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Research Article

q-Gaussian Distributions: Simplifications and Simulations

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We present some properties of measures (q-Gaussian) that orthogonalize the set of q-Hermite polynomials. We also present an algorithm for simulating i.i.d. sequences of random variables having q-Gaussian distribution.

1. Introduction

The paper is devoted to recollection of known and presentation of some new properties of a distribution called q-Gaussian. We propose also a method of simulation of i.i.d. sequences drawn from it.

q-Gaussian is in fact a family of distributions indexed by a parameter $q \in [-1, 1]$. It is defined as follows.

For $q = -1$, it is a discrete 2-point distribution, which assigns values $1/2$ to -1 and 1.

For $q \in (-1, 1)$, it has density given by

$$f_H(x \mid q) = \frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty}\left(\left(1+q^k\right)^2 - (1-q)x^2 q^k\right)\prod_{k=0}^{\infty}\left(1-q^{k+1}\right), \tag{1.1}$$

for $|x| \leq 2/\sqrt{1-q}$. In particular $f_H(x \mid 0) = (1/2\pi)\sqrt{4-x^2}$, for $|x| \leq 2$. Hence it is Wigner distribution with radius 2.

For $q = 1$, *q-Gaussian* distribution is the Normal distribution with parameters 0 and 1.

In Figure 1, we present plots of $f_H(x \mid -.4)$ in *blue*, $f_H(x \mid .1)$ in *orange*, $f_H(x \mid .8)$ in *red*, and standard *normal density* in *black*.

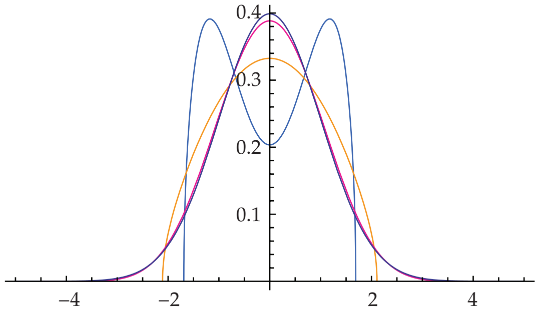

Figure 1: Plots of $f_H\ (\cdot \mid q)$ for different q.

This family of distributions was defined first in the paper of Bożejko et al., in 1997 [1] in noncommutative probability context. Later ([2, 3]) it appeared in quite classical context namely as a stationary distribution P_H of discrete time random field $\mathbf{X} = \{X_n\}_{n\in\mathbb{Z}}$ defined by the following relationships: $\mathbb{E}(X_i) = 0,\ \mathbb{E}(X_i^2) = 1,\ i \in \mathbb{Z}$,

$$\begin{aligned}
&\exists a \in \mathbb{R};\ \forall n \in \mathbb{Z} : \mathbb{E}(X_n \mid \mathcal{F}_{\neq n}) = a(X_{n-1} + X_{n+1}),\ \text{a.s.} \\
&\exists A, B, C \in \mathbb{R};\ \forall n \in \mathbb{Z} : \mathbb{E}\left(X_n^2 \mid \mathcal{F}_{\neq n}\right) \\
&\qquad = A\left(X_{n-1}^2 + X_{n+1}^2\right) + BX_{n-1}X_{n+1} + C,\ \text{a.s.},
\end{aligned} \tag{1.2}$$

where $\mathcal{F}_{\neq m} := \sigma(X_k : k \neq m)$. It turns out that parameters a, A, B, C are related to one another in such a way that there are two parameters $q \geq -1$ and $0 < |\rho| < 1$ and all others can be expressed through them:

$$\begin{aligned}
a &= \frac{\rho}{1+\rho^2}, \qquad A = \frac{\rho^2(1-q\rho^2)}{(\rho^2+1)(1-q\rho^4)}, \\
B &= \frac{\rho^2(1-\rho^2)(1+q)}{(\rho^2+1)(1-q\rho^4)}, \qquad C = \frac{(1-\rho^2)^2}{1-q\rho^4}.
\end{aligned} \tag{1.3}$$

Then, one proves that

$$\forall n \in \mathbb{Z}, k, i \geq 1 : \mathbb{E}(H_k(X_n \mid q) \mid \mathcal{F}_{\leq n-i}) = \rho^{ki} H_k(X_{n-i} \mid q),\ \text{a.s.}, \tag{1.4}$$

where $\mathcal{F}_{\leq m} := \sigma(X_k : k \leq m)$, (similarly one defines $\mathcal{F}_{\geq m} := \sigma(X_k : k \geq m)$) and $H_k(x \mid q) : k \geq -1$ are q-Hermite polynomials defined below. It turns out that for $q > 1$ the one-dimensional distribution of the process $\mathbf{X}$ is not defined by moments. This case is treated separately (e.g., in [4]).

As mentioned earlier, here we will consider only the case $|q| \leq 1$. We will denote family of q-Gaussian distributions by $P_H(q)$ or simply P_H.

It turns out that there is quite large literature where this distribution appears and is used to model different phenomena. See, for example, [5–9]. Random field defined above is a model of notions that first appeared in noncommutative context and hence establishes a link between noncommutative and classical probability theories.

Remark 1.1. In literature there exists another family of distributions under the same name. It appears in the context of (Boltzmann-Gibbs)-statistical mechanics. See, for example, [10] for applications and review.

Both families are indexed by basically one parameter $q \in [-1,1]$, and for $q = 1$ both include ordinary $N(0,1)$ distribution.

