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Interpolation of Compact Non-Linear Operators

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Let (E_0, E_1) and (F_0, F_1) be two Banach couples and let $T: E_0 + E_1 \to F_0 + F_1$ be a continuous map such that $T: E_0 \to F_0$ is a Lipschitz compact operator and $T: E_1 \to F_1$ is a Lipschitz operator. We prove that if $T: E_1 \to F_1$ is also compact or E_1 is continuously embedded in E_0 or F_1 is continuously embedded in F_0, then $T: (E_0, E_1)_{\theta,q} \to (F_0, F_1)_{\theta,q}$ is also a compact operator when $1 \leq q < \infty$ and $0 < \theta < 1$. We also investigate the behaviour of the measure of non-compactness under real interpolation and obtain best possible compactness results of Lions–Peetre type for non-linear operators. A two-sided compactness result for linear operators is also obtained for an arbitrary interpolation method when an approximation hypothesis on the Banach couple (F_0, F_1) is imposed.

Keywords: Interpolation; Compact non-linear operators; Measure of non-compactness

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1 INTRODUCTION

In 1960 Krasnoselskii [18] proved the following theorem: if $T: L_{p_0} \to L_{q_0}$ is a compact linear operator, $T: L_{p_1} \to L_{q_1}$ is a bounded linear operator, $1 \leq p_0, p_1, q_1 \leq \infty$ and $1 \leq q_0 < \infty$, then $T: L_p \to L_q$ is also a compact linear operator where $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $0 < \theta < 1$.

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With the development of the abstract interpolation theory, Krasnoselskii's theorem leads to the question if the result is also true if we replace the Banach couples (L_{p_0}, L_{p_1}) and (L_{q_0}, L_{q_1}) by general Banach couples (E_0, E_1) and (F_0, F_1).

The first abstract results were obtained in 1964 by Lions and Peetre [19] for the case $E_0 = E_1$ or $F_0 = F_1$ and by Persson [22] for $E_0 \neq E_1$ and $F_0 \neq F_1$ but with an approximation hypothesis on the Banach couple (F_0, F_1), corresponding to $q_0 < \infty$ in Krasnoselskii's result.

In 1969 Hayakawa [17] proved a general result for the real method without any approximation property. However it was necessary to impose an additional condition: both operators $T: E_0 \to F_0$ and $T: E_1 \to F_1$ are compact.

The paper by Cobos *et al.* [7] opened a new era in the research of this problem. After that there were several papers dealing with the same subject (see [4,5,8,12,13]).

In 1992 Cwikel [14] (see also [10]) showed that if $T: E_0 \to F_0$ is a compact linear operator and if $T: E_1 \to F_1$ is a bounded linear operator, then $T: (E_0, E_1)_{\theta,q} \to (F_0, F_1)_{\theta,q}$ is also a compact linear operator, $(E_0, E_1)_{\theta,q}$ and $(F_0, F_1)_{\theta,q}$ being the real interpolation spaces.

Related with this work is the behaviour under interpolation of the measure of non-compactness. The first results in this direction were obtained by Edmunds and Teixeira [16]. Their results are the analogues of the results of Lions–Peetre and that of Persson.

Recently, Cobos *et al.* [9] proved, for the real interpolation method, a logarithmic-convex inequality for the measure of non-compactness.

Using measures of non-compactness, Cobos *et al.* [6] obtained optimal compactness results of Lions–Peetre type for linear operators.

The behaviour of compact non-linear operators under interpolation did not receive much attention. The only paper dealing with this case of which we are aware is that of Cobos [5] where it is shown that the results of Lions and Peetre are also valid for Lipschitz operators.

In this paper we generalise some of the results proved by Cobos *et al.* [6,11] for non-linear operators.

We also prove that if (E_0, E_1) and (F_0, F_1) are two Banach couples and $T: E_0 + E_1 \to F_0 + F_1$ is a continuous map such that $T: E_0 \to F_0$ is a Lipschitz compact operator and $T: E_1 \to F_1$ is a Lipschitz operator, then $T: (E_0, E_1)_{\theta,q} \to (F_0, F_1)_{\theta,q}$ is also compact when $T: E_1 \to F_1$ is compact or

E_1 is continuously embedded in E_0 or F_1 is continuously embedded in F_0, with $1 \le q < \infty$ and $0 < \theta < 1$.

We close the paper with a two-sided compactness result for an arbitrary interpolation method. For this we need to impose an approximation property in the Banach couple (F_0, F_1). This approximation property is of the same kind as that required by Persson, but is a little stronger.

2 PRELIMINARIES

We start by recalling some notions of interpolation theory. The standard references are [1,2,23]. A pair $\bar{E} = (E_0, E_1)$ of Banach spaces E_0 and E_1 is called a Banach couple if E_0 and E_1 are continuously embedded in some Hausdorff topological vector space. Then $\bar{E}_\Delta = E_0 \cap E_1$ and $\bar{E}_\Sigma = E_0 + E_1$ are Banach spaces with the norms

$$\|x\|_{\bar{E}_\Delta} = \max\{\|x\|_{E_0}, \|x\|_{E_1}\}$$

and

$$\|x\|_{\bar{E}_\Sigma} = \inf\{\|x_0\|_{E_0} + \|x_1\|_{E_1}\colon x = x_0 + x_1,\ x_i \in E_i,\ i = 0, 1\},$$

respectively.

For each $t > 0$, we define

$$J(t, x) = J(t, x, \bar{E}) = \max\{\|x\|_{E_0}, t\|x\|_{E_1}\},$$

for every $x \in \bar{E}_\Delta$ and

$$\begin{aligned} K(t, x) &= K(t, x, \bar{E}) \\ &= \inf\{\|x_0\|_{E_0} + t\|x_1\|_{E_1}\colon x = x_0 + x_1,\ x_i \in E_i,\ i = 0, 1\}, \end{aligned}$$

for every $x \in \bar{E}_\Sigma$. Then $\{K(t, \cdot)\colon t > 0\}$ and $\{J(t, \cdot)\colon t > 0\}$ are families of equivalent norms in $\bar{E}_\Sigma$ and $\bar{E}_\Delta$, respectively.

A Banach space E is said to be intermediate with respect to a Banach couple $\bar{E} = (E_0, E_1)$ if

$$\bar{E}_\Delta \hookrightarrow E \hookrightarrow \bar{E}_\Sigma,$$

where $\hookrightarrow$ means continuous inclusion. To each intermediate space E, there are two other intermediate spaces related with E. The first is the closure of $E_0 \cap E_1$ in E. This space is called the clintersect of E and is denoted by E°. The second is the space of all $x \in E_0 + E_1$ for which there is a sequence $(x_n)_{n \in \mathbb{N}}$ in some bounded set of E which converges to x in $\bar{E}_\Sigma$. This space is denoted by $E^\sim$ and is called the Gagliardo completion of E. It is normed by

$$\|x\|_{E^\sim} = \inf\{\sup\{\|x_n\|_E\colon n \in \mathbb{N}\}\colon x_n \text{ converges to } x \text{ in } \bar{E}_\Sigma\}.$$

If E is an intermediate space with respect to $\bar{E} = (E_0, E_1)$, then for each $t > 0$ we set

$$\psi(t) = \psi(t, E, \bar{E}) = \sup\{K(t, x)\colon \|x\|_E = 1\}$$

and

$$\rho(t) = \rho(t, E, \bar{E}) = \inf\{J(t, x)\colon x \in E_0 \cap E_1,\ \|x\|_E = 1\}.$$

It is easy to prove that $\psi(t)$ and $\rho(t)$ are strictly positive and non-decreasing, while $\psi(t)/t$ and $\rho(t)/t$ are non-increasing.

An intermediate space E with respect to $\bar{E} = (E_0, E_1)$ is said to be of class $\mathscr{C}_K(\theta, \bar{E})$ (resp. $\mathscr{C}_J(\theta, \bar{E})$) if there is a constant C such that $\psi(t) \le Ct^\theta$ (resp. $\rho(t) \ge Ct^\theta$) for every $t > 0$.

Let $\bar{F} = (F_0, F_1)$ be another Banach couple. We denote by $\mathscr{L}(\bar{E}, \bar{F})$ the class of all linear operators $T\colon E_0 + E_1 \to F_0 + F_1$ such that the restriction of T to E_i is a bounded operator from E_i into F_i, $i = 0, 1$. The space $\mathscr{L}(\bar{E}, \bar{F})$ is a Banach space with the norm

$$\|T\|_{\bar{E},\bar{F}} = \max\{\|T\|_{E_0,F_0}, \|T\|_{E_1,F_1}\}.$$

The class of all continuous maps $T\colon \bar{E}_\Sigma \to \bar{F}_\Sigma$ such that the restriction of T to E_i is a continuous map from E_i into F_i, $i = 0, 1$, will be denoted by $\mathscr{C}(\bar{E}, \bar{F})$. If $E_0 = E_1 = E$ or $F_0 = F_1 = F$, then we write $\mathscr{L}(E, \bar{F})$ and $\mathscr{C}(E, \bar{F})$ or, respectively, $\mathscr{L}(\bar{E}, F)$ and $\mathscr{C}(\bar{E}, F)$.

An intermediate space E with respect to $\bar{E} = (E_0, E_1)$ is an interpolation space if for every $T \in \mathscr{L}(\bar{E}, \bar{E})$, the restriction of T to E is a bounded operator from E into itself. There is a constant C such that for every operator $T \in \mathscr{L}(\bar{E}, \bar{E})$,

$$\|T\|_{E,E} \leq C\|T\|_{\bar{E},\bar{E}}. \tag{2.1}$$

An intermediate space E with respect to $\bar{E} = (E_0, E_1)$ is a rank-one interpolation space or r.o. interpolation space if inequality (2.1) is verified for operators of rank one.

An interpolation method is a functor Φ that associates to every Banach couple $\bar{E} = (E_0, E_1)$ an intermediate space $\bar{E}_\Phi = (E_0, E_1)_\Phi$ with respect to $\bar{E}$ in such a way that given any other Banach couple $\bar{F} = (F_0, F_1)$ and any operator $T \in \mathscr{L}(\bar{E}, \bar{F})$, the restriction of T to $\bar{E}_\Phi$ is a bounded operator from $\bar{E}_\Phi$ into $\bar{F}_\Phi$.

Using the closed-graph theorem it can be proved that there exists a constant C such that for every operator $T \in \mathscr{L}(\bar{E}, \bar{F})$,

$$\|T\|_{\bar{E}_\Phi,\bar{F}_\Phi} \leq C\|T\|_{\bar{E},\bar{F}}. \tag{2.2}$$

One of the most important interpolation methods is that of real interpolation. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$. The real interpolation space $\bar{E}_{\theta,q} = (E_0, E_1)_{\theta,q}$ (realised as a K-space) is the collection of all $x \in E_0 + E_1$ for which the value of

$$\|x\|_{\bar{E}_{\theta,q}} = \begin{cases} \left(\sum_{m=-\infty}^{\infty} (2^{-\theta m} K(2^m, x))^q\right)^{1/q} & \text{if } q < \infty, \\ \sup_{m \in \mathbb{Z}} 2^{-\theta m} K(2^m, x) & \text{if } q = \infty \end{cases}$$

is finite.

Let $\bar{F} = (F_0, F_1)$ be another Banach couple. It is a well known fact that if $T \in \mathscr{L}(\bar{E}, \bar{F})$, then $T \in \mathscr{L}(\bar{E}_{\theta,q}, \bar{F}_{\theta,q})$ and

$$\|T\|_{\theta,q} \leq 2^\theta \|T\|_0^{1-\theta} \|T\|_1^\theta,$$

where $\|T\|_{\theta,q}$, $\|T\|_0$ and $\|T\|_1$ are the norms of the operators $T: \bar{E}_{\theta,q} \to \bar{F}_{\theta,q}$, $T: E_0 \to F_0$ and $T: E_1 \to F_1$, respectively.

