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Whitney Covers and Quasi-isometry of $L^s(\mu)$-averaging Domains

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This is a part of our series studies about the $L^s(\mu)$-averaging domains. In this paper, we first characterize $L^s(\mu)$-averaging domains using the Whitney covers. Then we prove the invariance of $L^s(\mu)$-averaging domains under some mappings, such as K-quasi-isometric mappings, φ-quasi-isometric mappings.

Keywords and Phrases: $L^s(\mu)$-averaging domain; Whitney covers; φ-quasi-isometry

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1. INTRODUCTION

Domains and mappings are studied and applied in many different fields in mathematics and engineering, such as ordinary and partial differential equations, potential theory and nonlinear elasticity, see [3, 5, 7, 8, 11, 13]. Gehring and Osgood study the uniform domains and the quasihyperbolic metric in [5]. As we know, uniform domains are John domains, while John domains are L^s-averaging domains and L^s (μ)-averaging domains are extensions of L^s-averaging domains. There has been remarkable progress made in studying these domains and

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their relationships, particularly, their properties and applications, see the references listed above. Recently, some results about A-harmonic tensors in John domains and $L^s(\mu)$-averaging domains are obtained in [1, 2 and 9]. In this paper we first characterize $L^s(\mu)$-averaging domains using the Whitney covers. Then we study the properties of $L^s(\mu)$-averaging domains under some mappings. We introduce the following definitions and theorems which we need later. We will always denote Ω as an open connected subset of R^n and we do not distinguish the balls from the cubes throughout this paper. The following Definition 1.1 appears in [3].

DEFINITION 1.1 We call a proper subdomain $\Omega \subset R^n$ an $L^s(\mu)$-averaging domain, if for $s \geq 1$ and $\mu(\Omega) < \infty$ there is a constant C such that

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega |u - u_{B_{0,\mu}}|^s d\mu\right)^{(1/s)} \leq C \sup_{2B\subset\Omega}\left(\frac{1}{\mu(B)}\int_B |u - u_{B,\mu}|^s d\mu\right)^{(1/s)} \tag{1.2}$$

for some ball $B_0 \subset \Omega$ and all $u \in L^s_{\text{loc}}(\Omega, \mu)$, where the measure μ is defined by $d\mu = w(x)dx$, $w(x)$ is a weight and the supremum is over all balls B with $2B \subset \Omega$.

DEFINITION 1.3 Let $\sigma > 1$. We say that w satisfies a weak reverse Hölder inequality and write $w \in WRH(\Omega)$ when there exist constants $\beta > 1$ and $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w^\beta dx\right)^{(1/\beta)} \leq C\frac{1}{|B|}\int_{\sigma B} w dx \tag{1.4}$$

for all balls $B \subset \Omega$ with $\sigma B \subset \Omega$. We say that w satisfies a reverse Hölder inequality when (1.4) holds with $\sigma = 1$ and we write $w \in RH(\Omega)$.

DEFINITION 1.5 We call w a doubling weight and write $w \in D(\Omega)$ if there exists a constant C such that $\mu(2B) \leq C\mu(B)$ for all balls B with $2B \subset \Omega$. If this condition holds only for all balls B with $4B \subset \Omega$, then w is weak doubling and denote $w \in WD(\Omega)$. The factor 4 here is for convenience and in fact these domains are independent of this expansion factor, see [3].

DEFINITION 1.6 The quasi-hyperbolic distance between x and y in a domain Ω is given by

$$k(x,y) = k(x,y;\Omega) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z,\partial\Omega)} ds,$$

where γ is any rectifiable curve in Ω joining x to y, $d(z, \partial\Omega)$ is the Euclidean distance between z and the boundary of Ω. Gehring and Osgood prove that for any two points x and y in Ω there is a quasi-hyperbolic geodesic arc joining them, see [5]. The quasi-hyperbolic metric provides a useful substitute for the hyperbolic metric. Applications can be found, for example, in [4–6, 10, 12]. We will show that it also plays an important role in describing the $L^s(\mu)$-averaging domains. The following Theorems 1.7 and 1.9 are given by Ding and Nolder [3].

THEOREM 1.7 *If w satisfies the reverse Hölder inequality and Ω is an $L^s(\mu)$-averaging domain, then there exists a constant A such that*

$$\left(\frac{1}{\mu(\Omega)} \int_{\Omega} k(x,x_0)^s d\mu\right)^{(1/s)} \leq A, \tag{1.8}$$

where A only depends on n, s, $\mu(\Omega)$, $\mu(B(x_0, d(x_0, \partial\Omega)/2))$ and the constant C in (1.2).