In the sequel we will use the following traditional notation used in so-called "q-series theory" $[0]_q = 0, [n]_q = 1 + q + \cdots + q^{n-1}$; $n \geq 1, [0]_q! = 1$, $[n]_q! = \prod_{i=1}^{n}[i]_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = [n]_q!/[k]_q![n-k]_q!$, for $n \geq 0$, $k = 0,\ldots,n$ and 0 otherwise, and $(a \mid q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$, for $n = 1,2,\ldots,\infty$ (so-called Pochhammer symbol). Sometimes $(a \mid q)_n$ will be abbreviated to $(a)_n$, if it will not cause misunderstanding. Notice that $(q)_n = (1-q)^n[n]_q!$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = (q)_n/(q)_k\ (q)_{n-k}$ and that $[n]_q$, $[n]_q!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ tend to $n, n!$, and $\binom{n}{k}$ (Newton's symbol), respectively, as $q->1$.

Remark 1.2. Introducing new variable z defined by the relationship $1 - (1-q)x^2/2 = \cos 2\pi z$ we can express q-Gaussian density through Jacobi θ functions defined, for example, in [11]. Namely, we have for $z \in [-1/2, 1/2]$

$$f_H\left(\frac{2\sin(\pi z)}{\sqrt{1-q}} \mid q\right) = C_q \theta_3(z \mid q)\theta_2(z \mid q), \tag{1.5}$$

with $C_q = \sqrt{1-q}\ (q \mid q^2)/2\pi q^{1/4}\ (q^2 \mid q^2)$ where $\theta_3(z \mid q)$ and $\theta_2(z \mid q)$ are so-called third and second Jacobi Theta functions.

Let us introduce family of polynomials (called q-Hermite) satisfying the following three-term recurrence relationship:

$$H_{n+1}(x \mid q) = xH_n(x \mid q) - [n]_q H_{n-1}(x \mid q), \tag{1.6}$$

with $H_{-1}(x \mid q) = 0$, $H_0(x \mid q) = 1$. Notice that $H_n(x \mid 0) = U_n(x/2)$, $n \geq -1$, where $U_n(x)$ are Chebyshev polynomials of the second kind defined by

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \tag{1.7}$$

and that $H_n(x \mid 1) = H_n(x),\ n \geq -1$ where $H_n(x)$ are ("probabilist") Hermite polynomials, that is, polynomials orthogonal with respect to Gaussian $N(0,1)$ measure.

It turns out that *q-Gaussian* is the distribution with respect to which *q-Hermite* polynomials are orthogonal. This fact can be easily deduced from (1.4).

Thus in particular using the condition

$$\forall n \geq 1;\ q \in (-1,1) : \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} H_n(x \mid q) f_H(x \mid q)\,dx = 0, \tag{1.8}$$

we can get all moments of *q-Gaussian* distribution. Hence in particular we have $\mathbb{E}X^{2n+1} = 0,\ n \geq 0,\ \mathbb{E}X^2 = 1,\ \mathbb{E}X^4 = 2+q,\ \mathbb{E}X^6 = 5+6q+3q^2+q^3$ if only $X \sim P_H(q)$.

The aim of this paper is to make *q-Gaussian* distribution more friendly by presenting an alternative form of the density f_H for $q \in (-1,1)$, more easy to deal with (in particular we find the c.d.f. of P_H), and suggest a method of simulation of i.i.d. sequences having density f_H.

2. Expansion of f_H

In this section we will prove the following expansion theorem.

Theorem 2.1. *For all* $|q| < 1,\ |x| \leq 2/\sqrt{1-q}$ *one has*

$$f_H(x \mid q) = \frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)x^2}\sum_{k=1}^{\infty}(-1)^{k-1} q^{\binom{k}{2}} U_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right), \tag{2.1}$$

where $\binom{n}{k} = n!/k!(n-k)!$.

As a corollary we get expression for the c.d.f. function of P_H.

Corollary 2.2. *The distribution function of q-Gaussian distribution is given by*

$$\begin{aligned} F_H(y \mid q) &= \frac{1}{2} + \frac{1}{\pi}\arcsin\left(\frac{y\sqrt{1-q}}{2}\right) \\ &\quad + \frac{1}{2\pi}\sqrt{4-(1-q)y^2}\sum_{k=1}^{\infty}(-1)^{k-1} q^{\binom{k}{2}}\left(1+q^k\right)\frac{U_{2k-1}\left(y\sqrt{1-q}/2\right)}{2k}. \end{aligned} \tag{2.2}$$

Identity (2.1) can be a source of many interesting identities, which may not be widely known outside the circle of researchers working in special functions.

Corollary 2.3. *For all $q \in (-1, 1)$*

(i) *for all $x \in [-2/\sqrt{1-q},\ 2/\sqrt{1-q}]$*

$$(q)_\infty \prod_{k=1}^{\infty}\left(\left(1+q^k\right)^2 - (1-q)x^2 q^k\right) = \sum_{k=1}^{\infty}(-1)^{k-1} q^{\binom{k}{2}} U_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right), \tag{2.3}$$

where the polynomials U_k are defined by (1.7).

In particular, one has

(ii)

$$(-q)_\infty \left(q^2 \mid q^2\right)_\infty = \sum_{k=1}^{\infty} q^{\binom{k}{2}}, \tag{2.4}$$

a particular case of so-called Jacobi's "triple product identity".