It was proved by Cobos in [5] that if $T \in \mathscr{C}(\bar{E}, \bar{F})$ and $T: E_0 \to F_0$ and $T: E_1 \to F_1$ are Lipschitz operators, then, for $q < \infty$, the restriction of

T to $\bar{E}_{\theta,q}$ is a Lipschitz operator from $\bar{E}_{\theta,q}$ into $\bar{F}_{\theta,q}$. Moreover,

$$\|T\|_{\theta,q} \leq 2^{\theta}\|T\|_0^{1-\theta}\|T\|_1^{\theta}.$$

Here we denote the Lipschitz constant of an operator T by $\|T\|$. In fact the Lipschitz constant is not a norm but only a semi-norm.

Let M be a bounded subset of a Banach space E. The nth entropy number, $\varepsilon_n(M)$, of M and the nth inner entropy number, $\varphi_n(M)$, of M are defined by

$$\varepsilon_n^E(M) = \varepsilon_n(M) = \inf\left\{\varepsilon > 0\colon M \subseteq \bigcup_{i=1}^{n}\{y_i + \varepsilon U_E\},\ y_1, \ldots, y_n \in E\right\}$$

and

$$\varphi_n^E(M) = \varphi_n(M)$$
$$= \sup\{\rho > 0\colon \exists x_1, \ldots, x_p \in M,\ p > n,\ \|x_i - x_j\| > 2\rho,\ i \neq j\},$$

respectively, where U_E is the closed unit ball of E. The inner entropy numbers and the entropy numbers are related by the following inequalities (see [3] pp. 7–8):

$$\varphi_n(M) \leq \varepsilon_n(M) \leq 2\varphi_n(M). \tag{2.3}$$

The measure of non-compactness, $\beta(M)$, of M is defined by

$$\beta_E(M) = \beta(M) = \lim_{n\to\infty} \varepsilon_n(M).$$

Let us list some elementary properties of the measure of non-compactness of a set (see [15] pp. 13–15):

(i) $\beta(M) = 0$ if, and only if, M is precompact;
(ii) if $M \subseteq N$, then $\beta(M) \leq \beta(N)$;
(iii) $\beta(cl(M)) = \beta(M)$, where $cl(M)$ is the closure of M;
(iv) $\beta(M \cup N) = \max\{\beta(M), \beta(N)\}$;
(v) $\beta(M + N) \leq \beta(M) + \beta(N)$;
(vi) $\beta(co(M)) = \beta(M)$, where $co(M)$ is the convex hull of M.

Let E and F be two Banach spaces and let $T: E \to F$ be a continuous map. If for every bounded subset $M \subseteq E$, $T(M)$ is a bounded subset of F and there is a constant $k \geq 0$ such that

$$\beta(T(M)) \leq k\beta(M),$$

for every bounded subset $M \subseteq E$, then T is called a k-ball-contraction. The (ball) measure of non-compactness, $\beta(T)$, of T is defined by

$$\beta_{E,F}(T) = \beta(T) = \inf\{k\colon T \text{ is a } k\text{-ball-contraction}\}.$$

We say that T is compact if $T(M)$ is relatively compact for every bounded subset $M \subseteq E$.

The measure of non-compactness of an operator has the following properties (see [15] p. 17):

(i) $\beta(T) = 0$ if, and only if, T is compact;
(ii) if T is a Lipschitz operator, then $\beta(T) \leq \|T\|$;
(iii) $\beta(T_1 + T_2) \leq \beta(T_1) + \beta(T_2)$;
(iv) $\beta(RS) \leq \beta(R)\beta(S)$;
(v) $\beta(T) = \beta(T(U_E))$, $T \in \mathscr{L}(E, F)$.

Nussbaum [21] proved that if $T \in \mathscr{L}(E, E)$ and $r_e(T) = r_e^E(T)$ is the radius of the essential spectrum, then

$$r_e(T) = \lim_{n\to\infty} \beta^{1/n}(T^n).$$

Let $\bar{E} = (E_0, E_1)$ and $\bar{F} = (F_0, F_1)$ be two Banach couples and let $T \in \mathscr{L}(\bar{E}, \bar{F})$. In [9] it is proved that there is a constant C, independent of the spaces and the operator, such that

$$\beta_{\theta,q}(T) \leq C\beta_0^{1-\theta}(T)\beta_1^{\theta}(T),$$

where $\beta_{\theta,q}(T)$, $\beta_0(T)$ and $\beta_1(T)$ are the measures of non-compactness of the operators $T: \bar{E}_{\theta,q} \to \bar{F}_{\theta,q}$, $T: E_0 \to F_0$ and $T: E_1 \to F_1$, respectively.

The following two Theorems will prove to be useful in the next sections.

THEOREM 2.1 *Let E and F be two Banach spaces and let $T: E \to F$ be a ball-contraction. Then*

$$\beta(T) = \sup\left\{\frac{\beta(T(\{x + rU_E\}))}{r} : x \in E,\ r > 0\right\}.$$

Proof If E has finite dimension, there is nothing to prove. Suppose that E has infinite dimension. Then

$$\beta(T) = \sup\left\{\frac{\beta(T(M))}{\beta(M)} : \beta(M) \neq 0\right\}.$$

Since $\beta(\{x + rU_E\}) = r$ when $\dim E = \infty$, we have

$$\beta(T) \geq \sup\left\{\frac{\beta(T(\{x + rU_E\}))}{r} : x \in E,\ r > 0\right\}.$$

Let M be a bounded set of E and suppose that $\sigma = \beta(M) > 0$. Then for every $\varepsilon > 0$ there exist $y_1, \ldots, y_n \in E$ such that

$$M \subseteq \bigcup_{i=1}^{n} \{y_i + (\sigma + \varepsilon)U_E\}.$$

It follows that

$$T(M) \subseteq \bigcup_{i=1}^{n} T(\{y_i + (\sigma + \varepsilon)U_E\})$$

and, consequently,

$$\beta(T(M)) \leq \max_{1 \leq i \leq n} \beta(T(\{y_i + (\sigma + \varepsilon)U_E\})).$$

Therefore

$$\frac{\beta(T(M))}{\beta(M)} \leq \max_{1 \leq i \leq n} \frac{\beta(T(\{y_i + (\sigma + \varepsilon)U_E\}))}{\sigma + \varepsilon} \frac{\sigma + \varepsilon}{\sigma},$$

and the theorem is proved.

THEOREM 2.2 *Let E and F be two Banach spaces and let $T: E \to F$ be a ball-contraction. If E_0 is a vector subspace dense in E, then*

$$\beta(T) = \sup\left\{\frac{\beta(T(\{x + rU_E\} \cap E_0))}{r} : x \in E_0, r > 0\right\}.$$

Proof Let $\{a + rU_E\}$ be a closed ball in E. Since E_0 is dense in E it follows that $cl(\{a + rU_E\} \cap E_0) = \{a + rU_E\}$ and, consequently, $T(\{a + rU_E\} \cap E_0) \subseteq T(\{a + rU_E\}) \subseteq cl(T(\{a + rU_E\} \cap E_0))$. By the properties of the measure of non-compactness it follows that $\beta(T(\{a + rU_E\})) = \beta(T(\{a + rU_E\} \cap E_0))$. By Theorem 2.1 we get

$$\beta(T) = \sup\left\{\frac{\beta(T(\{a + rU_E\} \cap E_0))}{r} : a \in E, r > 0\right\}.$$

On the other hand, for every $\varepsilon > 0$, there exists $x \in E_0$ such that $\|x - a\| \le \varepsilon$. Hence $T(\{a + rU_E\}) \subseteq T(\{x + (r + \varepsilon)U_E\})$ and

$$\frac{\beta(T(\{a + rU_E\} \cap E_0))}{r} \le \frac{\beta(T(\{x + (r + \varepsilon)U_E\} \cap E_0))}{r + \varepsilon}\frac{r + \varepsilon}{r}.$$

Therefore

$$\beta(T) = \sup\left\{\frac{\beta(T(\{x + rU_E\} \cap E_0))}{r} : x \in E_0, r > 0\right\}$$

and the proof is finished.

3 BEST POSSIBLE COMPACTNESS RESULTS OF LIONS–PEETRE TYPE: THE NON-LINEAR CASE

In 1964 Lions and Peetre [19] proved the following theorem:

THEOREM *Let (E_0, E_1) and (F_0, F_1) be two Banach couples and let $T \in \mathscr{L}(\bar{E}, \bar{F})$ be an operator such that $T: E_1 \to F_1$ is compact.*

(i) *If $F_0 = F_1 = F$ and E is a space of class $\mathscr{C}_K(\theta, \bar{E})$, then $T: E \to F$ is compact.*
(ii) *If $E_0 = E_1 = E$ and F is a space of class $\mathscr{C}_J(\theta, \bar{F})$, then $T: E \to F$ is compact.*

In [20] Mastyło noticed that this theorem is also valid under weaker conditions. Namely, if we substitute the hypothesis E is of class $\mathscr{C}_K(\theta, \bar{E})$ by $\lim_{t\to 0}\psi(t, E, \bar{E}) = 0$ and replace the assumption that F is of class $\mathscr{C}_J(\theta, \bar{F})$ by $\lim_{t\to\infty}\rho(t, F, \bar{F}) = \infty$ the result still holds. The result is also true if $T: E_0 \to F_0$ is also compact. After that, Cobos *et al.* [6], using the (ball) measure of non-compactness, proved that these hypotheses are also necessary in a great number of cases.

In this section we prove that the results in Cobos *et al.* [6] still hold for non-linear operators.

THEOREM 3.1 *Let $\bar{E} = (E_0, E_1)$ be a Banach couple, let E be an intermediate space with respect to $\bar{E}$ such that $E_0 \cap E_1$ is dense in E and let F be another Banach space. If $T \in \mathscr{C}(\bar{E}, F)$ is an operator such that $T: E_0 \to F$ and $T: E_1 \to F$ are Lipschitz operators, then $T: E \to F$ is also a Lipschitz operator. Furthermore,*

(i) *if $\beta_{E_0,F}(T) = 0$, then $\beta_{E,F}(T) \leq \|T\|_{E_1,F} \cdot \lim_{t\to\infty}\psi(t, E, \bar{E})/t$;*
(ii) *if $\beta_{E_1,F}(T) = 0$, then $\beta_{E,F}(T) \leq \|T\|_{E_0,F} \cdot \lim_{t\to 0}\psi(t, E, \bar{E})$;*
(iii) *if $\beta_{E_i,F}(T) \neq 0$, $i = 0, 1$, then*

$$\beta_{E,F}(T) \leq \beta_{E_0,F}(T)\psi\left(\frac{\beta_{E_1,F}(T)}{\beta_{E_0,F}(T)}, E, \bar{E}\right)\left(1 + \frac{\|T\|_{E_0,F} + \|T\|_{E_1,F}}{\beta_{E_0,F}(T) + \beta_{E_1,F}(T)}\right).$$

Proof We first show that $T: E \to F$ is also a Lipschitz operator. Let $x, y \in E_0 \cap E_1$ and choose any decomposition $x - y = x_0 + x_1$ with $x_i \in E_i$, $i = 0, 1$. Then

$$\begin{aligned}\|Tx - Ty\|_F &\leq \|Tx - T(x - x_0)\|_F + \|T(x - x_0) - Ty\|_F \\ &\leq \|T\|_{E_0,F}\|x_0\|_{E_0} + \|T\|_{E_1,F}\|x_1\|_{E_1} \\ &\leq \max\{\|T\|_{E_0,F}, \|T\|_{E_1,F}\}(\|x_0\|_{E_0} + \|x_1\|_{E_1}).\end{aligned}$$

Therefore, for any $x, y \in E_0 \cap E_1$

$$\|Tx - Ty\|_F \leq \max\{\|T\|_{E_0,F}, \|T\|_{E_1,F}\}\|x - y\|_{E_0+E_1}.$$

Hence

$$T: (E_0 \cap E_1, \|\cdot\|_{E_0+E_1}) \to F$$

is a Lipschitz operator. Since $E \hookrightarrow \bar{E}_\Sigma$,

$$T : (E_0 \cap E_1, \|\cdot\|_E) \to F$$

is also a Lipschitz operator. Therefore $T : E \to F$ is also a Lipschitz operator because $E_0 \cap E_1$ is dense in E.

