1}$ is a (minimal) geodesic between a_0 and a_1. Then the geodesic triangle $\Delta(a_0, a_1, c)$ also satisfies the $CAT^*(0)$-inequality.*

LEMMA 3 [1, Corollary 5] *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be geodesic spaces satisfying $CAT^*(0)$-inequality. Suppose $A_1 \subset \mathcal{X}_1$ and $A_2 \subset \mathcal{X}_2$ are closed and convex subsets such that there is an isometry ι of A_1 onto A_2. Then the glueing $\mathcal{X}$ of $\mathcal{X}_1$ and $\mathcal{X}_2$ under the identification of A_1 and A_2 via the isometry ι is also a geodesic space satisfying $CAT^*(0)$-inequality with the canonical glueing metric given by*

$$d_{\mathcal{X}}(p, q) = \begin{cases} d_{\mathcal{X}_j}(p, q) & \text{if } p, q \in \mathcal{X}_j \ (j = 1, 2) \\ \inf_{r \in A_1} [d_{\mathcal{X}_1}(p, r) + d_{\mathcal{X}_2}(q, \iota(r))] & \text{if } p \in \mathcal{X}_1 \text{ and } q \in \mathcal{X}_2. \end{cases}$$

A geodesic space $\mathcal{X}$ is said to be nonpositively curved in the sense of Gromov if CAT(0)-inequality is satisfied, locally. It was recently proved by Cao and Esocobar that the Euclidean isoperimetric inequality remains true in a simply-connected, complete, piecewise-linear space which is nonpositively curved in the sense of Gromov.

THEOREM 1 [2] *Let $\mathcal{M}^n$ be a simply-connected, complete, piecewise-linear manifold which is nonpositively curved in the sense of Gromov. Then, for any compact domain $\Omega \subset \mathcal{M}$ with rectifiable boundary $\partial\Omega$, the Euclidean isoperimetric inequality*

$$\mathrm{Vol}_{n-1}(\partial\Omega) \geq n\omega_n^{1/n} \mathrm{Vol}_n(\Omega)^{(n-1)/n}$$

holds, and the equality occurs if and only if Ω is isometric to a Euclidean ball.

3. PROOF OF THE MAIN THEOREM

First note that the convex function f defining the cylindrical domain $\mathcal{W}$ may be assumed to be piecewise-linear by simple approximation arguments. $\mathcal{W}$ is clearly a geodesic space with respect to the intrinsic metric i.e. for any $p, q \in \mathcal{W}$,

$$d_{\mathcal{W}}(p, q) = \inf \mathrm{Length}(c),$$

where the infimum is taken over all continuous paths c lying in $\mathcal{W}$ and connecting p to q. With respect to this metric the boundary $\partial\mathcal{W}$ is a closed and convex subset of $\mathcal{W}$. Note also that $\partial\mathcal{W}$ is isometric to $\mathbb{R}^{n-1}$, because $\partial\mathcal{W}$ is the product of the graph of $f: \mathbb{R} \to \mathbb{R}$ and $\mathbb{R}^{n-2}$. Now we consider the piecewise-linear geodesic space $\mathcal{M}^n$ obtained from glueing $\mathcal{W}$ with itself under the identification along $\partial\mathcal{W}$. We claim that $\mathcal{M}$ is nonpositively curved in the sense of Gromov. In view of Lemmas 1 and 3 it suffices to check the validity of CAT*(0)-inequality in $\mathcal{W}$.

Let $\Delta = \Delta(a, b, c)$ be a geodesic triangle in $\mathcal{W}$. If Δ is an Euclidean triangle, there is nothing to prove. If Δ lies entirely in $\partial\mathcal{W}$, CAT*(0)-inequality is also trivial because $\partial\mathcal{W}$ is isometric to $\mathbb{R}^{n-1}$. The remaining case can be handled with the aid of Lemma 2. In fact, since the function f is piecewise-linear and the closed domain $\mathcal{W}$ is concave (i.e. the complement of a convex domain in $\mathbb{R}^n$), we can decompose Δ into the union of finite number of Euclidean triangles $\Delta_j, j = 1, 2, \ldots, l$. Hence, applying Lemma 2 successively, we conclude that Δ satisfies CAT*(0)-inequality and thereby we have proved the claim.

Let Ω be any compact domain in $\mathcal{W}$ with rectifiable boundary $\partial\Omega$. Note that the space $\mathcal{M}$ has the canonical isometric reflection (with respect to $\partial\mathcal{W}$) and thus there is a compact domain $\tilde{\Omega} \subset \mathcal{M}$, the union of Ω with itself under the identification along $\partial\Omega \cap \partial\mathcal{W}$, satisfying the following volume relation

$$\begin{aligned} \mathrm{Vol}_n(\tilde{\Omega}) &= 2\mathrm{Vol}_n(\Omega), \\ \mathrm{Vol}_{n-1}(\partial\tilde{\Omega}) &= 2\mathrm{Vol}_{n-1}(\partial\Omega - \partial\mathcal{W}). \end{aligned}$$

Since $\mathcal{M}$ is a simply-connected, complete, piecewise-linear and nonpositively curved space we may apply the isoperimetric inequality by Cao and Escobar and then, substituting the above volume relation, we obtain the required relative isoperimetric inequality. The equality case follows easily from the equality case of [2] and the proof is completed.

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