THEOREM 1.9 *Let w be weak doubling over Ω, see* [3]. *If*

$$\left(\frac{1}{\mu(\Omega)} \int_{\Omega} k(x,x_0)^s d\mu\right)^{(1/s)} \leq A \tag{1.10}$$

for some fixed point x_0 in Ω and a constant A, then Ω is an $L^s(\mu)$-averaging domain and inequality (1.2) *holds with constant C depending on n, s and A.*

DEFINITION 1.11 We say that a weight w satisfies the A_r-condition, where $r > 1$, and write $w \in A_r(\Omega)$ when

$$\sup_{B} \left(\frac{1}{B} \int_{B} w dx\right) \left(\frac{1}{B} \int_{B} w^{1/(1-r)} dx\right)^{r-1} < \infty. \tag{1.12}$$

The following Lemma 1.13 was proved in [3] without using Theorem 1.7 and the fact that a ball B is an $L^s(\mu)$-averaging domain.

LEMMA 1.13 *Let B be any ball in Ω with center x_1 and radius r, and the measure μ is defined by $d\mu = w(x)dx$ with $w \in WRH(\Omega)$. Then*

$$\left(\frac{1}{\mu(B)}\int_B k(x,x_1)^s d\mu\right)^{(1/s)} \leq \alpha,$$

where α is a constant independent of B, $s \geq 1$ and the supremum is over all balls $B \subset \Omega$.

2. WHITNEY COVERS OF $L^s(\mu)$-AVERAGING DOMAINS

We will need the following lemma appeared in [9].

LEMMA 2.1 *Each Ω has a modified Whitney cover of cubes $W = \{Q_i\}$ which satisfy*

$$\bigcup_i Q_i = \Omega,$$

$$\sum_{Q \in W} \chi_{\sqrt{\frac{5}{4}}Q} \leq N\chi_\Omega$$

for all $x \in R^n$ and some $N > 1$ and if $Q_i \cap Q_j \neq \phi$, then there exists a cube $R(\notin W)$ in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover if Ω is δ-John, then there is a distinguished cube $Q_0 \in W$ which can be connected with every cube $Q \in W$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from W and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.

Now we show that the $L^s(\mu)$-averaging domains can be characterized in terms of the Whitney covers.

THEOREM 2.2 *Let Ω be an $L^s(\mu)$-averaging domain with measure μ such that $d\mu = w(x)dx$, where the weight function w satisfies the weak reverse Hölder inequality in Ω (i.e., $w \in WRH(\Omega)$). If the Whitney cover $\mathcal{F}$ of Ω consists of cubes Q_j with centers x_j, then the following two conditions are equivalent:*

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega k(x,x_0)^s d\mu\right)^{(1/s)} < \infty, \tag{2.3}$$

$$\left(\frac{1}{\mu(\Omega)}\sum_{Q_j\in\mathcal{F}} k(x_j,x_0)^s\mu(Q_j)\right)^{(1/s)} < \infty, \tag{2.4}$$

where x_0 is a fixed point of Ω.

Proof Assume (2.4) holds. By Definition 1.5, for any $x, x_0, x_1 \in \Omega$, $k(x,x_0) \leq k(x,x_1)+k(x_1,x_0)$. Let $\mathcal{F}$ be a Whitney cover of Ω consisting of cubes Q_j with centers x_j. Then for any x_j, due to Minkowski's inequality and an elementary inequality, $|\sum t_\alpha|^r \leq \sum |t_\alpha|^r$, where $0 \leq r \leq 1$, and Lemma 1.13, we have