Let $\{a + rU_{E_0 \cap E_1}\}$ be a closed ball of $(E_0 \cap E_1, \|\cdot\|_E)$. For every $x \in \{a + rU_{E_0 \cap E_1}\}$ and every $t, \varepsilon > 0$, there exist $x_0 \in E_0$ and $x_1 \in E_1$ such that $x - a = x_0 + x_1$ and

$$\|x_0\|_{E_0} + t\|x_1\|_{E_1} \leq (1+\varepsilon)K(t, x-a) \leq (1+\varepsilon)\psi(t)\|x-a\|_E,$$

which implies

$$\|x_0\|_{E_0} \leq (1+\varepsilon)\psi(t)r \quad \text{and} \quad \|x_1\|_{E_1} \leq (1+\varepsilon)\frac{\psi(t)}{t}r.$$

Let $\sigma_0 > \beta_{E_0,F}(T)$ and $\sigma_1 > \beta_{E_1,F}(T)$. Then there exist $y_1, y_2, \ldots, y_k \in F$ and $z_1, z_2, \ldots, z_n \in F$ such that

$$\min_{1 \leq i \leq k} \|Ty - y_i\|_F \leq (1+\varepsilon)\psi(t)r\sigma_0,$$

for every $y \in \{a + (1+\varepsilon)\psi(t)rU_{E_0}\}$ and

$$\min_{1 \leq j \leq n} \|Tz - z_j\|_F \leq (1+\varepsilon)\frac{\psi(t)}{t}r\sigma_1,$$

for every $z \in \{a + (1+\varepsilon)(\psi(t)/t)rU_{E_1}\}$. Hence there exist y_{i_0} and z_{j_0} such that

$$\|T(a + x_0) - y_{i_0}\|_F \leq (1+\varepsilon)\psi(t)\sigma_0 r$$

and

$$\|T(a + x_1) - z_{j_0}\|_F \leq (1+\varepsilon)\frac{\psi(t)}{t}\sigma_1 r.$$

Then, putting $\alpha_0 = \sigma_0/(\sigma_0+\sigma_1)$ and $\alpha_1 = \sigma_1/(\sigma_0+\sigma_1)$, we have

$$\begin{aligned}
&\|Tx - \alpha_1 y_{i_0} - \alpha_0 z_{j_0}\|_F \\
&\quad \le \alpha_1 \|Tx - T(a+x_0)\|_F + \alpha_0 \|Tx - T(a+x_1)\|_F \\
&\qquad + \alpha_1 \|T(a+x_0) - y_{i_0}\|_F + \alpha_0 \|T(a+x_1) - z_{j_0}\|_F \\
&\quad \le \alpha_1 \|T\|_{E_1,F} \|x_1\|_{E_1} + \alpha_0 \|T\|_{E_0,F} \|x_0\|_{E_0} + (1+\varepsilon)\psi(t) r \left(\alpha_1 \sigma_0 + \frac{\alpha_0 \sigma_1}{t}\right) \\
&\quad \le (1+\varepsilon)\psi(t) r \left(\alpha_0 \|T\|_{E_0,F} + \alpha_1 \sigma_0 + \frac{\alpha_1 \|T\|_{E_1,F} + \alpha_0 \sigma_1}{t}\right),
\end{aligned}$$

which implies

$$\begin{aligned}
&\beta(T(\{a + rU_{E_0 \cap E_1}\})) \\
&\quad \le (1+\varepsilon)\psi(t) r \left(\alpha_0 \|T\|_{E_0,F} + \alpha_1 \sigma_0 + \frac{\alpha_1 \|T\|_{E_1,F} + \alpha_0 \sigma_1}{t}\right),
\end{aligned}$$

for every closed ball $\{a + rU_{E_0 \cap E_1}\}$. By Theorem 2.2, we have

$$\beta_{E,F}(T) \le (1+\varepsilon)\psi(t) \left(\alpha_0 \|T\|_{E_0,F} + \alpha_1 \sigma_0 + \frac{\alpha_1 \|T\|_{E_1,F} + \alpha_0 \sigma_1}{t}\right).$$

If $\beta_{E_0,F}(T) = 0$, letting $\sigma_0, \varepsilon \to 0$, it follows that

$$\beta_{E,F}(T) \le \|T\|_{E_1,F} \frac{\psi(t)}{t}$$

and, since $\psi(t)/t$ is non-increasing,

$$\beta_{E,F}(T) \le \|T\|_{E_1,F} \cdot \lim_{t\to\infty} \frac{\psi(t)}{t}.$$

In the case $\beta_{E_1,F}(T) = 0$, similarly, we obtain

$$\beta_{E,F}(T) \le \|T\|_{E_0,F} \cdot \lim_{t\to 0} \psi(t).$$

Finally, if $\beta_{E_i,F}(T) \neq 0$, $i=0,1$, letting $\sigma_i \to \beta_{E_i,F}(T)$, $i=0,1$, and $\varepsilon \to 0$ and putting $t = \beta_{E_1,F}(T)/\beta_{E_0,F}(T)$ we have

$$\beta_{E,F}(T) \leq \beta_{E_0,F}(T)\psi\left(\frac{\beta_{E_1,F}(T)}{\beta_{E_0,F}(T)}\right)\left(1 + \frac{\|T\|_{E_0,F} + \|T\|_{E_1,F}}{\beta_{E_1,F}(T) + \beta_{E_0,F}(T)}\right),$$

and the theorem is proved.

Using Lemma 3.3 of Cobos *et al.* [6] we obtain immediately the following corollary.

COROLLARY 3.2 *Let $\bar{E} = (E_0, E_1)$ be a Banach couple, let E be an r.o. interpolation space with respect to $\bar{E}$ such that $E_0 \cap E_1$ is dense in E, let F be another Banach space and let $T \in \mathscr{C}(\bar{E}, F)$ be an operator such that $T: E_0 \to F$ is a Lipschitz operator and $T: E_1 \to F$ is a Lipschitz compact operator. Then at least one of the following conditions must hold:*

(i) *$T: E \to F$ is compact;*
(ii) *$E_0^\circ \hookrightarrow E$.*

If, in addition the couple $\bar{E}$ satisfies $E_0^\circ = E_0$, then $T: E \to F$ is compact implies at least one of the following conditions:

(i′) $\lim_{t\to 0} \psi(t, E, \bar{E}) = 0$;
(ii′) *$T: E_0 \to F$ is compact.*

THEOREM 3.3 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple, let F be an intermediate space with respect to $\bar{F}$. Then every bounded subset M of $\bar{F}_\Delta$ is a bounded subset of F and*

(i) *if $\varepsilon_k^{F_0}(M) = 0$, then $\varepsilon_{kn}^{F}(M) \leq 2\varepsilon_n^{F_1}(M) \cdot \lim_{t\to 0} t/\rho(t, F, \bar{F})$;*
(ii) *if $\varepsilon_n^{F_1}(M) = 0$, then $\varepsilon_{kn}^{F}(M) \leq 2\varepsilon_k^{F_0}(M) \cdot \lim_{t\to\infty} 1/\rho(t, F, \bar{F})$;*
(iii) *if $\varepsilon_k^{F_0}(M) \cdot \varepsilon_n^{F_1}(M) \neq 0$, then*

$$\varepsilon_{kn}^{F}(M) \leq \frac{2\varepsilon_k^{F_0}(M)}{\rho(\varepsilon_k^{F_0}(M)/\varepsilon_n^{F_1}(M), F, \bar{F})}.$$

Proof For $\sigma_0 > \varepsilon_k^{F_0}(M)$ and $\sigma_1 > \varepsilon_n^{F_1}(M)$, there exist $y_1, \ldots, y_k \in F_0$ and $z_1, \ldots, z_n \in F_1$ such that

$$M \subseteq \bigcup_{i=1}^{k} \{y_i + \sigma_0 U_{F_0}\} \quad \text{and} \quad M \subseteq \bigcup_{j=1}^{n} \{z_j + \sigma_1 U_{F_1}\}.$$

Let $x_1, x_2, \ldots, x_m \in M$ where $m > kn$ and put

$$I_i = \{h\colon x_h \in \{y_i + \sigma_0 U_{F_0}\}\},$$

$i = 1, 2, \ldots, n$. Since $\sum_{i=1}^{n} |I_i| \geq m > kn$, there is i_0 such that $|I_{i_0}| > n$. Hence, there are $r, s \in I_{i_0}$ such that $x_r, x_s \in \{z_{j_0} + \sigma_1 U_{F_1}\}$ for some positive integer $j_0 \leq n$. It follows that

$$\begin{aligned}\|x_r - x_s\|_F &\leq \frac{1}{\rho(t, F, \bar{F})} J(t, x_r - x_s)\\ &\leq \frac{1}{\rho(t, F, \bar{F})} \max\{\|x_r - x_s\|_{F_0}, t\|x_r - x_s\|_{F_1}\}\\ &\leq \frac{2}{\rho(t, F, \bar{F})} \max\{\sigma_0, t\sigma_1\}\end{aligned}$$

and letting $\sigma_0 \to \varepsilon_k^{F_0}(M)$ and $\sigma_1 \to \varepsilon_n^{F_1}(M)$ we have

$$\|x_r - x_s\|_F \leq \frac{2}{\rho(t, F, \bar{F})} \max\{\varepsilon_k^{F_0}(M), t\varepsilon_n^{F_1}(M)\}.$$

Therefore

$$\varphi_{kn}^F(M) \leq \frac{1}{\rho(t, F, \bar{F})} \max\{\varepsilon_k^{F_0}(M), t\varepsilon_n^{F_1}(M)\}.$$

By the inequality (2.3) we have

$$\varepsilon_{kn}^F(M) \leq \frac{2}{\rho(t, F, \bar{F})} \max\{\varepsilon_k^{F_0}(M), t\varepsilon_n^{F_1}(M)\}. \tag{3.1}$$

Since $\rho(t)/t$ is non-increasing and $\rho(t)$ is non-decreasing, from inequality (3.1) we obtain (i) and (ii) when $\varepsilon_k^{F_0}(M) = 0$ and $\varepsilon_n^{F_1}(M) = 0$, respectively. For (iii) we put $t = \varepsilon_k^{F_0}(M)/\varepsilon_n^{F_1}(M)$.