(iii)

$$\left(q^3 \mid q^3\right)_\infty = 1 + \sum_{k=1}^{\infty}(-1)^{3k}\left(q^{\binom{3k}{2}} + q^{\binom{3k+1}{2}}\right), \tag{2.5}$$

(iv)

$$(q)_\infty^3 = 1 + \sum_{k=2}^{\infty}(-1)^{k+1}(2k-1)q^{\binom{k}{2}}, \tag{2.6}$$

(v)

$$\forall n \in \mathbb{N}: \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{k=1}^{n}(-1)^{k-1}\left(1+q^k\right)q^{\binom{k}{2}}\begin{bmatrix} 2n \\ n-k \end{bmatrix}_q, \tag{2.7}$$

Corollary 2.4. *$f_H(x \mid q)$ is bimodal for $q \in (-1, q_0)$, where $q_0 (\cong -.107)$ is the largest real root of the equation $\sum_{k=0}^{\infty}(2k+1)^2 q^{k(k+1)/2} = 0$.*

Lemma 2.5. *For all $q \in (-1, 1)$ and $n \geq 4$ one has*

(i)

$$\begin{aligned}\sup_{|x|<2/\sqrt{1-q}}&\left| f_H(x \mid q) - \frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)x^2}\left(\sum_{k=1}^{n-1}(-1)^{k-1}q^{\binom{k}{2}} U_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right)\right)\right| \\ &\leq \frac{n|q|^{(n-1)(n-2)/2}}{\pi\left(1-q^2\right)^2},\end{aligned} \tag{2.8}$$

(ii)

$$\sup_{|x|<2/\sqrt{1-q}}\left|F_H(y\mid q)-\frac{1}{2}-\frac{1}{\pi}\arcsin\left(\frac{y\sqrt{1-q}}{2}\right)\right.$$
$$\left.-\frac{1}{2\pi}\sqrt{4-(1-q)y^2}\sum_{k=1}^{n-1}(-1)^{k-1}q^{\binom{k}{2}}\left(1+q^k\right)\frac{U_{2k-1}(y\sqrt{1-q}/2)}{2k}\right| \qquad (2.9)$$
$$\le\frac{|q|^{n(n-1)/2}+|q|^{n(n+1)/2}}{2\pi(1-|q|^n)}.$$

Remark 2.6. Using the assertion of the above corollary one can approximate the density f_H as well as function $F_H(y \mid q)-1/2-1/\pi\arcsin(y\sqrt{1-q}/2)$ of P_H by expressions of the type $\sqrt{4-(1-q)x^2}\times$ *polynomial in x* with great accuracy. This expression is simple to analyze, simulate, and calculate interesting characteristics. Of course one should be aware that for small values of $n(\sqrt{1-q}/2\pi)\sqrt{4-(1-q)x^2}\times(\sum_{k=1}^{n}(-1)^{k-1}q^{\binom{k}{2}}U_{2k-2}(x\sqrt{1-q}/2)$ is not nonnegative for all $(1-q)x^2\le 4$! To give a scent of how many n's are needed to obtain the given accuracy we solved numerically (using program Mathematica) the equation

$$\frac{n|q|^{(n-1)(n-2)/2}}{\pi(1-q^2)^2}=\varepsilon, \qquad (2.10)$$

for several values q and ε. Let us denote by $N(q,\varepsilon)$ the solution of this equation:

we have (2.11)

$\varepsilon=\backslash q=$	.1	.4	.7	.9	.99.
.01	3.59	4.97	7.71	14.93	56.73
.001	4.04	5.67	8.73	16.53	60.86
.0001	4.76	6.26	9.61	17.97	64.70.

We also performed similar calculations for equation

$$\frac{|q|^{n(n-1)/2}+|q|^{n(n+1)/2}}{2\pi(1-|q|^n)}=\varepsilon,$$

obtaining : (2.12)

$\varepsilon=\backslash q=$	.1	.4	.7	.9	.99.
.01	2.3	3.3	5.1	9.5	33
.001	2.8	4.2	6.3	11.5	39
.0001	3.2	4.73	7.3	13.3	44.

3. Simulation

There is an interesting problem of quick simulation of i.i.d. sequences drawn from *q-Gaussian* distribution, using few realizations of i.i.d. standard uniform variates. One possibility is the *rejection method* (see, e.g., [12]). It is not optimal in the sense that it uses least realizations of independent, uniform on $[0,1]$ variates. But, as one can see below, it works.

To apply this method one has to compare density of the generated variates with another density that has the property of being "easy generated" or another words i.i.d. sequences of variables having this control density are easily obtainable. In the case of density f_H such natural candidate is $\sqrt{(1-q)(4-(1-q)x^2)}/2\pi$. However this density is unimodal, while the densities f_H for q below certain negative value are bimodal. This would lead to inefficient simulation method requiring many trial observations to be generated from $\sqrt{(1-q)(4-(1-q)x^2)}/2\pi$ to obtain one observation from f_H for sufficiently small q. That is why we decided to take as "easy generated" density the following one:

$$f_E(x \mid q) = \frac{\sqrt{(1-q)(4-(1-q)x^2)}\prod_{j=1}^{3}\left(\left(1+q^j\right)^2-(1-q)q^j x^2\right)}{2\pi[9]_q[5]_q}, \tag{3.1}$$

defined for $x \in [-2/\sqrt{1-q}, 2/\sqrt{1-q}]$. However to be sure that this distribution can be used one has to prove the following inequalities presented by the following Lemma.