$$\begin{aligned}
\left(\int_\Omega k(x,x_0)^s d\mu\right)^{(1/s)} &\leq \left(\int_{\cup_{Q_j\in\mathcal{F}}Q_j} (k(x,x_0))^s d\mu\right)^{(1/s)} \\
&\leq \left(\int_{\cup_{Q_j\in\mathcal{F}}Q_j} (k(x,x_j)+k(x_j,x_0))^s d\mu\right)^{(1/s)} \\
&\leq \left(\int_{\cup_{Q_j\in\mathcal{F}}Q_j} k(x,x_j)^s d\mu\right)^{(1/s)} \\
&\quad + \left(\int_{\cup_{Q_j\in\mathcal{F}}Q_j} k(x_j,x_0)^s d\mu\right)^{(1/s)} \\
&\leq \left[\sum_{Q_j\in\mathcal{F}}\int_{Q_j} (k(x,x_j))^s d\mu\right]^{(1/s)} \\
&\quad + \left[\sum_{Q_j\in\mathcal{F}}\int_{Q_j} k(x_j,x_0)^s d\mu\right]^{(1/s)} \\
&\leq \sum_{Q_j\in\mathcal{F}}\left[\int_{Q_j} (k(x,x_j))^s d\mu\right]^{(1/s)} \\
&\quad + \left[\sum_{Q_j\in\mathcal{F}}\int_{Q_j} k(x_j,x_0)^s d\mu\right]^{(1/s)} \\
&\leq \sum_{Q_j\in\mathcal{F}}\left[\int_{Q_j} (k(x,x_j))^s d\mu\right]^{(1/s)} \chi_{\sqrt{\frac{5}{4}}Q_j} \\
&\quad + \left[\sum_{Q_j\in\mathcal{F}}\int_{Q_j} k(x_j,x_0)^s d\mu\right]^{(1/s)}
\end{aligned}$$

$$\leq \sum_{Q_j \in \mathcal{F}} \mu(Q_j)^{(1/s)} \left[\frac{1}{\mu(Q_j)} \int_{Q_j} (k(x, x_j))^s d\mu\right]^{(1/s)} \chi_{\sqrt{\frac{5}{4}}Q_j}$$

$$+ \left[\sum_{Q_j \in \mathcal{F}} k(x_j, x_0)^s \mu(Q_j)\right]^{(1/s)}$$

$$\leq \sum_{Q_j \in \mathcal{F}} \mu(Q_j)^{(1/s)} \cdot C \cdot \chi_{\sqrt{\frac{5}{4}}Q_j}$$

$$+ \mu(\Omega)^{(1/s)} \left[\frac{1}{\mu(\Omega)} \sum_{Q_j \in \mathcal{F}} k(x_j, x_0)^s \mu(Q_j)\right]^{(1/s)}$$

$$\leq C \cdot \mu(\Omega)^{1/s} \sum_{Q_j \in \mathcal{F}} \chi_{\sqrt{\frac{5}{4}}Q_j}$$

$$+ \mu(\Omega)^{(1/s)} \left[\frac{1}{\mu(\Omega)} \sum_{Q_j \in \mathcal{F}} k(x_j, x_0)^s \mu(Q_j)\right]^{(1/s)}$$

$$\leq C\mu(\Omega)^{(1/s)} \cdot N_{\chi_\Omega}$$

$$+ \mu(\Omega)^{1/s} \left[\frac{1}{\mu(\Omega)} \sum_{Q_j \in \mathcal{F}} k(x_j, x_0)^s \mu(Q_j)\right]^{(1/s)} < \infty,$$

where $N > 1$ is some constant and the second inequality to the last is due to Lemma 2.1.

On the other hand, assume (2.3) holds. We show that (2.4) is also true. Note that

$$k(x_j, x_0)^s \leq (k(x_j, x) + k(x, x_0))^s \leq 2^s (k(x_j, x)^s + k(x, x_0)^s).$$

Integrating over Q_j gives

$$k(x_j, x_0)^s \mu(Q_j) = \int_{Q_j} (k(x_j, x_0)^s d\mu$$

$$\leq 2^s \int_{Q_j} (k(x_j, x))^s d\mu + 2^s \int_{Q_j} k(x, x_0)^s d\mu.$$

Summing and using Lemma 1.13 yields

$$\sum_{Q_j \in \mathcal{F}} k(x_j, x_0)^s \mu(Q_j) \leq 2^s \sum_{Q_j \in \mathcal{F}} \int_{Q_j} k(x_j, x)^s d\mu$$