THEOREM 3.4 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple, let F be an intermediate space with respect to $\bar{F}$ and let M be a bounded subset of $\bar{F}_\Delta$.*

(i) *If $\beta_{F_0}(M) = 0$, then $\beta_F(M) \leq 2\beta_{F_1}(M) \cdot \lim_{t\to 0} t/\rho(t, F, \bar{F})$.*
(ii) *If $\beta_{F_1}(M) = 0$, then $\beta_F(M) \leq 2\beta_{F_0}(M) \cdot \lim_{t\to\infty} 1/\rho(t, F, \bar{F})$.*
(iii) *If $\beta_{F_0}(M) \cdot \beta_{F_1}(M) \neq 0$, then*

$$\beta_F(M) \leq \frac{2\beta_{F_0}(M)}{\rho(\beta_{F_0}(M)/\beta_{F_1}(M), F, \bar{F})}.$$

Proof Letting $k, n \to \infty$ in the inequality (3.1), it follows that

$$\beta_F(M) \leq \frac{2}{\rho(t, F, \bar{F})} \max\{\beta_{F_0}(M), t\beta_{F_1}(M)\}. \tag{3.2}$$

As in the proof of Theorem 3.3 we have (i), (ii) and (iii).

THEOREM 3.5 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple, let F be an intermediate space with respect to $\bar{F}$ and let E be a Banach space. If $T \in \mathscr{C}(E, \bar{F})$ is an operator such that $T: E \to F_0$ and $T: E \to F_1$ are ball-contractions, then $T: E \to F$ is also a ball-contraction. Furthermore,*

(i) *if $\beta_{E,F_0}(T) = 0$, then $\beta_{E,F}(T) \leq 2\beta_{E,F_1}(T) \cdot \lim_{t\to 0} t/\rho(t, F, \bar{F})$;*
(ii) *if $\beta_{E,F_1} = 0$, then $\beta_{E,F}(T) \leq 2\beta_{E,F_0}(T) \cdot \lim_{t\to\infty} 1/\rho(t, F, \bar{F})$;*
(iii) *if $\beta_{E,F_0}(T) \cdot \beta_{E,F_1}(T) \neq 0$, then*

$$\beta_{E,F}(T) \leq \frac{2\beta_{E,F_0}(T)}{\rho(\beta_{E,F_0}(T)/\beta_{E,F_1}(T), F, \bar{F})}.$$

Proof First we prove that $T: E \to F$ is continuous. Since $T: E \to F_0$ and $T: E \to F_1$ are continuous, $T: E \to \bar{F}_\Delta$ is continuous and this implies that $T: E \to F$ is continuous.

Let M be a bounded subset of E. By inequality (3.2) it follows that

$$\begin{aligned}\beta_F(T(M)) &\leq \frac{2}{\rho(t, F, \bar{F})} \max\{\beta_{F_0}(T(M)), t\beta_{F_1}(T(M))\} \\ &\leq \frac{2}{\rho(t, F, \bar{F})} \max\{\beta_{E,F_0}(T), t\beta_{E,F_1}(T)\}\beta(M).\end{aligned}$$

Hence

$$\beta_{E,F}(T) \leq \frac{2}{\rho(t, F, \bar{F})} \max\{\beta_{E,F_0}(T), t\beta_{E,F_1}(T)\}.$$

Now using the same arguments as in the proof of Theorem 3.3 we obtain (i), (ii) and (iii).

Using Lemma 3.4 of Cobos *et al.* [6] we have immediately the following Corollary.

COROLLARY 3.6 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple, let F be an r.o. interpolation space with respect to $\bar{F}$, let E be another Banach space and let $T \in \mathscr{C}(E, \bar{F})$ be an operator such that $T: E \to F_0$ is a ball-contraction* *and $T: E \to F_1$ is compact. Then at last one of the following conditions must hold:*

(i) *$T: E \to F$ is compact;*
(ii) *$F \hookrightarrow F_0^{\sim}$.*

If, in addition the couple $\bar{F}$ satisfies $F_0^{\sim} = F_0$, then $T: E \to F$ is compact if and only if at least one of the following conditions hold:

(i′) $\lim_{t\to\infty} \rho(t, F, \bar{F}) = \infty$;
(ii′) *$T: E \to F_0$ is compact.*

Let ℓ_1 be the Banach space of the absolutely summable sequences $(u_n)_{n\in\mathbb{N}}$ and let ℓ_∞ be the Banach space of the bounded sequences $(u_n)_{n\in\mathbb{N}}$ equipped with the usual norms.

The following two theorems are generalisations of the Theorems 3.9 and 3.10 of Cobos *et al.* [6] for non-linear operators. Since the proofs are essentially the same we omit them.

THEOREM 3.7 *Let $\bar{E} = (E_0, E_1)$ be a Banach couple and let E be an intermediate space with respect to $\bar{E}$ such that $E_0 \cap E_1$ is dense in E. Suppose that $E \cap E_1$ is dense in E, or $E_0 \cap E_1$ is dense in E_0, or that*

$$\lim_{t\to 0} K(t, x, \bar{E}) = 0 \quad \text{for all } x \in E.$$

Then the following are equivalent:

(i) $\lim_{t\to 0} \psi(t, E, \bar{E}) = 0$;
(ii) *for every Banach space F, if $T \in \mathscr{C}(\bar{E}, F)$ is an operator such that $T: E_0 \to F$ is a Lipschitz operator, and $T: E_1 \to F$ is a compact Lipschitz operator, then $T: E \to F$ is a compact operator;*
(iii) *if $T \in \mathscr{C}(\bar{E}, \ell_\infty)$ is an operator such that $T: E_0 \to \ell_\infty$ is a Lipschitz operator and $T: E_1 \to \ell_\infty$ is a compact Lipschitz operator, then $T: E \to \ell_\infty$ is a compact operator.*

THEOREM 3.8 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple and let F be an intermediate space with respect to $\bar{F}$. Then the following are equivalent:*

(i) $\lim_{t\to\infty} \rho(t, F, \bar{F}) = \infty$;
(ii) *for every Banach space E, if $T \in \mathscr{C}(E, \bar{F})$ is an operator such that $T: E \to F_0$ is a ball-contraction operator and $T: E \to F_1$ is compact, then $T: E \to F$ is a compact;*
(iii) *if $T \in \mathscr{C}(\ell_1, \bar{F})$ is an operator such that $T: \ell_1 \to F_0$ is a ball-contraction and $T: \ell_1 \to F_1$ is compact, then $T: \ell_1 \to F$ is compact.*

4 FURTHER RESULTS IN THE CASES $E_0 = E_1$ AND $F_0 = F_1$

In this section we generalise some of the results obtained by Cobos *et al.* [11] for the measure of non-compactness of Lipschitz operators.

THEOREM 4.1 *Let $\bar{E} = (E_0, E_1)$ be a Banach couple, let E be an intermediate space with respect to $\bar{E}$ such that $E_0 \cap E_1$ is dense in E and let F be another Banach space. Assume that $T \in \mathscr{C}(\bar{E}, F)$ is an operator such that $T: E_0 \to F$ and $T: E_1 \to F$ are Lipschitz operators.*

(i) *If $\beta_{\bar{E}_\Delta, F}(T) = 0$, then*

$$\beta_{E,F}(T) \le 2\|T\|_{\bar{E},F} \cdot \max\left\{ \lim_{t\to 0} \psi(t, E, \bar{E}),\ \lim_{t\to\infty} \frac{\psi(t, E, \bar{E})}{t} \right\}.$$

(ii) *If $\beta_{\bar{E}_\Delta, F}(T) \ne 0$, then*

$$\beta_{E,F}(T) \le 6\|T\|_{\bar{E},F}\, \eta\left(\frac{\beta_{\bar{E}_\Delta, F}(T)}{\|T\|_{\bar{E},F}}, E, \bar{E} \right),$$

where $\eta(t, E, \bar{E}) = \max\{\psi(t, E, \bar{E}),\ \psi(t^{-1}, E, \bar{E})/t^{-1}\}$.

Proof Let $\{a + rU_{E_0 \cap E_1}\}$ be a closed ball of $(E_0 \cap E_1, \|\cdot\|_E)$ and put, for every $t > 0$,

$$\eta(t) = \max\left\{ \psi(t),\ \frac{\psi(t^{-1})}{t^{-1}} \right\}.$$

For every $x \in \{a + rU_{E_0 \cap E_1}\}$ and every $t, \varepsilon > 0$, there are $x_0, x_0' \in E_0$ and $x_1, x_1' \in E_1$ such that $x - a = x_0 + x_1 = x_0' + x_1'$,

$$\begin{aligned} \|x_0\|_{E_0} + t\|x_1\|_{E_1} &\le (1+\varepsilon) K(t, x-a) \\ &\le (1+\varepsilon)\psi(t)\|x-a\|_E \\ &\le (1+\varepsilon) r\, \eta(t) \end{aligned}$$

and

$$\begin{aligned} \|x_0'\|_{E_0} + t^{-1}\|x_1'\|_{E_1} &\le (1+\varepsilon) K(t^{-1}, x-a) \\ &\le (1+\varepsilon)\psi(t^{-1})\|x-a\|_E \\ &\le (1+\varepsilon) r t^{-1} \eta(t). \end{aligned}$$

It follows that

$$\|x_0\|_{E_0} \leq (1+\varepsilon)r\eta(t), \quad \|x_1\|_{E_1} \leq (1+\varepsilon)rt^{-1}\eta(t),$$
$$\|x_0'\|_{E_0} \leq (1+\varepsilon)rt^{-1}\eta(t), \quad \|x_1'\|_{E_1} \leq (1+\varepsilon)r\eta(t).$$

Putting $y = x_1 - x_1' = x_0' - x_0 \in \bar{E}_\Delta$, we have

$$\begin{aligned}\|y\|_{\bar{E}_\Delta} &\leq \max\{\|x_0\|_{E_0} + \|x_0'\|_{E_0}, \|x_1\|_{E_1} + \|x_1'\|_{E_1}\} \\ &\leq (1+\varepsilon)r\eta(t)(1+t^{-1})\end{aligned}$$

and

$$\begin{aligned}\|x-(y+a)\|_{\bar{E}_\Sigma} &\leq \|x_0\|_{E_0} + \|x_1'\|_{E_1} \\ &\leq (1+\varepsilon)r\eta(t) + (1+\varepsilon)r\eta(t) \\ &= 2(1+\varepsilon)r\eta(t).\end{aligned}$$

Let $\sigma > \beta_{\bar{E}_\Delta,F}(T)$. Then there are $z_1, \ldots, z_n \in \{a + (1+\varepsilon)r\eta(t) \times (1+t^{-1})U_{\bar{E}_\Delta}\}$ such that

$$\min_{1\leq j\leq n} \|Tz - Tz_j\|_F \leq 2(1+\varepsilon)r\eta(t)(1+t^{-1})\sigma,$$

for every $z \in \{a + (1+\varepsilon)r\eta(t)(1+t^{-1})U_{\bar{E}_\Delta}\}$. In particular there is z_j such that

$$\|T(y+a) - Tz_j\|_F \leq 2(1+\varepsilon)r\eta(t)(1+t^{-1})\sigma.$$

Therefore, for every $x \in \{a + rU_{E_0\cap E_1}\}$, there is $z_j \in \{z_1, \ldots, z_n\}$ such that

$$\begin{aligned}\|Tx - Tz_j\|_F &\leq \|Tx - T(y+a)\|_F + \|T(y+a) - Tz_j\|_F \\ &\leq \|T\|_{\bar{E},F}\|x-(y+a)\|_{\bar{E}_\Sigma} + 2(1+\varepsilon)r\eta(t)(1+t^{-1})\sigma \\ &\leq 2(1+\varepsilon)r\eta(t)\|T\|_{\bar{E},F} + 2(1+\varepsilon)r\eta(t)(1+t^{-1})\sigma \\ &= 2(1+\varepsilon)r\eta(t)\left[\|T\|_{\bar{E},F} + (1+t^{-1})\sigma\right]\end{aligned}$$

and this implies

$$\beta(T(\{a + rU_{E_0\cap E_1}\})) \leq 2(1+\varepsilon)r\eta(t)\left[\|T\|_{\bar{E},F} + (1+t^{-1})\sigma\right],$$

for every closed ball $\{a + rU_{E_0 \cap E_1}\}$ in $(E_0 \cap E_1, \|\cdot\|_E)$. By Theorem 2.2, it follows that

$$\beta_{E,F}(T) \leq 2(1+\varepsilon)\eta(t)\big[\|T\|_{\bar{E},F} + (1+t^{-1})\sigma\big].$$

In the case $\beta_{\bar{E}_\Delta,F}(T) = 0$, letting $\varepsilon, \sigma \to 0$ we have

$$\beta_{E,F}(T) \leq 2\eta(t)\|T\|_{\bar{E},F},$$

for every $t > 0$, and, consequently,

$$\beta_{E,F}(T) \leq 2\inf_{t>0} \eta(t)\|T\|_{\bar{E},F}.$$

Since $\psi(t)$ and $\psi(t^{-1})/t^{-1}$ are non-decreasing,

$$\inf_{t>0} \eta(t) = \max\left\{\lim_{t\to 0}\psi(t),\ \lim_{t\to 0}\frac{\psi(t^{-1})}{t^{-1}}\right\} = \max\left\{\lim_{t\to 0}\psi(t),\ \lim_{t\to\infty}\frac{\psi(t)}{t}\right\},$$

and (i) is proved.