Lemma 3.1. *For $-1 < q < 1$ and $x \in [-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ one has*

$$\frac{f_H(x \mid q)}{f_E(x \mid q)} \le M(q), \tag{3.2}$$

where

$$M(q) = \begin{cases} (1+q)(1-q^3)(1-q^5)(1-q^9)\displaystyle\prod_{k=4}^{\infty}(1-q^{2k})(1+q^k) & \text{if } q \in (0,1), \\ (1+q)(1-q^3)(1-q^5)(1-q^9)\displaystyle\prod_{k=4}^{\infty}\left(1+|q|^k\right)^2(1-q^k) & \text{if } q \in (-1,0). \end{cases} \tag{3.3}$$

Function $M(q)$ has the plot in Figure 2

Now following [12] we can simulate sequences of independent random variables with *q-Gaussian* distribution. If $q = \pm 1$ then such simulation is trivial.

For $q \in (-1,1)$ we use Lemma 3.1 and program Mathematica. We generated sequence of independent random variables from density f_E by inversion method (see [12]), since $f_E(x \mid q)$ can be integrated leading to cumulative distribution function (c.d.f.):

$$\begin{aligned}\int_{-2/\sqrt{1-q}}^{x} f_E(y \mid q)dy &= D(q)x\sqrt{(1-q)(4-(1-q)x^2)}Q_2\left(x^3,q\right) \\ &\quad + C(q)\arcsin\left(\frac{x\sqrt{1-q}}{2}\right) \overset{\text{def}}{=} F_E(x \mid q),\end{aligned} \tag{3.4}$$

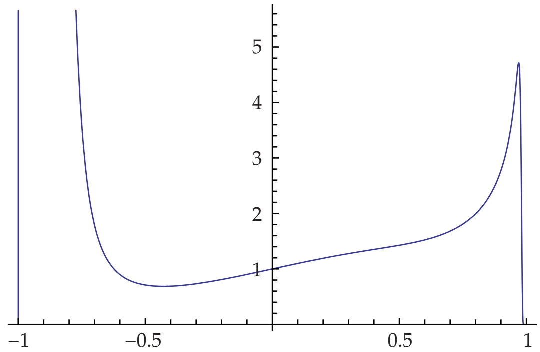

Figure 2: $M(q)$ $(0 \div 5)$ versus q $(-1 \div 1)$.

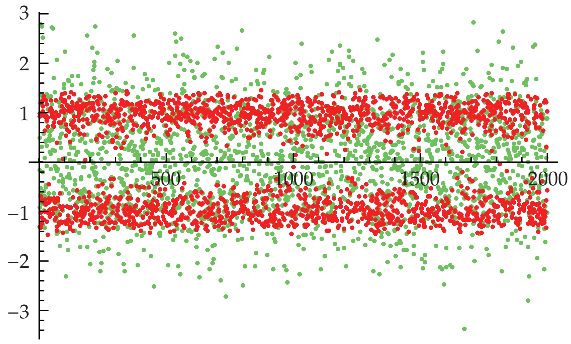

Figure 3: Simulation of i.i.d. sequences from P_H.

for $|x| \le 2/\sqrt{1-q}$, where Q_2 denotes quadratic polynomial in x^3 with coefficients depending on q, while the constants D and C are known functions of q. Recall that the inversion method requires solving numerically the sequence of equations $F_E(x \mid q) = r_i$, where r_i are observations drawn from standard uniform distribution.

Since the function $F_E(x \mid q)$ is strictly increasing on its support and its derivative is known, there are no numerical problems in solving this equation. Due to efficient procedure "FindRoot" of Mathematica solving this equation is quick.

Now let us recall how rejection method works in case $q \in (-1, 1)$.

Applying algorithm described in [12], the rule to get one realization of random variable having density f_H is as follows.

(1) We generate two variables: $X \sim f_E(.)$ and $Y \sim U(0,1)$.

(2) Set $T = M(q) f_E(X \mid q)/f_H(X \mid q)$.

(3) If $YT > 1$ then set $Z = X$ otherwise repeat (1) and (2).

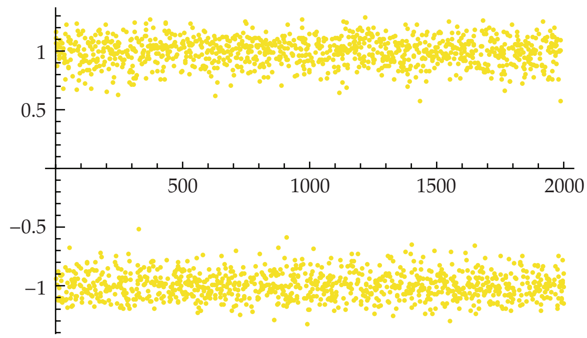

Figure 4: i.i.d. sequence from P_H by inversion method for $q = -.95$.

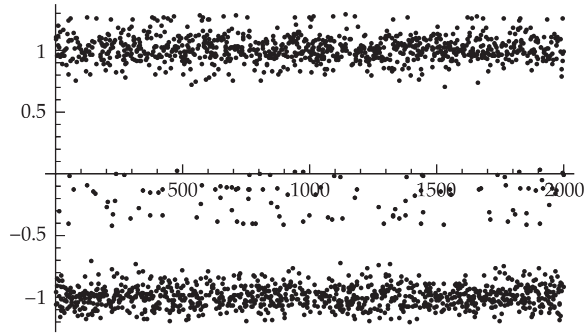

Figure 5: i.i.d. sequence from P_H by inversion method for $q = -.97$.

For details of *Mathematica* program realizing the above mentioned algorithm, see Appendix.

To see how this algorithm works, we present two simulation results performed (consisting of 2000 simulations) with $q = -.8$ (red dots) and $q = .8$ (green dots) in Figure 3.