$$+ 2^s \sum_{Q_j \in \mathcal{F}} \int_{Q_j} k(x, x_0)^s d\mu$$

$$
\begin{aligned}
&\le 2^s \sum_{Q_j \in \mathcal{F}} \left(\int_{Q_j} k(x_j, x)^s d\mu \right) \chi_{\sqrt{\frac{5}{4}}Q_j} \\
&\quad + 2^s \sum_{Q_j \in \mathcal{F}} \int_{\Omega} k(x, x_0)^s d\mu \chi_{\sqrt{\frac{5}{4}}Q_j} \\
&\le 2^s \sum_{Q_j \in \mathcal{F}} \mu(Q_j) \cdot M \cdot \chi_{\sqrt{\frac{5}{4}}Q_j} \\
&\quad + 2^s \cdot N_{\chi_\Omega} \cdot \int_{\Omega} k(x, x_0)^s d\mu \\
&\le 2^s M \cdot \mu(\Omega) \cdot N_{\chi_\Omega} \\
&\quad + 2^s \cdot N_{\chi_\Omega} \cdot \int_{\Omega} k(x, x_0)^s d\mu < \infty
\end{aligned}
$$

which says that (2.4) is true. The proof of Theorem 2.2 is completed.

COROLLARY 2.5 *Either* (2.3) *or* (2.4) *is a sufficient condition for a domain* Ω *to be an* L^s (μ)*-averaging domain if* Ω *with a Whitney cover* $\mathcal{F}$ *and the measure* μ *is defined as in Theorem* 2.2.

Proof As the matter of fact, (2.3) is sufficient by Theorem 1.9. From the first part of the proof of Theorem 2.2, where $((1/\mu(Q_j)) \int_{Q_j} k(x_j, x)^s d\mu)^{(1/s)} < \infty$ is from Lemma 1.13, (2.4) implies (2.3), so that (2.4) is also sufficient for Ω being an $L^s(\mu)$-averaging domain.

3. SOME MAPPINGS OF $L^s(\mu)$-AVERAGING DOMAINS

We first prove that the $L^s(\mu)$-averaging domains are preserved under K-quasi-isometric mappings.

DEFINITION 3.1 A mapping f defined in Ω is said to be a K-quasi-isometry, $K > 1$, if

$$\frac{1}{K} \le \frac{|f(x) - f(y)|}{|x - y|} \le K \tag{3.2}$$

for all x, $y \in \Omega$.

LEMMA 3.3 *Let* $f : \Omega \to \Omega'$ *be a K-quasi-isometric mapping. Then*

$$\frac{1}{K^n} |B| \le |B'| \le K^n |B|, \tag{3.4}$$

where $B' = f(B)$ *and* $B \subset \Omega$ *is any ball or cube.*

Proof *If f is a K*-quasi-isometric mapping, then

$$\frac{1}{K^n} \le J(f) \le K^n \quad \text{a.e.} \tag{3.5}$$

where $J(f)$ is the Jacobian of f. Therefore,

$$|B'| = \int_{B'} dx = \int_B J(f)dx \le K^n|B|,$$

and

$$\frac{1}{K^n}|B| = \frac{1}{K^n}\int_B dx = \int_B \frac{1}{K^n}dx \le \int_B J(f)dx = \int_{B'} dx = |B'|.$$

THEOREM 3.6 *Let $f: \Omega \to \Omega'$ be a K-quasi-isometric mapping. If $w \in A_r$, then $w(f(x)) \in A_r$.*

Proof Due to the Definition 1.11, we will show

$$\left(\frac{1}{|B'|}\int_{B'} w(f(x))dx\right)\left(\frac{1}{|B'|}\int_{B'}\left(\frac{1}{w(f(x))}\right)^{(1/(r-1))}dx\right)^{r-1} < \infty.$$

Let $w \in A_r$, $r > 1$. Then using (3.4) and the inequality (3.5), we have

$$\begin{aligned}
&\left(\frac{1}{|B'|}\int_{B'} w(f(x))dx\right)\left(\frac{1}{|B'|}\int_{B'}\left(\frac{1}{w(f(x))}\right)^{(1/(r-1))}dx\right)^{r-1}\\
&\le \left(\frac{K^n}{|B|}\int_{B'} w(f(x))dx\right)\left(\frac{K^n}{|B|}\int_{B'}\left(\frac{1}{w(f(x))}\right)^{(1/(r-1))}dx\right)^{r-1}\\
&\le K^n \cdot K^{n(r-1)}\left(\frac{1}{|B|}\int_B w(x)J(f)dx\right)\\
&\quad \left(\frac{1}{|B|}\int_B\left(\frac{1}{w(x)}\right)^{(1/(r-1))}J(f)dx\right)^{r-1}\\
&\le K^{2n} \cdot K^{2n(r-1)}\left(\frac{1}{|B|}\int_B w(x)dx\right)\\
&\quad \left(\frac{1}{|B|}\int_B\left(\frac{1}{w(x)}\right)^{(1/(r-1))}dx\right)^{r-1}\\
&\le K^{2nr} \cdot M < \infty.
\end{aligned}$$

Therefore, $w(f(x)) \in A_r$.