If $\beta_{\bar{E}_\Delta,F}(T) \neq 0$, putting $t = \beta_{\bar{E}_\Delta,F}(T)/\|T\|_{\bar{E},F}$ and letting $\varepsilon \to 0$ and $\sigma \to \beta_{\bar{E}_\Delta,F}(T)$, we have

$$\begin{aligned}\beta_{E,F}(T) &\leq 2\eta\left(\frac{\beta_{\bar{E}_\Delta,F}(T)}{\|T\|_{\bar{E},F}}\right)\left[\|T\|_{\bar{E},F} + \left(1 + \frac{\|T\|_{\bar{E},F}}{\beta_{\bar{E}_\Delta,F}(T)}\right)\beta_{\bar{E}_\Delta,F}(T)\right]\\ &\leq 6\|T\|_{\bar{E},F}\,\eta\left(\frac{\beta_{\bar{E}_\Delta,F}(T)}{\|T\|_{\bar{E},F}}\right),\end{aligned}$$

and the theorem is proved.

An immediate consequence of Theorem 4.1 is the following corollary.

COROLLARY 4.2 *Let $\bar{E} = (E_0, E_1)$ be a Banach couple, let E be an intermediate space with respect to $\bar{E}$ such that $E_0 \cap E_1$ is dense in E, let F be another Banach space and let $T \in \mathscr{C}(\bar{E}, F)$ such that $T: E_0 \to F$ and $T: E_1 \to F$ are Lipschitz operators. If $\lim_{t\to 0}\psi(t, E, \bar{E}) = \lim_{t\to\infty}\psi(t, E, \bar{E})/t = 0$, then $T: E \to F$ is compact if and only if $T: \bar{E}_\Delta \to F$ is compact.*

In particular, if E is a space of class $\mathscr{C}_K(\theta, \bar{E})$ the last corollary takes the following form.

COROLLARY 4.3 *Let $\bar{E}=(E_0,E_1)$ be a Banach couple, let E be an intermediate space of class $\mathscr{C}_K(\theta,\bar{E})$ such that $E_0\cap E_1$ is dense in E, let F be another Banach space and let $T\in\mathscr{C}(\bar{E},F)$ such that $T\colon E_0\to F$ and $T\colon E_1\to F$ are Lipschitz. Then $T\colon E\to F$ is compact if and only if $T:\bar{E}_\Delta\to F$ is compact.*

Using Lemma 3.3 of Cobos *et al.* [6] we obtain immediately the following corollary.

COROLLARY 4.4 *Let $\bar{E}=(E_0,E_1)$ be a Banach couple, let E be an r.o. interpolation space with respect to $\bar{E}$ such that $E_0\cap E_1$ is dense in E, let F be another Banach space and let $T\in\mathscr{C}(\bar{E},F)$ be an operator such that $T\colon E_0\to F$ and $T\colon E_1\to F$ are Lipschitz operators and $T:\bar{E}_\Delta\to F$ is compact. Then at least one of the following conditions must hold:*

(i) *$T\colon E\to F$ is compact;*
(ii) *$E_0^\circ\hookrightarrow E$;*
(iii) *$E_1^\circ\hookrightarrow E$.*

THEOREM 4.5 *Let $\bar{F}=(F_0,F_1)$ be a Banach couple, let F be an intermediate space with respect to $\bar{F}$ and let E be a Banach space. Assume that $T\in\mathscr{C}(E,\bar{F})$ is an operator such that $T\colon E\to F_0$ and $T\colon E\to F_1$ are Lipschitz operators.*

(i) *If $\beta_{E,\bar{F}_\Sigma}(T)=0$, then*

$$\beta_{E,F}(T)\le 2\|T\|_{E,\bar{F}}\cdot\left(\lim_{t\to 0}\frac{t}{\rho(t,F,\bar{F})}+\lim_{t\to\infty}\frac{1}{\rho(t,F,\bar{F})}\right).$$

(ii) *If $\beta_{E,\bar{F}_\Sigma}(T)\neq 0$, then*

$$\beta_{E,F}(T)\le\frac{4\beta_{E,\bar{F}_\Sigma}(T)}{\rho\big(\beta_{E,\bar{F}_\Sigma}(T)/\|T\|_{E,\bar{F}},F,\bar{F}\big)}+\frac{4\|T\|_{E,\bar{F}}}{\rho\big(\|T\|_{E,\bar{F}}/\beta_{E,\bar{F}_\Sigma}(T),F,\bar{F}\big)}.$$

Proof Let $\{a+rU_E\}$ be a closed ball in E. For any $\sigma>\beta_{E,\bar{F}_\Sigma}(T)$ there are $z_1,\ldots,z_n\in\{a+rU_E\}$ such that

$$\min_{1\le j\le n}\|Tx-Tz_j\|_{\bar{F}_\Sigma}\le 2r\sigma,$$

for every $x\in\{a+rU_E\}$. Let $x\in\{a+rU_E\}$ and choose z_j such that

$$\|Tx-Tz_j\|_{\bar{F}_\Sigma}\le 2r\sigma.$$

For every $\varepsilon > 0$, there are $y_0 \in F_0$ and $y_1 \in F_1$ such that $Tx - Tz_j = y_0 + y_1$ and

$$\| y_0 \|_{F_0} + \| y_1 \|_{F_1} \leq (1+\varepsilon) \| Tx - Tz_j \|_{\bar{F}_\Sigma} \leq 2(1+\varepsilon) r\sigma.$$

It follows that

$$\begin{aligned} \| y_0 \|_{F_1} &= \| Tx - Tz_j - y_1 \|_{F_1} \\ &\leq \| Tx - Tz_j \|_{F_1} + \| y_1 \|_{F_1} \\ &\leq \| T \|_{E,\bar{F}} \| x - z_j \|_E + 2(1+\varepsilon) r\sigma \\ &\leq 2r \| T \|_{E,\bar{F}} + 2(1+\varepsilon) r\sigma \end{aligned}$$

and

$$\begin{aligned} \| y_1 \|_{F_0} &= \| Tx - Tz_j - y_0 \|_{F_0} \\ &\leq \| Tx - Tz_j \|_{F_0} + \| y_1 \|_{F_0} \\ &\leq \| T \|_{E,\bar{F}} \| x - z_j \|_E + 2(1+\varepsilon) r\sigma \\ &\leq 2r \| T \|_{E,\bar{F}} + 2(1+\varepsilon) r\sigma. \end{aligned}$$

Therefore

$$\begin{aligned} \| Tx - Tz_j \|_F &\leq \| y_0 \|_F + \| y_1 \|_F \\ &\leq \frac{J(t^{-1}, y_0)}{\rho(t^{-1})} + \frac{J(t, y_1)}{\rho(t)} \\ &\leq \frac{2r}{\rho(t^{-1})} \max\left\{ (1+\varepsilon)\sigma, t^{-1} \left[\| T \|_{E,\bar{F}} + (1+\varepsilon)\sigma \right] \right\} \\ &\quad + \frac{2r}{\rho(t)} \max\left\{ \| T \|_{E,\bar{F}} + (1+\varepsilon)\sigma, t(1+\varepsilon)\sigma \right\}. \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} &\| Tx - Tz_j \|_F \\ &\quad \leq \frac{2r}{\rho(t^{-1})} \max\left\{ \sigma, t^{-1} \left[\| T \|_{E,\bar{F}} + \sigma \right] \right\} + \frac{2r}{\rho(t)} \max\left\{ \| T \|_{E,\bar{F}} + \sigma, t\sigma \right\} \end{aligned}$$

and this implies

$$\beta_{E,F}(T) \le \frac{2}{\rho(t^{-1})}\max\left\{\sigma, t^{-1}\left[\|T\|_{E,\bar{F}} + \sigma\right]\right\} + \frac{2}{\rho(t)}\max\{\|T\|_{E,\bar{F}} + \sigma, t\sigma\}.$$

If $\beta_{E,\bar{F}_\Sigma}(T) = 0$, letting $\sigma \to 0$, we obtain

$$\beta_{E,F}(T) \le 2\|T\|_{E,\bar{F}}\left(\frac{t^{-1}}{\rho(t^{-1})} + \frac{1}{\rho(t)}\right),$$

for every $t > 0$. Because $t^{-1}/\rho(t^{-1}) + 1/\rho(t)$ is non-increasing, it follows

$$\beta_{E,F}(T) \le 2\|T\|_{E,\bar{F}}\left(\lim_{t\to 0}\frac{t}{\rho(t)} + \lim_{t\to\infty}\frac{1}{\rho(t)}\right).$$

If $\beta_{E,\bar{F}_\Sigma}(T) \neq 0$, then putting $t = \|T\|_{E,\bar{F}}/\beta_{E,\bar{F}_\Sigma}(T)$ and letting $\sigma \to \beta_{E,\bar{F}_\Sigma}(T)$, we have

$$\beta_{E,F}(T) \le \frac{2}{\rho(\beta_{E,\bar{F}_\Sigma}(T)/\|T\|_{E,\bar{F}})}\max\left\{\beta_{E,\bar{F}_\Sigma}(T), \beta_{E,\bar{F}_\Sigma}(T) + \frac{\beta^2_{E,\bar{F}_\Sigma}(T)}{\|T\|_{E,\bar{F}}}\right\}$$
$$+ \frac{2}{\rho(\|T\|_{E,\bar{F}}/\beta_{E,\bar{F}_\Sigma}(T))}\max\{\|T\|_{E,\bar{F}} + \beta_{E,\bar{F}_\Sigma}(T), \|T\|_{E,\bar{F}}\},$$

and this proves (ii).

COROLLARY 4.6 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple, let F be an intermediate space with respect to $\bar{F}$, let E be a Banach space and let $T \in \mathscr{C}(E, \bar{F})$ such that $T: E \to F_0$ and $T: E \to F_1$ are Lipschitz. If $\lim_{t\to 0} t/\rho(t, F, \bar{F}) = \lim_{t\to\infty} 1/\rho(t, F, \bar{F}) = 0$, then $T: E \to F$ is compact if and only if $T: E \to \bar{F}_\Sigma$ is compact.*

COROLLARY 4.7 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple, let F be an intermediate space of class $\mathscr{C}_J(\theta, \bar{F})$, let E be a Banach space and let $T \in \mathscr{C}(E, \bar{F})$ such that $T: E \to F_0$ and $T: E \to F_1$ are Lipschitz. Then $T: E \to \bar{F}_\Sigma$ is compact if and only if $T: E \to F$ is compact.*

Using Lemma 3.4 of Cobos *et al.* [6] we have the following corollary.