Unfortunately this algorithm turns out to be very inefficient for q close to -1, more practically less than say $-.85$. One can see this by examining Figure 2. Values of $M(q)$ are very large then, showing that one needs very large number of observations from density f_E to obtain one observation from f_H. Thus there is still an open question to generate efficiently observations from f_H for values close to -1.

One might be inclined to use formula (2.2) and inversion method applied to its finite approximation and again use procedure "FindRoot". Well we applied this idea to simulate 2000 observations from P_H for $q = -.95$. It worked giving the results in Figure 4:

We used procedure "Findroot" of Mathematica. It worked, as one can see, however it lasted quite a time to get the result.

Besides, when we tried to get 2000 observations from P_H for $q = -.97$, numerical errors seemed to play an important role as one can notice judging from black dots that appeared between levels 0 and $-.5$ on the picture in Figure 5.

4. Proofs

Proof of Theorem 2.1. Let us denote $z = x\sqrt{1-q}/2$. Hence $|z| < 1$. We have

$$f_H(z \mid q) = \frac{(q)_\infty \sqrt{1-q}}{4\pi\sqrt{1-z^2}} \prod_{k=0}^{\infty} \left(\left(1+q^k\right)^2 - 4z^2 q^k \right). \tag{4.1}$$

Now let us notice that

$$\begin{aligned}\left(1+q^k\right)^2 - 4z^2q^k &= \left(1+q^k-2zq^{k/2}\right)\left(1+q^k+2zq^{k/2}\right) \\ &= \left(\left(\sqrt{1-z^2}+iz\right)^2+q^k\right)\left(\left(\sqrt{1-z^2}-iz\right)^2+q^k\right).\end{aligned} \tag{4.2}$$

Now notice that since $|z| < 1$, we see that $|\sqrt{1-z^2}+iz| = 1$. Thus we can write $\sqrt{1-z^2}+iz = \exp(i\theta)$ where $\theta = \arcsin z$ and also $\sqrt{1-z^2} = \cos\theta$. Hence we can write

$$\left(1+q^k\right)^2 - 4z^2q^k = \left(1+e^{2i\theta}q^k\right)\left(1+e^{-2i\theta}q^k\right), \tag{4.3}$$

and consequently

$$f_H(z \mid q) = \frac{(q)_\infty\sqrt{1-q}}{4\pi\cos\theta}\left(-e^{2i\theta}\right)_\infty\left(1+e^{-2i\theta}\right)\left(-qe^{-2i\theta}\right)_\infty. \tag{4.4}$$

We will now use so-called "triple product identity" (see [13, Theorem 10.4.1., page 497]) that states in our setting, that

$$\begin{aligned}(q)_\infty\left(-e^{2i\theta}\right)_\infty\left(-qe^{-2i\theta}\right)_\infty &= \sum_{k=-\infty}^{\infty}(-1)^k q^{\binom{k}{2}}\left(-e^{2i\theta}\right)^k \\ &= \sum_{k=-\infty}^{\infty} q^{k(k-1)/2}e^{2ik\theta} \\ &= 1+\sum_{k=1}^{\infty}q^{k(k-1)/2}e^{2ik\theta}+\sum_{j=1}^{\infty}q^{j(j+1)/2}e^{-2ij\theta} \\ &= 1+\sum_{k=1}^{\infty}q^{\binom{k}{2}}e^{2ik\theta}+\sum_{k=2}^{\infty}q^{\binom{k}{2}}e^{-2i(k-1)\theta}.\end{aligned} \tag{4.5}$$

Now notice that

$$\frac{\left(1+e^{-2i\theta}\right)}{\cos\theta} = \frac{2\cos\theta}{e^{i\theta}\cos\theta} = 2e^{-i\theta}. \tag{4.6}$$

Hence,

$$
\begin{aligned}
f_H(z \mid q) &= \frac{\sqrt{1-q}}{2\pi} e^{-i\theta}\left(1+\sum_{k=1}^{\infty} q^{\binom{k}{2}} e^{2ik\theta}+\sum_{k=2}^{\infty} q^{\binom{k}{2}} e^{-2i(k-1)\theta}\right) \\
&= \frac{\sqrt{1-q}}{2\pi} e^{-i\theta} \\
&\quad \times\left(1+e^{2\theta}+\sum_{k=2}^{\infty} q^{\binom{k}{2}}(\cos 2k\theta+\cos 2(k-1)\theta)+i\sum_{k=2}^{\infty} q^{\binom{k}{2}}(\sin 2k\theta-\sin 2(k-1)\theta)\right) \\
&= \frac{\sqrt{1-q}}{\pi}\cos\theta+\frac{\sqrt{1-q}}{\pi}\sum_{k=2}^{\infty} q^{\binom{k}{2}}\cos(2k-1)\theta.
\end{aligned}
\tag{4.7}
$$

To return to variable z we have to recall definition of Chebyshev polynomials. Namely, we have

$$
\begin{aligned}
\cos((2k-1)\theta) &= \cos((2k-1)\arcsin z) \\
&= \cos\left((2k-1)\left(\frac{\pi}{2}-\arccos z\right)\right) \\
&= (-1)^{k+1}\sin((2k-1)\arccos z) \\
&= (-1)^{k+1}\sqrt{1-z^2}\,U_{2k-2}(z),
\end{aligned}
\tag{4.8}
$$

where $U_n(z)$ is the Chebyshev polynomial of the second kind. More precisely we have here

$$
U_n(z) = \frac{\sin((n+1)\arccos z)}{\sqrt{1-z^2}}. \tag{4.9}
$$

It is well known, that sequence $\{U_n\}$ satisfies three-term recurrence equation