LEMMA 3.7 *Let $f : \Omega \to \Omega'$ be a K-quasi-isometric mapping. If μ and ν are measures defined by $d\nu = w(f(x))dx$ and $d\mu = w(x)dx$, respectively, then*

$$\frac{1}{K^n}\mu(D) \le \nu(D') \le K^n \mu(D), \tag{3.8}$$

where $D \subset \Omega$ and $D' = f(D) \subset \Omega'$.

Proof The result is immediately from

$$\begin{aligned}\nu(D') &= \int_{D'} d\nu = \int_{D'} w(f(x))dx \\ &= \int_D w(x)J(f)dx \quad \text{and} \quad \frac{1}{K^n} \le J(f) \le K^n.\end{aligned}$$

THEOREM 3.9 *If $f : \Omega \to \Omega'$ is a K-quasi-isometric mapping and Ω is an $L^s(\mu)$-averaging domain, then Ω' is an $L^s(\nu)$-averaging domain.*

Proof Let γ be a quasi-hyperbolic geodesic arc joining x to y in Ω and set $\gamma' = f(\gamma)$. By the virtue of (3.2), (3.4), (3.8) and the inequality (3.5), and note that

$$\begin{aligned}d(f(x), \partial\Omega') &= \inf_{u \in \partial\Omega'} |f(x) - u| = \inf_{t \in \partial\Omega} |f(x) - f(t)| \\ &\ge \inf_{t \in \partial\Omega} \left(\frac{1}{K}|x - t|\right) = \frac{1}{K} \inf_{t \in \partial\Omega} |x - t| = \frac{1}{K} d(x, \partial\Omega),\end{aligned}$$

we have

$$k(f(x), f(y); \Omega') \le \int_{\gamma'} \frac{ds'}{d(f(z), \partial\Omega')} \le \int_{\gamma} \frac{K^2 ds}{d(z, \partial\Omega)} = K^2 k(x, y; \Omega).$$

Therefore

$$\begin{aligned}&\left(\frac{1}{\nu(\Omega')}\int_{\Omega'} k(f(x), f(x_0); \Omega')^s d\nu\right)^{(1/s)} \\ &\quad = \left(\frac{1}{\nu(\Omega')}\int_{\Omega'} k(f(x), f(x_0); \Omega')^s w(f(x))dx\right)^{(1/s)} \\ &\quad \le \left(\frac{C_1}{\mu(\Omega)}\int_{\Omega} K^{2s} k(x, x_0; \Omega)^s w(x)J(f)dx\right)^{(1/s)} \\ &\quad \le \left(\frac{C_1}{\mu(\Omega)}\int_{\Omega} K^{2s} k(x, x_0; \Omega)^s w(x)K^n dx\right)^{(1/s)}\end{aligned}$$

$$\leq (C_1 K^{2s+n})^{(1/s)} \left(\frac{1}{\mu(\Omega)} \int_\Omega k(x, x_0; \Omega)^s d\mu \right)^{(1/s)}$$
$$\leq (C_1 K^{2s+n})^{(1/s)} \cdot A < \infty.$$

Thus, Ω' is an $L^s(\nu)$-averaging domain with the measure ν defined by $d\nu = w(f(x))dx$ due to Theorem 1.9.

We can also extend Theorem 3.9 to a class of more general mappings so called φ-quasi-isometric mapping, see [13].

DEFINITION 3.10 Let φ: $[0, \infty) \to [0, \infty)$ be a homeomorphism with $\varphi(t) \geq t$, X and Y be metrical spaces. An embedding $f: X \to Y$ is said to be a φ-quasi-isometry if

$$\varphi^{-1}(|x-y|) \leq |f(x) - f(y)| \leq \varphi(|x-y|) \quad \text{for all } x, y \in X.$$

Obviously, Theorem 3.9 is a special case of φ-quasi-isometric mapping as $\varphi(t) = Kt$, $K \geq 1$. (f is also called a K-bilipschitz map.) We prove the following result.