COROLLARY 4.8 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple, let F be an r.o. interpolation space with respect to $\bar{F}$, let E be another Banach space and let $T \in \mathscr{C}(E, \bar{F})$ be an operator such that $T: E \to F_0$ and $T: E \to F_1$ are Lipschitz operators and $T: E \to \bar{F}_\Sigma$ is compact. Then at least one of the following conditions must hold:*

(i) $T: E \to F$ *is compact*;
(ii) $F \hookrightarrow F_0^{\sim}$;
(iii) $F \hookrightarrow F_1^{\sim}$.

The following theorems are generalisations of Theorems 2.7 and 2.8 of Cobos *et al.* [11] for non-linear operators. Since the proofs are essentially the same, we omit them.

THEOREM 4.9 *Let $\bar{E} = (E_0, E_1)$ be a Banach couple and let E be an intermediate space with respect to $\bar{E}$ such that $E_0 \cap E_1$ is dense in E and*

$$\lim_{t\to 0} K(t, x, \bar{E}) = \lim_{t\to\infty} \frac{K(t, x, \bar{E})}{t} = 0 \quad \textit{for all } x \in E.$$

Then the following are equivalent:

(i) $\lim_{t\to 0} \psi(t, E, \bar{E}) = \lim_{t\to\infty} \psi(t, E, \bar{E})/t = 0$;
(ii) *for every Banach space F, if $T \in \mathscr{C}(\bar{E}, F)$ is an operator such that $T: E_0 \to F$ and $T: E_1 \to F$ are Lipschitz operator and $T: \bar{E}_\Delta \to F$ is compact, then $T: E \to F$ is a compact operator*;
(iii) *if $T \in \mathscr{C}(\bar{E}, \ell_\infty)$ is an operator such that $T: E_0 \to \ell_\infty$ and $T: E_1 \to \ell_\infty$ are Lipschitz operators and $T: \bar{E}_\Delta \to \ell_\infty$ is compact, then $T: E \to \ell_\infty$ is a compact.*

THEOREM 4.10 *Let $\bar{F} = (F_0, F_1)$ be a Banach couple and let F be an intermediate space with respect to $\bar{F}$. Then the following are equivalent:*

(i) $\lim_{t\to 0} 1/\rho(t, F, \bar{F}) = \lim_{t\to\infty} t/\rho(t, F, \bar{F}) = 0$;
(ii) *for every Banach space E, if $T \in \mathscr{C}(E, \bar{F})$ is an operator such that $T: E \to F_0$ and $T: E \to F_1$ are Lipschitz operators and $T: E \to \bar{F}_\Sigma$ is compact, then $T: E \to F$ is compact*;
(iii) *if $T \in \mathscr{C}(\ell_1, \bar{F})$ is an operator such that $T: \ell_1 \to F_0$ and $T: \ell_1 \to F_1$ are Lipschitz operators and $T: \ell_1 \to \bar{F}_\Sigma$ is compact, then $T: \ell_1 \to F$ is compact.*

5 MAIN RESULTS

Given a sequence of Banach spaces $(W_m)_{m\in\mathbb{Z}}$ and a sequence of non-negative numbers $(\lambda_m)_{m\in\mathbb{Z}}$, we write $\ell_q(\lambda_m W_m)$ to designate the vector-valued space

$$\ell_q(\lambda_m W_m) = \{w = (w_m)\colon w_m \in W_m,\ \|w\|_{\ell_q(\lambda_m W_m)} < \infty\},$$

where

$$\|w\|_{\ell_q(\lambda_m W_m)} = \begin{cases} \left(\sum_{m=-\infty}^{\infty}(\lambda_m\|w_m\|_{W_m})^q\right)^{1/q} & \text{if } q<\infty,\\ \sup_{m\in\mathbb{Z}}\lambda_m\|w_m\|_{W_m} & \text{if } q=\infty. \end{cases}$$

THEOREM 5.1 *Let $0<\theta<1$ and $1\leq q<\infty$, let $\bar{E}=(E_0,E_1)$ and $\bar{F}=(F_0,F_1)$ be two Banach couples and let $T\in\mathscr{C}(\bar{E},\bar{F})$ be an operator such that $T\colon E_0\longrightarrow F_0$ and $T\colon E_1\longrightarrow F_1$ are Lipschitz operators.*

(i) *If $T\colon E_0\longrightarrow F_0$ and $T\colon E_1\longrightarrow F_1$ are compact, then $T:\bar{E}_{\theta,q}\longrightarrow\bar{F}_{\theta,q}$ is compact.*
(ii) *If $\beta_0(T)\neq 0$ or $\beta_1(T)\neq 0$, then there is a constant $c=c(\theta)>0$ such that*

$$\beta_{\theta,q}\leq 2^{\theta+1}\beta_0^{1-\theta}\beta_1^{\theta}\left(1+\frac{\|T\|_0+\|T\|_1}{\beta_0+\beta_1}\right)+c\beta_0^{1-\theta}\|T\|_1^{\theta}+c\beta_1^{\theta}\|T\|_0^{1-\theta},$$

where $\beta_{\theta,q}=\beta_{\theta,q}(T)$, $\beta_0=\beta_0(T)$ and $\beta_1=\beta_1(T)$.

Proof Put $W_m := (F_0+F_1, K(2^m,\cdot,\bar{F}))$, $m\in\mathbb{Z}$, and consider the operator j that associates to every $y\in F_0+F_1$ the constant sequence $j(y)=(\ldots,y,y,y,\ldots)$. The restriction of j to $\bar{F}_{\theta,q}$ is a metric injection from $\bar{F}_{\theta,q}$ into $\ell_q(2^{-\theta m}W_m)$. Moreover, the restrictions of j to F_0 (resp. F_1) is a bounded operator from F_0 (resp. F_1) into $\ell_\infty(W_m)$ (resp. $\ell_\infty(2^{-m}W_m)$) with norm less than or equal to one. Furthermore,

$$(\ell_\infty(W_m),\ell_\infty(2^{-m}W_m))_{\theta,q}=\ell_q(2^{-\theta m}W_m)$$

with equivalence of norms and the embedding

$$(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{\theta,q} \hookrightarrow \ell_q(2^{-\theta m}W_m)$$

has norm less than or equal to one.

Let $\hat{T} = jT$. We have the following diagram of operators:

$$\begin{array}{c} E_0 \xrightarrow{T} F_0 \xrightarrow{j} \ell_\infty(W_m) \\ \underline{E_1 \xrightarrow{T} F_1 \xrightarrow{j} \ell_\infty(2^{-m}W_m)} \\ \hline \bar{E}_{\theta,q} \xrightarrow{T} \bar{F}_{\theta,q} \xrightarrow{j} \ell_q(2^{-\theta m}W_m) \end{array}$$

Since j is a metric injection, we have

$$\beta_{\theta,q}(T) \leq 2\beta_{\theta,q}(\hat{T}).$$

For every $n \in \mathbb{N}$, let P_n, Q_n^+ and Q_n^- be linear operators on the Banach couple $(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))$ defined by

$$\begin{aligned} P_n(u_m) &= (\ldots, 0, 0, u_{-n}, u_{-n+1}, \ldots, u_{n-1}, u_n, 0, 0, \ldots), \\ Q_n^+(u_m) &= (\ldots, 0, 0, \ldots, u_{n+1}, u_{n+2}, \ldots), \\ Q_n^-(u_m) &= (\ldots, u_{-n-2}, u_{-n-1}, 0, 0, \ldots). \end{aligned}$$

These operators have the following properties:

(I) the identity operator on $\ell_\infty(W_m) + \ell_\infty(2^{-m}W_m)$ can be decomposed as

$$I = P_n + Q_n^+ + Q_n^-, \quad n = 1, 2, \ldots;$$

(II) they are uniformly bounded

$$\|P_n\|_{\ell_\infty(W_m),\ell_\infty(W_m)} = \|P_n\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(2^{-m}W_m)} = 1$$

and similarly for Q_n^+ and Q_n^-;

(III) the operator Q_n^+ maps $\ell_\infty(W_m)$ boundedly into $\ell_\infty(2^{-m}W_m)$, the operator Q_n^- maps $\ell_\infty(2^{-m}W_m)$ boundedly into $\ell_\infty(W_m)$ and

$$\|Q_n^+\|_{\ell_\infty(W_m),\ell_\infty(2^{-m}W_m)} = \|Q_n^-\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(W_m)} = 2^{-n-1}.$$

The operator $\hat{T}$ can be decomposed as

$$\hat{T} = P_n\hat{T} + Q_n^+\hat{T} + Q_n^-\hat{T}$$

and this implies

$$\beta_{\theta,q}(\hat{T}) \leq \beta_{\theta,q}(P_n\hat{T}) + \beta_{\theta,q}(Q_n^+\hat{T}) + \beta_{\theta,q}(Q_n^-\hat{T}).$$

We will now estimate each one of the terms on the right hand side of the last inequality. For that, let $\sigma_0 > \beta_0(T)$ and $\sigma_1 > \beta_1(T)$.

Let us start with $\beta_{\theta,q}(Q_n^-\hat{T})$. Let E_0° and E_1° be the closures of $E_0 \cap E_1$ in E_0 and in E_1, respectively, and put $\bar{E}^\circ = (E_0^\circ, E_1^\circ)$. Since $\bar{E}_{\theta,q}^\circ = \bar{E}_{\theta,q}$ with equivalence of norms, we have

$$\begin{aligned}
\beta_{\theta,q}(Q_n^-\hat{T}) &= \beta_{\bar{E}_{\theta,q},\ell_q(2^{-\theta m}W_m)}(Q_n^-\hat{T}) \\
&\leq c_1\beta_{\bar{E}_{\theta,q}^\circ,\ell_q(2^{-\theta m}W_m)}(Q_n^-\hat{T}) \\
&\leq c_1\|Q_n^-\hat{T}\|_{\bar{E}_{\theta,q}^\circ,\ell_q(2^{-m\theta}W_m)} \\
&\leq c_2\|Q_n^-\hat{T}\|_{E_0^\circ,\ell_\infty(W_m)}^{1-\theta}\|Q_n^-\hat{T}\|_{E_1^\circ,\ell_\infty(2^{-m}W_m)}^{\theta} \\
&\leq c_2\|Q_n^-\hat{T}\|_{E_0^\circ,\ell_\infty(W_m)}^{1-\theta}\|T\|_{E_1,F_1}^{\theta}.
\end{aligned}$$

Given $\varepsilon > 0$, choose $x, y \in E_0^\circ$ such that

$$\|Q_n^-\hat{T}\|_{E_0^\circ,\ell_\infty(W_m)} \leq \frac{\|Q_n^-\hat{T}x - Q_n^-\hat{T}y\|_{\ell_\infty(W_m)}}{\|x-y\|_{E_0^\circ}} + \frac{\varepsilon}{2}.$$

Put $z = (x+y)/2$ and $r = \|x-y\|_{E_0^\circ}/2$. Since $\sigma_0 > \beta_0(T) \geq \beta_{E_0^\circ,\ell_\infty(W_m)}(\hat{T})$, there are $x_1, \ldots, x_k \in \{z + rU_{E_0^\circ}\} \cap E_1$ such that

$$\min_{1\leq i\leq k}\|\hat{T}w - \hat{T}x_i\|_{\ell_\infty(W_m)} \leq 2r\sigma_0,$$

for every $w \in \{z + rU_{E_0^\circ}\}$. In particular, there are x_i and x_j such that

$$\|\hat{T}x - \hat{T}x_i\|_{\ell_\infty(W_m)} \leq 2r\sigma_0 \ \text{ and } \ \|\hat{T}y - \hat{T}x_j\|_{\ell_\infty(W_m)} \leq 2r\sigma_0.$$