$$
U_{n+1}(z) - 2xU_n(z) + U_{n-1}(z) = 0, \tag{4.10}
$$

with $U_{-1}(z) = 0, U_0(z) = 1$, and can be calculated directly (see [14, Theorem 7.2, page 188]) as in (1.7). Thus we have shown that

$$
f_H(z \mid q) = \frac{\sqrt{1-q}}{\pi}\sqrt{1-z^2}\left(\sum_{k=1}^{\infty}(-1)^{k-1} q^{\binom{k}{2}} U_{2k-2}(z)\right), \tag{4.11}
$$

or equivalently

$$f_H(x \mid q) = \frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)x^2}\left(\sum_{k=1}^{\infty}(-1)^{k-1}q^{\binom{k}{2}}U_{2k-2}\left(x\sqrt{1-q}/2\right)\right). \tag{4.12}$$

□

Proof of Corollary 2.2. We have $F_H(y \mid q) = \int_{-2/\sqrt{1-q}}^{y} f_H(x)dx$. Now we change variable to $z = x\sqrt{1-q}/2$ and use (2.1). Thus

$$F_H(y \mid q) = \frac{2}{\pi}\left(\int_{-1}^{y\sqrt{1-q}/2}\sqrt{1-z^2}\,dz + \sum_{k=2}^{\infty}(-1)^{k-1}q^{\binom{k}{2}}\int_{-1}^{y\sqrt{1-q}/2}\sqrt{1-z^2}U_{2k-2}(z)dz\right). \tag{4.13}$$

We use now the following, easy to prove, formulae: $\int_{-1}^{y}\sqrt{1-z^2}U_{2n}(z)dz = \sqrt{1-y^2}(U_{2n+1}(y)/(4n+4) - U_{2n-1}(y)/(4n))$ and $\int_{-1}^{y}\sqrt{1-x^2}dx = (\pi/2)+(1/2)\arcsin y+(1/2)y\sqrt{1-y^2}$ and after rearranging terms get (2.2). □

Proof of Corollary 2.3. Let $f_U(x \mid q) = (\sqrt{1-q}/2\pi)\sqrt{4-(1-q)x^2}$ for $|x|\sqrt{1-q} \le 2$. Assertion (i) is obtained directly after noting that

$$f_H(x \mid q) = f_U(x \mid q)\prod_{k=1}^{\infty}\left(\left(1+q^k\right)^2 - (1-q)x^2q^k\right)\left(1-q^k\right). \tag{4.14}$$

Following (2.1), we get

$$f_H(x \mid q) = f_U(x \mid q)\left(\sum_{k=1}^{\infty}(-1)^{k-1}q^{\binom{k}{2}}U_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right)\right). \tag{4.15}$$

(ii) and (iii) are obtained by inserting $x = 0$ and $x = 1/\sqrt{1-q}$ in (2.1) and canceling out common factors. From (2.1) it follows also that values $U_{2n}(0)$ and $U_{2n}(1/2)$ will be needed. Keeping in mind (4.9) we see that $U_{2n}(0) = \cos(n\pi) = (-1)^n$ and

$$U_{2n}\left(\frac{1}{2}\right) = \begin{cases} 1 & \text{if } n = 3m, \\ 0 & \text{if } n = 3m+1, \\ -1 & \text{if } n = 3m+2. \end{cases} \tag{4.16}$$

On the other hand we see that

$$\prod_{k=1}^{\infty}\left(\left(1+q^k\right)^2-(1-q)x^2q^k\right)\prod_{k=0}^{\infty}(1-q^{k+1})\Big|_{x=0}=\prod_{k=1}^{\infty}\left(1-q^{2k}\right)\left(1+q^k\right),$$

$$\begin{aligned}\prod_{k=1}^{\infty}\left(\left(1+q^k\right)^2-(1-q)x^2q^k\right)\prod_{k=0}^{\infty}(1-q^{k+1})\Big|_{x=1/\sqrt{1-q}}&=\prod_{k=1}^{\infty}\left(\left(1+q^k\right)^2-q^k\right)\prod_{k=1}^{\infty}\left(1-q^k\right)\\&=\prod_{k=1}^{\infty}\left(1+q^k+q^{2k}\right)\prod_{k=1}^{\infty}\left(1-q^k\right)\\&=\prod_{k=1}^{\infty}\left(1-q^{3k}\right).\end{aligned}\tag{4.17}$$

(iv) Putting $x=\pm 2/\sqrt{1-q}$ in (2.3) we get

$$\prod_{k=1}^{\infty}\left(1-q^k\right)^3=\sum_{k=1}^{\infty}(-1)^{k-1}q^{\binom{k}{2}}U_{2k-2}(1).\tag{4.18}$$

Now recall that $U_{2k}(1)=2k+1$.