THEOREM 3.11 *Theorem 3.9 is still true if $f: \Omega \to \Omega'$ is an φ-quasi-isometric mapping with $m \leq |\varphi'(t)| \leq M$ and $\varphi(0) = 0$, where m, M are positive real numbers.*

Proof For any $x, y \in \Omega$, $|x-y| < \infty$. By the virtue of Mean Value Theorem, there exists $\xi \in (0, |x-y|)$, such that $\varphi(|x-y|) - \varphi(0) = \varphi'(\xi)(|x-y| - 0)$. Thus,

$$m|x-y| \leq \varphi(|x-y|) \leq M|x-y| \tag{3.12}$$

because of $m \leq |\varphi'(t)| \leq M$ for all $t \in [0, \infty)$.

On the other hand, $(1/M) \leq |(\varphi^{-1})'(t)| \leq (1/m)$ due to homeomorphism property of φ. Thus, we also have

$$\frac{1}{M}|x-y| \leq \varphi^{-1}(|x-y|) \leq \frac{1}{m}|x-y|. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain the inequality

$$\frac{1}{M}|x-y| \leq |f(x) - f(y)| \leq M|x-y|,$$

which is just the case of Theorem 3.9.

According to Väisälä [13], a homeomorphism $f: \Omega \subset R^n \to \Omega'$ is a k-quasi-isometric mapping implies that it is a K-quasi-conformal mapping. Theorems 3.6, 3.9 show that the K-quasi-isometric mappings preserve the A_r weights and $L^s(\mu)$-averaging domains. Then naturally, one would ask that if K-quasi-conformal mappings also preserve those properties. The answer is No. Staples [11] shows that L^s-averaging domains are *not invariant* with respect to quasi-conformal self-mappings of R^n. Therefore *neither* are $L^s(\mu)$-averaging domains, since we can choose weight $w(x) = 1$ for the measure μ defined by $d\mu = w(x)dx$. Ding and Nolder [3] show that if $f: \Omega \subset R^n \to \Omega'$ is a K-quasi-conformal mapping and Ω' is an $L^s(m)$-averaging domain, where m is n-dimensional Lebesgue measure, then Ω is an $L^s(\mu)$-averaging domain with $d\mu = J(f)dm$, and $J(f)$ is the Jacobian determinant of f. Now we proof that the inverse of this result is also true.

THEOREM 3.14 *Let f be a K-quasi-conformal mapping of an $L^s(\mu)$-averaging domain $\Omega \subset R^n$ onto a proper subdomain $\Omega' \subset R^n$ for $s \geq 1$, where μ is a measure defined by $d\mu = J(f)dx$ and $J(f)$ is the Jacobian of f. Then Ω' is an L^s-averaging domain.*

Proof Let $x_0, x \in \Omega$ and write $y = f(x)$, $y_0 = f(x_0)$. By Theorem 3 in [5], we have

$$k(f(x), f(x_0); \Omega') \leq C_1 \max(k(x, x_0; \Omega), k(f(x), f(x_0); \Omega')^{\alpha}),$$

where $\alpha = K^{1/(1-n)} \leq 1$. So that we have

$$k(y, y_0; \Omega')^s \leq C_2(k(y, y_0; \Omega')^s + k(y, y_0; \Omega')^{\alpha s}).$$

We may assume that $\alpha < 1$. Then by the generalized Hölder's inequality,

$$\begin{aligned}\left(\int_\Omega k(x, x_0; \Omega)^{\alpha s} d\mu\right)^{(1/\alpha s)} &\leq \left(\int_\Omega d\mu\right)^{((s-\alpha s)/\alpha s^2)} \\ &\quad \left(\int_\Omega k(x, x_0; \Omega)^s d\mu\right)^{(1/s)} \\ &= (\mu(\Omega))^{((1-\alpha)/\alpha s)} \\ &\quad \left(\int_\Omega k(x, x_0; \Omega)^s d\mu\right)^{(1/s)}.\end{aligned} \tag{3.15}$$

Thus,

$$\int_{\Omega'} k(y,y_0;\Omega')^s dy \le C_2\left(\int_\Omega k(x,x_0;\Omega)^s J(f)dx + \int_\Omega k(x,x_0;\Omega)^{\alpha s} J(f)dx\right)$$
$$\le C_2\left(\int_\Omega k(x,x_0;\Omega)^s d\mu + \int_\Omega k(x,x_0;\Omega)^{\alpha s} d\mu\right)$$
$$\le C_2\left(\int_\Omega k(x,x_0;\Omega)^s d\mu + \mu(\Omega)^{1-\alpha}\left(\int_\Omega k(x,x_0;\Omega)^s d\mu\right)^\alpha\right) < \infty.$$

Therefore, Ω' is an L^s-averaging domain (or $L^s(m)$-averaging domain where m is the Lebesgue measure).