By property (III), it follows that

$$\begin{aligned}\|Q_n^- \hat{T}x_i - Q_n^- \hat{T}x_j\|_{\ell_\infty(W_m)} &\leq \|Q_n^-\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(W_m)} \|\hat{T}x_i - \hat{T}x_j\|_{\ell_\infty(2^{-m}W_m)} \\ &\leq 2^{-n-1} \|\hat{T}x_i - \hat{T}x_j\|_{\ell_\infty(2^{-m}W_m)}.\end{aligned}$$

Hence, there is $N_1 \in \mathbb{N}$ such that, for every $n \geq N_1$,

$$\|Q_n^- \hat{T}x_i - Q_n^- \hat{T}x_j\|_{\ell_\infty(W_m)} \leq r\varepsilon.$$

Therefore, for every $n \geq N_1$,

$$\begin{aligned}&\|Q_n^- \hat{T}x - Q_n^- \hat{T}y\|_{\ell_\infty(W_m)} \\ &\quad\leq \|Q_n^- \hat{T}x - Q_n^- \hat{T}x_i\|_{\ell_\infty(W_m)} + \|Q_n^- \hat{T}x_i - Q_n^- \hat{T}x_j\|_{\ell_\infty(W_m)} \\ &\qquad + \|Q_n^- \hat{T}x_j - Q_n^- \hat{T}y\|_{\ell_\infty(W_m)} \\ &\quad\leq 4r\sigma_0 + r\varepsilon,\end{aligned}$$

and this implies, for every $n \geq N_1$,

$$\|Q_n^- \hat{T}\|_{E_0^\circ,\ell_\infty(W_m)} \leq 2\sigma_0 + \varepsilon.$$

Consequently,

$$\beta_{\theta,q}(Q_n^- \hat{T}) \leq c_2 \|T\|_1^\theta (2\sigma_0 + \varepsilon)^{1-\theta}.$$

Similarly, for every $\varepsilon > 0$, there is $N_2 \in \mathbb{N}$ such that, for every $n \geq N_2$,

$$\beta_{\theta,q}(Q_n^+ \hat{T}) \leq c_2 \|T\|_0^{1-\theta} (2\sigma_1 + \varepsilon)^\theta.$$

We now estimate $\beta_{\theta,q}(P_n \hat{T})$. Let ℓ_q^{2n+1} be $\mathbb{R}^{2n+1}$ with the ℓ_q-norm. Since ℓ_q^{2n+1} is finite dimensional, given any $\varepsilon > 0$, there exist $\mu_1, \ldots, \mu_k \in \ell_q^{2n+1}$ such that

$$U_{\ell_q^{2n+1}} \subseteq \bigcup_{i=1}^{k} \{\mu_i + \varepsilon U_{\ell_q^{2n+1}}\}.$$

Let $\{a + rU_{E_0 \cap E_1}\}$ be a closed ball in $(E_0 \cap E_1, \|\cdot\|_{\bar{E}_{\theta,q}})$ and take $\nu \in \mathbb{Z}$ such that $2^{\nu-1} < \sigma_1/\sigma_0 \leq 2^{\nu}$. Then, for every $x \in \{a + rU_{E_0 \cap E_1}\}$,

$$\left(\sum_{m=-n}^{n} \left(2^{-\theta(m+\nu)} K(2^{m+\nu}, x-a)\right)^q\right)^{1/q} \leq \|x-a\|_{\bar{E}_{\theta,q}} \leq r$$

and this implies that there is some $\mu_i \in \{\mu_1, \ldots, \mu_k\}$ such that

$$2^{-\theta(m+\nu)} K(2^{m+\nu}, x-a) \leq r(\mu_m^{(i)} + \varepsilon), \quad m = -n, \ldots, n,$$

where $\mu_i = (\mu_{-n}^{(i)}, \ldots, \mu_n^{(i)})$. It follows that

$$\begin{aligned} K\left(2^m \frac{\sigma_1}{\sigma_0}, x-a\right) &\leq K(2^{m+\nu}, x-a) \\ &\leq r2^{\theta(m+\nu)}(\mu_m^{(i)} + \varepsilon) \\ &< r2^{\theta}\left(2^m \frac{\sigma_1}{\sigma_0}\right)^{\theta} (\mu_m^{(i)} + \varepsilon), \end{aligned}$$

$m = -n, \ldots, n$. By the definition of the K-functional there exist $x_m^{(0)} \in E_0$ and $x_m^{(1)} \in E_1$ such that $x - a = x_m^{(0)} + x_m^{(1)}$ and

$$\|x_m^{(0)}\|_{E_0} + 2^m \frac{\sigma_1}{\sigma_0} \|x_m^{(1)}\|_{E_1} \leq r2^{\theta}\left(2^m \frac{\sigma_1}{\sigma_0}\right)^{\theta} (\mu_m^{(i)} + \varepsilon),$$

$m = -n, \ldots, n$. From the last inequality we get

$$\|x_m^{(0)}\|_{E_0} \leq r2^{\theta} 2^{m\theta} \sigma_0^{-\theta} \sigma_1^{\theta} (\mu_m^{(i)} + \varepsilon)$$

and

$$\|x_m^{(1)}\|_{E_1} \leq r2^{\theta} 2^{m(\theta-1)} \sigma_0^{1-\theta} \sigma_1^{\theta-1} (\mu_m^{(i)} + \varepsilon),$$

$m = -n, \ldots, n$. Because $\sigma_0 > \beta_{E_0,F_0}(T)$ and $\sigma_1 > \beta_{E_1,F_1}(T)$ there exist $y_{1,m}^{(i)}, \ldots, y_{p(i),m}^{(i)} \in F_0$ and $z_{1,m}^{(i)}, \ldots, z^{(i)}{}_{t(i),m} \in F_1$ such that

$$\min_{1 \leq w \leq p(i)} \|Ty - y_{w,m}^{(i)}\|_{F_0} \leq r2^{\theta} 2^{m\theta} \sigma_0^{1-\theta} \sigma_1^{\theta} (\mu_m^{(i)} + \varepsilon),$$

for every $y \in \{a + r2^{\theta}2^{m\theta}\sigma_0^{-\theta}\sigma_1^{\theta}(\mu_m^{(i)} + \varepsilon)U_{E_0}\}$ and

$$\min_{1\le s\le t(i)} \|Tz - z_{s,m}^{(i)}\|_{F_1} \le r2^{\theta}2^{m(\theta-1)}\sigma_0^{1-\theta}\sigma_1^{\theta}(\mu_m^{(i)} + \varepsilon),$$

for every $z \in \{a + r2^{\theta}2^{m(\theta-1)}\sigma_0^{1-\theta}\sigma_1^{\theta-1}(\mu_m^{(i)} + \varepsilon)U_{E_1}\}$. Let $\alpha_0 = \sigma_0/(\sigma_0+\sigma_1)$, let $\alpha_1 = \sigma_1/(\sigma_0+\sigma_1)$ and let $u_{w,s}^{(i)} = (u_{w,s,m}^{(i)})$ be the vector-valued sequence defined by

$$u_{w,s,m}^{(i)} = \begin{cases} 0 & \text{if } m > n \text{ or } m < -n, \\ \alpha_1 y_{w,m}^{(i)} + \alpha_0 z_{s,m}^{(i)} & \text{if } -n \le m \le n, \end{cases}$$

$i = 1, \ldots, k$, $w = 1, \ldots, p(i)$ and $s = 1, \ldots, t(i)$. Given any $x \in \{a + rU_{E_0\cap E_1}\}$, there exists a $u_{w,s}^{(i)}$ such that $u_{w,s,m}^{(i)} = \alpha_1 y_{w,m}^{(i)} + \alpha_0 z_{s,m}^{(i)}$,

$$\|T(a + x_m^{(0)}) - y_{w,m}^{(i)}\|_{F_0} \le r2^{\theta}2^{m\theta}\sigma_0^{1-\theta}\sigma_1^{\theta}(\mu_m^{(i)} + \varepsilon)$$

and

$$\|T(a + x_m^{(1)}) - z_{s,m}^{(i)}\|_{F_1} \le r2^{\theta}2^{m(\theta-1)}\sigma_0^{1-\theta}\sigma_1^{\theta}(\mu_m^{(i)} + \varepsilon)$$

with $x - a = x_m^{(0)} + x_m^{(1)}$, $m = -n, \ldots, n$. From

$$\begin{aligned} &K(2^m, Tx - u_{w,s,m}^{(i)}) \\ &\quad = K(2^m, Tx - \alpha_1 y_{w,m}^{(i)} - \alpha_0 z_{s,m}^{(i)}) \\ &\quad \le \alpha_0\|Tx - T(a + x_m^{(1)})\|_{F_0} + 2^m\alpha_1\|Tx - T(a + x_m^{(0)})\|_{F_1} \\ &\qquad + \alpha_1\|T(a + x_m^{(0)}) - y_{w,m}^{(i)}\|_{F_0} + 2^m\alpha_0\|T(a + x_m^{(1)}) - z_{s,m}^{(i)}\|_{F_1} \\ &\quad \le \alpha_0\|T\|_0\|x_m^{(0)}\|_{E_0} + 2^m\alpha_1\|T\|_1\|x_m^{(1)}\|_{E_1} \\ &\qquad + (\alpha_0 + \alpha_1)r2^{\theta}2^{m\theta}\sigma_0^{1-\theta}\sigma_1^{\theta}(\mu_m^{(i)} + \varepsilon) \\ &\quad \le r2^{\theta}2^{m\theta}\sigma_0^{1-\theta}\sigma_1^{\theta}(\mu_m^{(i)} + \varepsilon)\left(1 + \frac{\|T\|_0 + \|T\|_1}{\sigma_0 + \sigma_1}\right), \end{aligned}$$

it follows that

$$\|P_n\hat{T}x - u_{w,s}^{(i)}\| = \left(\sum_{m=-n}^{n} \left(2^{-\theta m} K(2^m, Tx - u_{w,s,m}^{(i)}) \right)^q \right)^{1/q}$$

$$\leq r 2^\theta \sigma_0^{1-\theta} \sigma_1^\theta \left(1 + \frac{\|T\|_0 + \|T\|_1}{\sigma_0 + \sigma_1} \right) \left(\sum_{m=-n}^{n} (\mu_m^{(i)} + \varepsilon)^q \right)^{1/q}$$

$$\leq r 2^\theta \sigma_0^{1-\theta} \sigma_1^\theta \left(1 + \frac{\|T\|_0 + \|T\|_1}{\sigma_0 + \sigma_1} \right) (1 + (2n+1)^{1/q} \varepsilon).$$

Using Theorem 2.2, we obtain

$$\beta_{\theta,q}(P_n\hat{T}) \leq 2^\theta \sigma_0^{1-\theta} \sigma_1^\theta \left(1 + \frac{\|T\|_0 + \|T\|_1}{\sigma_0 + \sigma_1} \right) (1 + (2n+1)^{1/q} \varepsilon)$$

and this implies

$$\beta_{\theta,q}(P_n\hat{T}) \leq 2^\theta \sigma_0^{1-\theta} \sigma_1^\theta \left(1 + \frac{\|T\|_0 + \|T\|_1}{\sigma_0 + \sigma_1} \right),$$

for every $n \in \mathbb{N}$.