(v) To see this notice that q-Hermite polynomials are orthogonal with respect to the measure with density f_H. Thus we have

$$\forall m>0:\int_{-2/\sqrt{1-q}}^{-2/\sqrt{1-q}}H_m(x\mid q)f_H(x\mid q)dx=0.\tag{4.19}$$

Using (2.1) know that $\forall m>0$

$$\int_{-2/\sqrt{1-q}}^{-2/\sqrt{1-q}}H_m(x\mid q)\sqrt{4-(1-q)x^2}\left(1+\sum_{k=2}^{\infty}(-1)^{k+1}q^{\binom{k}{2}}U_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right)\right)dx=0.\tag{4.20}$$

Observing that function f_H is symmetric and q-Hermite polynomials of odd order are odd functions, we deduce that above mentioned identities are trivial for odd m. Thus, let us concentrate on even m. Introducing new variable $z=x\sqrt{1-q}/2$, and multiplying both sides of this identity by $(1-q)^{m/2}$ we get

$$\forall m>1:\int_{-1}^{1}h_m(z\mid q)\left(1+\sum_{k=2}^{\infty}(-1)^{k+1}q^{\binom{k}{2}}U_{2k-2}(z)\right)\sqrt{1-z^2}dz=0,\tag{4.21}$$

where $h_m(z \mid q) = (1-q)^{m/2} H_m(2z/\sqrt{1-q} \mid q)$. Polynomials h_m are called continuous q-Hermite polynomials. It can be easily verified (following (1.6)) that they satisfy the following three-term recurrence equation:

$$h_{n+1}(t \mid q) = 2th_n(t \mid q) - (1-q^n)h_{n-1}(t \mid q), \tag{4.22}$$

with $h_{-1} = 0$, $h_0(z \mid q) = 1$. Moreover, it is also known that (see, e.g., [13]),

$$h_n(\cos\theta \mid q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \cos((n-2k)\theta). \tag{4.23}$$

Let us change once more variables in (4.21) and put $z = \cos\tau$. Then, for all $m > 1$,

$$\int_0^{\pi} h_m(\cos\tau \mid q) \times \left(1 + \sum_{k=2}^{\infty} (-1)^{k+1} q^{\binom{k}{2}} U_{2k-2}(\cos\tau)\right) \times \sin\tau \, d\tau = 0, \tag{4.24}$$

or for all $m > 1$,

$$\int_0^{\pi} h_m(\cos\tau \mid q) \left(\sum_{k=1}^{\infty} (-1)^{k-1} q^{\binom{k}{2}} \sin(2k-1)\tau \sin\tau \right) d\tau = 0. \tag{4.25}$$

Keeping in mind that $2\sin(2k-1)\tau\sin\tau = \cos(2k-2)\tau - \cos(2k)\tau$, we see that

$$\forall m > 1 : \int_0^{\pi} h_{2m}(\cos\tau \mid q) \left(\sum_{k=1}^{\infty} (-1)^{k-1} q^{\binom{k}{2}} (\cos(2k-2)\tau - \cos(2k)\tau) \right) d\tau = 0. \tag{4.26}$$

Now keeping in mind that $\int_0^{\pi} \cos 2m\tau d\tau = 0$ for $m > 0$ we see that

$$\forall m > 1 : \int_0^{\pi} h_{2m}(\cos\tau \mid q) \left(\sum_{k=m+1}^{\infty} (-1)^{k+1} q^{\binom{k}{2}} (\cos(2k-2)\tau - \cos(2k)\tau) \right) d\tau = 0. \tag{4.27}$$

On the other hand taking into account (4.23) we see that

$$\int_0^{\pi} h_{2m}(\cos\tau \mid q) \cos 2k\tau d\tau = \pi \begin{bmatrix} 2m \\ m-k \end{bmatrix}_q, \tag{4.28}$$

for $k = 0, 1, \ldots, m,$. Hence we have (2.7). □

Proof of Corollary 2.4. Keeping in mind that f_H is symmetric with respect to x we deduce that the point of change of modality of f_H must be characterized by the condition $f_H''(0 \mid q_0) = 0$. Calculating second derivative of the right hand side of (2.1) and remembering that

$(\sqrt{4-(1-q)x^2})'\big|_{x=0} = 0$ we end up with an equation $0 = -(1-q)/2\sum_{k=0}^{\infty} q^{k(k+1)/2} + 2(1-q) \times \sum_{k=1}^{\infty}(-1)^k k(k+1)q^{k(k+1)/2}(-1)^{k+1}$. Now since $4k(k+1)+1 = (2k+1)^2$ we get equation in Corollary 2.4 defining q_0. □

To prove Lemma 2.5 we need the following lemma.

Lemma 4.1. *Suppose* $0 < r < 1$ *and* $n \geq 3$. *Then*

$$\sum_{k\geq n}(2k-1)r^{\binom{k}{2}} \leq \frac{2nr^{n(n-1)/2}}{(1-r^2)^2}. \tag{4.29}$$

Proof. Recall that for $|\rho| < 1$ we have: $\sum_{i\geq 1} i\rho^{i-1} = (1/(1-\rho))' = 1/(1-\rho)^2$ and that $\sum_{i=0}^{m}\rho^i = (1-\rho^{m+1})/(1-\rho)$. Thus we have

$$\begin{aligned}\sum_{k\geq n}(2k-1)r^{\binom{k}{2}} &= r^{(n-1)(n-4)/2}\sum_{k\geq n}(2k-1)r^{k(k-1)/2-(n-1)(n-4)/2}\\ &= r^{(n-1)(n-4)/2}\times\sum_{k\geq n}(2k-1)r^{2k-2}r^{k(k-1)/2-(n-1)(n-4)/2-2(k-1)}.\end{aligned} \tag{4.30}$$