Now we construct an example of $L^s(\mu)$-averaging domain by the similar method used in [11].

Example We consider a domain $\Omega = Q \cup S \subset R^n$, where Q is the cube $Q = \{(x_1, x_2, \dots, x_n) : |x_1 - 2|, |x_2|, \dots, |x_n| < 1\}$, and S is a spire $S = \{(x_1, x_2, \dots, x_n) : \sum_{i=2}^n (x_i)^2 < g(x_1)^2,\ 0 \le x_1 < 1\}$, where $g(x)$ satisfies the following properties:

(i) $g(0) = 0$, $g(1) \le 1$,
(ii) $0 < g'(x) \le M$, for $0 < x \le 1$,
(iii) $g''(x) \ge 0$, for $0 \le x \le 1$.

Then Ω is an $L^s(\mu)$-averaging domain with $w \in WRH(\Omega)$ if

$$\int_0^1 g(x)^{n-1}\left(\int_x^1 \frac{1}{g(t)}dt\right)^{sp} dx < \infty, \quad p > 1. \tag{3.16}$$

Proof Let $z_0 = (1, 0, \dots, 0)$ be our fixed point. We will estimate $k(z, z_0)$ for $z = (z_1, z_2, \dots, z_n) \in S$ as follows. Let $y = (z_1, 0, \dots, 0)$. Then $k(z, z_0) \le k(z, y) + k(y, z_0)$. For the upper bound of $k(z, y)$, we examine the cross section of S when $x_1 = z_1$, see [11 and 3], and have

$$k(z,y) \le \log\frac{g(z_1)}{g(z_1) - r}, \quad \text{where } r^2 = \sum_{i=2}^n (z_i)^2. \tag{3.17}$$

For upper bound of $k(y, z_0)$, we consider the distance of any point $y = (x_1, 0, \ldots, 0)$ to the boundary of Ω, which satisfies

$$g(x_1) \geq d(y, \partial\Omega) \geq g(x_1)\cos\theta, \quad \text{where } \tan\theta = g'(x_1).$$

Then, by (ii),

$$\frac{1}{g(x_1)} \leq \frac{1}{d(y, \partial\Omega)} \leq \frac{1}{g(x_1)}(g'(x_1)^2 + 1)^{1/2} \leq \frac{c}{g(x_1)}, \tag{3.18}$$

therefore,

$$\left(\int_{z_1}^1 \frac{1}{g(t)}dt\right)^s \leq k(z, z_0)^s \leq C\left(\int_{z_1}^1 \frac{1}{g(t)}dt\right)^s. \tag{3.19}$$

Since $k(z, z_0)^s \leq 2^s(k(z, y)^s + k(y, z_0)^s)$, using (3.17) and applying (3.19) to $k(y, z_0)$ yields

$$\begin{aligned}\left(\int_{z_1}^1 \frac{1}{g(t)}dt\right)^s &\leq k(z, z_0)^s \\ &\leq 2^s\left[\left(\log\frac{g(z_1)}{g(z_1) - r}\right)^s + \left(C\int_{z_1}^1 \frac{1}{g(t)}dt\right)^s\right].\end{aligned} \tag{3.20}$$

Thus,

$$\begin{aligned}\int_S \left(\int_{z_1}^1 \frac{1}{g(t)}dt\right)^s d\mu &\leq \int_S k(z, z_0)^s d\mu \\ &\leq C_1 \int_S \left(\log\frac{g(z_1)}{g(z_1) - r}\right)^s d\mu \\ &\quad + C_2 \int_S \left(\int_{z_1}^1 \frac{1}{g(t)}dt\right)^s d\mu.\end{aligned}$$

Note that $d\mu = w(x)dx$ with $w \in WRH(\Omega)$, then w is a weak doubling weight, *i.e.*, there is a constant C such that $\mu(2B) \leq C\mu(B)$ for all balls B with $2B \subset \Omega$. Let B be the unit ball with the center at origin, then $S \subset B$. By the virtue of Hölder inequality and weak reverse Hölder inequality and $(1/p) + (1/q) = 1$ for some $p > 1$ and