Therefore, for every $\sigma_0 > \beta_0(T)$, every $\sigma_1 > \beta_1(T)$ and every $\varepsilon > 0$, we have

$$\beta_{\theta,q}(T) \leq 2^{\theta+1} \sigma_0^{1-\theta} \sigma_1^\theta \left(1 + \frac{\|T\|_0 + \|T\|_1}{\sigma_0 + \sigma_1} \right)$$
$$+ c_2 \|T\|_1^\theta (2\sigma_0 + \varepsilon)^{1-\theta} + c_2 \|T\|_0^{1-\theta} (2\sigma_1 + \varepsilon)^\theta.$$

If $\beta_0(T) = \beta_1(T) = 0$, then letting first $\sigma_0 \to 0$ and after $\sigma_1, \varepsilon \to 0$ we have $\beta_{\theta,q}(T) = 0$. If $\beta_0(T) \neq 0$ or $\beta_1(T) \neq 0$, then letting $\sigma_i \to \beta_i(T)$, $i = 0, 1$, and $\varepsilon \to 0$ we obtain (ii).

THEOREM 5.2 *Let* $0 < \theta < 1$ *and* $1 \leq q < \infty$, *let* $\bar{E} = (E_0, E_1)$ *and* $\bar{F} = (F_0, F_1)$ *be two Banach couples and let* $T \in \mathscr{C}(\bar{E}, \bar{F})$ *be an operator such that* $T: E_0 \to F_0$ *and* $T: E_1 \to F_1$ *are Lipschitz operators. Suppose that*

E_1 is continuously embedded in E_0 or F_1 is continuously embedded in F_0.

(i) *If $T: E_0 \to F_0$ is compact, then $T: \bar{E}_{\theta,q} \to \bar{F}_{\theta,q}$ is compact.*
(ii) *If $\beta_0(T) \neq 0$, then there is a constant $c = c(\theta)$ such that*

$$\beta_{\theta,q}(T) \leq 2^{\theta+1}\beta_0^{1-\theta}(T)\beta_1^{\theta}(T)\left(1+\frac{\|T\|_0+\|T\|_1}{\beta_0(T)+\beta_1(T)}\right)+c\beta_0^{1-\theta}(T)\|T\|_1^{\theta}.$$

Proof As in the proof of Theorem 5.1, for every $\sigma_0 > \beta_0(T)$, every $\sigma_1 > \beta_1(T)$ and every $\varepsilon > 0$, there is $N_1 \in \mathbb{N}$ such that

$$\beta_{\theta,q}(Q_n^- \hat{T}) \leq c_1 \|T\|_1^{\theta}(2\sigma_0+\varepsilon)^{1-\theta},$$

for any $n \geq N_1$ and

$$\beta_{\theta,q}(P_n \hat{T}) \leq 2^{\theta}\sigma_0^{1-\theta}\sigma_1^{\theta}\left(1+\frac{\|T\|_0+\|T\|_1}{\sigma_0+\sigma_1}\right),$$

for any $n \in \mathbb{N}$. For $\beta_{\theta,q}(Q_n^+ \hat{T})$, we have

$$\begin{aligned}\beta_{\theta,q}(Q_n^+ \hat{T}) &\leq \|Q_n^+ \hat{T}\|_{\bar{E}_{\theta,q},\ell_q(2^{-m\theta}W_m)} \\ &\leq 2^{\theta}\|Q_n^+ \hat{T}\|_{E_0,\ell_\infty(W_m)}^{1-\theta}\|Q_n^+ \hat{T}\|_{E_1,\ell_\infty(2^{-m}W_m)}^{\theta} \\ &\leq 2^{\theta}\|T\|_{E_0,F_0}^{1-\theta}\|Q_n^+ \hat{T}\|_{E_1,\ell_\infty(2^{-m}W_m)}^{\theta}.\end{aligned}$$

In the case $E_1 \hookrightarrow E_0$, let $I: E_1 \to E_0$ be the embedding from E_1 into E_0. Then

$$\begin{aligned}\|Q_n^+ \hat{T}\|_{E_1,\ell_\infty(2^{-m}W_m)} &\leq \|Q_n^+\|_{\ell_\infty(W_m),\ell_\infty(2^{-m}W_m)}\|\hat{T}\|_{E_0,\ell_\infty(W_m)}\|I\|_{E_1,E_0} \\ &\leq 2^{-n-1}\|T\|_{E_0,F_0}\|I\|_{E_1,E_0}.\end{aligned}$$

If $F_1 \hookrightarrow F_0$, we have

$$\begin{aligned}\|Q_n^+ \hat{T}\|_{E_1,\ell_\infty(2^{-m}W_m)} &\leq \|Q_n^+\|_{\ell_\infty(W_m),\ell_\infty(2^{-m}W_m)}\|j\|_{F_0,\ell_\infty(W_m)}\|J\|_{F_1,F_0}\|T\|_{E_1,F_1} \\ &\leq 2^{-n-1}\|J\|_{F_1,F_0}\|T\|_{E_1,F_1},\end{aligned}$$

where J is the embedding from F_1 into F_0. In both cases we have

$$\|Q_n^+ \hat{T}\|_{E_1,\ell_\infty(2^{-m}W_m)} \to 0,$$

when $n \to \infty$ and this implies

$$\beta_{\theta,q}(Q_n^+ \hat{T}) \to 0,$$

when $n \to \infty$. As in the proof of Theorem 5.1 we conclude (i) and (ii).

6 REMARKS IN THE LINEAR CASE

The following theorem is mentioned in the introduction of [6].

THEOREM 6.1 *Let $\bar{E} = (E_0, E_1)$ be a Banach couple, let F be Banach space, let E be an intermediate space with respect to $\bar{E}$ and let $T \in \mathscr{L}(\bar{E}, F)$. If $T: E_0 \to F$ and $T: E_1 \to F$ are compact, then $T: E \to F$ is compact.*

We say that a Banach couple $\bar{F} = (F_0, F_1)$ has the approximation property H_1 if there is a positive constant c such that given any $\varepsilon > 0$ and any finite sets $K_0 \subset F_0$ and $K_1 \subset F_1$, there is an operator $P \in \mathscr{L}(\bar{F}, \bar{F})$ such that

(i) $P(F_i) \subseteq F_0 \cap F_1$, $i = 0, 1$;
(ii) $\|I - P\|_{F_i,F_i} \leq c$, $i = 0, 1$;
(iii) $\|x - Px\|_{F_i} < \varepsilon$ for all $x \in K_i$, $i = 0, 1$.

We say that the Banach couple $\bar{F} = (F_0, F_1)$ has the approximation property H_2 if has the approximation property H_1 and
(iv) $P: F_i \to F_i$ is compact, $i = 0, 1$.

Remark 6.2 In [16] it is proved that if X is a locally compact space endowed with a positive measure μ, then the Banach couple $(L^p(X, \mu), L^q(X, \mu))$ satisfies the approximation property H_2 for $p, q \in [1, \infty)$.

We shall need the following lemma from [16]:

LEMMA 6.3 *Let $\bar{E} = (E_0, E_1)$ and $\bar{F} = (F_0, F_1)$ be two Banach couples, suppose that $\bar{F}$ has the approximation property H_1, let Φ be an interpolation method and let $T \in \mathscr{L}(\bar{E}, \bar{F})$. Then given any $\varepsilon > 0$, there exists*

$P \in \mathscr{L}(\bar{F}, \bar{F})$ *verifying* (i), (ii), (iii) *and*

$$\|T - PT\|_{E_i,F_i} \le c\beta_{E_i,F_i}(T) + \varepsilon, \quad i = 0, 1.$$

Moreover, if $\bar{F}$ *has the approximation property* H_2, *then* P *also verifies* (iv).

THEOREM 6.4 *Let* $\bar{E} = (E_0, E_1)$ *be a Banach couple, let* $\bar{F} = (F_0, F_1)$ *be a Banach couple satisfying the approximation property* H_1, *let* Φ *be an interpolation method and let* $T \in \mathscr{L}(\bar{E}, \bar{F})$. *If* $T: E_0 \to F_0$ *and* $T: E_1 \to F_1$ *are compact, then* $T: \bar{E}_\Phi \to \bar{F}_\Phi$ *is compact.*

Proof Let $\varepsilon > 0$ and let c be the constant in inequality (2.2). By Lemma 6.3 there is $P \in \mathscr{L}(\bar{F}, \bar{F})$ satisfying (i), (ii), (iii) and

$$\|T - PT\|_{E_i,F_i} \le \frac{\varepsilon}{c}, \quad i = 0, 1.$$

By inequality (2.2) we have

$$\|T - PT\|_{\bar{E}_\Phi,\bar{F}_\Phi} \le \varepsilon,$$

i.e., $T: \bar{E}_\Phi \to \bar{F}_\Phi$ can be approximated uniformly by operators $PT: \bar{E}_\Phi \to \bar{F}_\Phi$. If we prove that the operators $PT: \bar{E}_\Phi \to \bar{F}_\Phi$ are compact then the result follows immediately. Using the following diagram:

$$\begin{array}{ccccc} E_0 & \xrightarrow{T} & F_0 & \searrow^{P} & & & \\ & & & & F_0 \cap F_1 & \overset{J}{\hookrightarrow} & \bar{F}_\Phi \\ E_1 & \xrightarrow{T} & F_1 & \nearrow_{P} & & & \end{array}$$

we see that $PT: E_0 \to \bar{F}_\Phi$ and $PT: E_1 \to \bar{F}_\Phi$ are compact. By Theorem 6.1 it follows that $PT: \bar{E}_\Phi \to \bar{F}_\Phi$ is compact.

THEOREM 6.5 *Let* $\bar{E} = (E_0, E_1)$ *and* $\bar{F} = (F_0, F_1)$ *be two Banach couples, let* Φ *be an interpolation method and let* $T \in \mathscr{L}(\bar{E}, \bar{F})$. *If* $\bar{F} = (F_0, F_1)$ *has the approximation property* H_2, *then*

$$\beta_{\bar{E}_\Phi,\bar{F}_\Phi}(T) \le c \max\{\beta_{E_0,F_0}(T), \beta_{E_1,F_1}(T)\}.$$

Proof Let $\varepsilon > 0$. By Lemma 6.3 there exists $P \in \mathscr{L}(\bar{F}, \bar{F})$ satisfying (i), (ii), (iii), (iv) and

$$\|T - PT\|_{E_i,F_i} \leq c\beta_{E_i,F_i}(T) + \varepsilon, \quad i = 0, 1.$$

Since $PT: E_0 \to F_0$ and $PT: E_1 \to F_1$ are compact, by Theorem 6.4 $PT: \bar{E}_\Phi \to \bar{F}_\Phi$ is compact. By Lemma 6.3, it follows that

$$\begin{aligned}\beta_{\bar{E}_\Phi,\bar{F}_\Phi}(T) &\leq \beta_{\bar{E}_\Phi,\bar{F}_\Phi}(PT) + \|T - PT\|_{\bar{E}_\Phi,\bar{F}_\Phi} \\ &\leq c_1 \max\{\|T - PT\|_{E_0,F_0}, \|T - PT\|_{E_1,F_1}\} \\ &\leq c_2 \max\{\beta_{E_0,F_0}(T), \beta_{E_1,F_1}(T)\},\end{aligned}$$

and the proof is finished.

COROLLARY 6.6 *Let $\bar{E} = (E_0, E_1)$ be a Banach couple, let Φ be an interpolation method and let $T \in \mathscr{L}(\bar{E}, \bar{E})$. If $\bar{E} = (E_0, E_1)$ has the approximation property H_2, then*

$$r_{\mathrm{e}}^{\bar{E}_\Phi}(T) \leq \max\{r_{\mathrm{e}}^{E_0}(T), r_{\mathrm{e}}^{E_1}(T)\}.$$

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