Now notice that $k(k-1)/2-(n-1)(n-4)/2-2(k-1) = (1/2)(k-n)(k+n-5) \geq 0$ for $k \geq n \geq 3$. Hence

$$\begin{aligned}\sum_{k\geq n}(2k-1)r^{\binom{k}{2}} &\leq r^{(n-1)(n-4)/2}\sum_{k\geq n}(2k-1)r^{2k-2}\\ &\leq 2r^{(n-1)(n-4)/2}\sum_{k\geq n}k\left(r^2\right)^{k-1}\\ &= 2r^{(n-1)(n-4)/2}\frac{d}{dr^2}\left(\frac{1}{1-r^2}-\frac{1-r^{2n}}{1-r^2}\right)\\ &= 2r^{(n-1)(n-4)/2}\frac{d}{dr^2}\left(\frac{\left(r^2\right)^n}{1-r^2}\right)\\ &= 2r^{(n-1)(n-4)/2}\frac{n\left(r^2\right)^{n-1}-(n-1)\left(r^2\right)^n}{(1-r^2)^2}\\ &\leq 2\frac{nr^{n(n-1)/2}}{(1-r^2)^2}.\end{aligned} \tag{4.31}$$

□

Proof of Lemma 2.5. (i) We have

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{4.32}$$

From this fact we deduce that $\sup_{|x|\le 1}|U_n(x)| = (n+1)$. Now using Lemma (4.1) we have

$$\begin{aligned}
&\left| f_H(x \mid q) - \frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)x^2}\left(1+\sum_{k=2}^{n-1}(-1)^{k+1}q^{\binom{k}{2}}U_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right)\right)\right| \\
&\quad = \left|\frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)x^2}\sum_{k\ge n}(-1)^{k+1}q^{\binom{k}{2}}U_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right)\right| \\
&\quad \le \frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)x^2}\sum_{k\ge n}|q|^{\binom{k}{2}}(2k-1) \\
&\quad \le \frac{\sqrt{1-q}}{\pi}\frac{2n|q|^{n(n-1)/2}}{(1-q^2)^2}.
\end{aligned} \tag{4.33}$$

(ii)

$$\begin{aligned}
&\left| F_H(y \mid q) - \frac{1}{2} - \frac{1}{\pi}\arcsin\left(\frac{y\sqrt{1-q}}{2}\right) - \frac{\sqrt{4-(1-q)y^2}}{2\pi}\sum_{k=1}^{n-1}(-1)^{k-1}q^{\binom{k}{2}}U_{2n-2}\left(\frac{y\sqrt{1-q}}{2}\right)\right| \\
&\quad \le \frac{\sqrt{4-(1-q)y^2}}{2\pi}\sum_{k=n}^{\infty}|q|^{\binom{k}{2}}\left|U_{2n-2}\left(\frac{y\sqrt{1-q}}{2}\right)\right| \\
&\quad \le \frac{1}{2\pi}\sum_{k=n}^{\infty}|q|^{\binom{k}{2}}\left(1+|q|^k\right),
\end{aligned} \tag{4.34}$$

since $\sup_{|x|\le 1}\sqrt{1-x^2}|U_n(x)| = 1$. Now to get (ii) we use routine transformations and sum two geometric series. □

Proof of Lemma 3.1. Notice that comparing definitions of f_H and f_E we have

$$\begin{aligned}
f_H(x \mid q) &= f_E(x \mid q) \times \frac{1-q^9}{1-q}\frac{1-q^5}{1-q} \times \prod_{k=4}^{\infty}\left(\left(1+q^k\right)^2 - (1-q)x^2q^k\right) \times \prod_{k=1}^{\infty}\left(1-q^k\right) \\
&= (1+q)\left(1-q^3\right)\left(1-q^5\right)\left(1-q^9\right) \times \prod_{k=4}^{\infty}\left(\left(\left(1+q^k\right)^2 - (1-q)x^2q^k\right)\left(1-q^k\right)\right).
\end{aligned} \tag{4.35}$$

Now if $q \in [0,1)$ we have

$$\begin{aligned}
\left(\left(1+q^k\right)^2 - (1-q)x^2q^k\right)\left(1-q^k\right) &\le \left(1+q^k\right)^2\left(1-q^k\right) \\
&= \left(1-q^{2k}\right)\left(1+q^k\right).
\end{aligned} \tag{4.36}$$

If $q \in (-1, 0)$ then

$$\left(\left(1+q^k\right)^2 - (1-q)x^2q^k\right)\left(1-q^k\right) \le \left(1-q^k\right)\begin{cases}\left((1+q^k)^2 - 4q^k\right) & \text{if } k \text{ is odd}\\ (1+q^k)^2 & \text{if } k \text{ is even}\end{cases} \\ = \left(1-q^k\right)\left(1+|q|^k\right)^2, \tag{4.37}$$

since then $\sup_{|x|\le 2/\sqrt{1-q}}((1+q^k)^2 - (1-q)x^2q^k) = ((1+q^k)^2 - 4q^k) = (1-q^k)^2$. □

Appendix

Program in Mathematica that generates i.i.d. sequences from f_H.

$QN[q\_, M\_] := (Label[pocz];\ Y = y/.FindRoot[F[y,q]\text{-}RandomReal[],\ \{y,0\}];$
$u = RandomReal[];\ t = newMM[q,M]/R[Y,q,M];\ If[t\ u <= 1, Y, Goto[pocz]]);$

However it requires definition of function F which is in fact function F_E of this paper. It is quite lengthy. newMM denotes function M of this paper. Further function R denotes the ratio f_H/f_E. Parameter M denotes number that we insert instead of ∞ in the above mentioned formulae. The above procedure produces 1 observation from f_H.

Now $AA[q\_, M\_, h\_] := ListPlot[Table[QN[q,M],\{2000\}], PlotStyle\text{-}>Hue[h]];$ produces table of 2000 observation from $f_H(\cdot \mid q)$ and plots it in color h. Then $AA[.8, 100, .4]$ and $AA[-.8, 100, 0]$ produce plots for $q = .8$ in color.4 (green) and $q = -.8$ in color 0 (red).

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