$q > 1$, we yield

$$\begin{aligned}
&\int_S \left(\log \frac{g(z_1)}{g(z_1)-r}\right)^s d\mu \\
&\le \left(\int_S \left(\log \frac{g(z_1)}{g(z_1)-r}\right)^{sp} dx\right)^{1/p} \left(\int_S w^q dx\right)^{1/q} \\
&\le C\left(\int_S \left(\log \frac{g(z_1)}{g(z_1)-r}\right)^{sp} dx\right)^{1/p} |B|^{1/q}\left(\frac{1}{|B|}\int_B w^q dx\right)^{1/q} \\
&\le C_1\left(\int_S \left(\log \frac{g(z_1)}{g(z_1)-r}\right)^{sp} dx\right)^{1/p} |B|^{1/q-1}\int_{2B} w dx \\
&\le C_2\left(\int_S \left(\log \frac{g(z_1)}{g(z_1)-r}\right)^{sp} dx\right)^{1/p} \mu(B) \\
&\le C_3\left(\int_S \left(\log \frac{g(z_1)}{g(z_1)-r}\right)^{sp} dx\right)^{1/p}. \\
&\le C_3\left(\int_0^1 \int_0^{g(x)} \left(\log \frac{g(z_1)}{g(z_1)-r}\right)^{sp} r^{n-2} dr dx\right)^{1/p} < \infty,
\end{aligned}$$

for all $n \ge 2$ and $s \ge 1$.

Similarly,

$$\begin{aligned}
\int_S \left(\int_{z_1}^1 \frac{1}{g(t)}dt\right)^s d\mu &\le C_4\left(\int_0^1 \int_0^{g(x)} \left(\int_x^1 \frac{1}{g(t)}dt\right)^{sp} r^{n-2} dr dx\right)^{1/p} \\
&\le C_5\left(\int_0^1 g(x)^{n-1}\left(\int_x^1 \frac{1}{g(t)}dt\right)^{sp} dx\right)^{1/p}.
\end{aligned}$$

Thus, the conclusion of Example 3 holds.

Considering a special case of the Example as $g(x_1) = x_1^\alpha$ for $\alpha > 1$ and $sp + 1 > n$, the spire S is an $L^s(\mu)$-averaging domain if $\alpha < ((sp+1)/(sp+1-n))$.

References

[1] Ding, S. (1997). Weighted Hardy-Littlewood inequality for A-harmonic tensors. *Proc. Amer. Math. Soc.*, **125**(6), 1727–1735.

[2] Ding, S. and Liu, B., Generalized Poincaré inequalities for solutions to the A-harmonic equation in certain domains, *J. of Math. Anal. & Appl.*, to appear.

[3] Ding, S. and Nolder, C. A., $L^s(\mu)$-*averaging domains and their applications, preprint.*

[4] Gehting, F. W. and Martio, O. (1985). Lipschitz classes and quasi-conformal mappings. *Ann. Acad. Sci. Fenn. Ser. A.I. Math.*, **10**, 203–219.
[5] Gehring, F. W. and Osgood, B. G. (1987). Lipschitz classes and quasi-conformal extension domains. *Complex Variables*, **9**, 175–188.
[6] Gehring, F. W. and Palka, B. P. (1976). Quasiconformally homogeneous domains. *J. Analyse Math.*, **30**, 172–199.
[7] Iwaniec, T. and Martin, G. (1993). Quasiregular mappings in even dimensions. *Acta Math.*, **170**, 29–81.
[8] Liu, B. and Ding, S. (1999). The Monotonic property of $L^s(\mu)$-averaging domains and weighted weak reverse Hölder inequality. *I. Math. Anal. Appl.*, **237**, 730–739.
[9] Nolder, C., (1999). Hardy-Littlewood theorems for A-harmonic tensors. *Illinois Journal of Mathematics*, **43**, 613–631.
[10] Stein, E. M., *Singular integrals and differentiability properties of functions*. Princton University Press, Princton, 1970.
[11] Staples, S. G. (1989). L^p-averaging domains and the Poincaré inequality. *Ann. Acad. Sci. Fenn, Ser. A.I. Math.*, **14**, 103–127.
[12] Smith, W. S. and Stegenga, D. A. (1991). Exponential integrability of the quasi-hyperbolic metric on Hölder domains. *Ann. Acad. Sci. Fenn. Ser. A.I. Math.*, **16**, 345–360.
[13] Väisälä, J. (1992). Domains and maps. *Lecture Notes in Mathematics*, Springer-Verlag, **1508**, 119–131.