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Research Article

Convergence of Locally Square Integrable Martingales to a Continuous Local Martingale

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Let for each $n \in \mathbb{N}$ X_n be an $\mathbb{R}^d$-valued locally square integrable martingale w.r.t. a filtration $(\mathcal{F}_n(t),\ t \in \mathbb{R}_+)$ (probability spaces may be different for different n). It is assumed that the discontinuities of X_n are in a sense asymptotically small as $n \to \infty$ and the relation $\mathsf{E}(f(\langle zX_n\rangle(t))|\mathcal{F}_n(s)) - f(\langle zX_n\rangle(t)) \xrightarrow{\mathsf{P}} 0$ holds for all $t > s > 0$, row vectors z, and bounded uniformly continuous functions f. Under these two principal assumptions and a number of technical ones, it is proved that the X_n's are asymptotically conditionally Gaussian processes with conditionally independent increments. If, moreover, the compound processes $(X_n(0), \langle X_n\rangle)$ converge in distribution to some $(\overset{\circ}{X}, H)$, then a sequence (X_n) converges in distribution to a continuous local martingale X with initial value $\overset{\circ}{X}$ and quadratic characteristic H, whose finite-dimensional distributions are explicitly expressed via those of $(\overset{\circ}{X}, H)$.

1. Introduction

The theory of functional limit theorems for martingales may appear finalized in the monographs [1, 2]. This paper focuses at two points, where the classical results can be refined.

(1) The convergence in distribution to a local martingale with $\mathcal{G}$-conditional increments has been studied hitherto in the model, where the σ-algebra $\mathcal{G}$ enters the setting along with the prelimit processes. This assumption is worse than restrictive—it is simply unnatural when one studies the convergence in distribution, not in probability. In the present paper, conditions ensuring asymptotic conditional independence of increments for a sequence of locally square integrable martingales are formulated in terms of quadratic characteristics of the prelimit processes (Theorem 4.5). Our approach to the proving of this property is based on the idea to combine the Stone-Weierstrass theorem (actually its slight

modification—Lemma 2.2) with an elementary probabilistic result—Lemma 2.4, which issues in Corollaries 2.7 and 2.8. These corollaries, as well as Lemma 2.4 itself and the cognate Lemma 2.5, will be our tools.

(2) The main object of study in [1, 2] is semimartingale. So, some specific for local martingales facts are passed by. Thus, Theorem VI.6.1 and Corollary VI.6.7 in [2] assert that under appropriate assumptions about semimartingales Z_n, the relation

$$Z_n \xrightarrow{\text{law}} Z, \tag{$*$}$$

where Z also is a semimartingale, entails the stronger one $(Z_n, [Z_n]) \xrightarrow{\text{law}} (Z, [Z])$ (below, the notation of convergence in law will be changed). For locally square integrable martingales, one can modify the problem as follows. Let relation $(*)$ be fulfilled. What extra assumptions ensure that Z is a continuous local martingale and

$$(Z_n, \langle Z_n \rangle) \xrightarrow{\text{law}} (Z, \langle Z \rangle)? \tag{$**$}$$

There is neither an answer nor even the question in [1, 2]. A simple set of sufficient conditions is provided by Corollary 5.2 (weaker but not so simple conditions are given by Corollary 5.5). Recalling that the quadratic variation of a continuous local martingale coincides with its quadratic characteristic, we see that the last two relations imply together asymptotic proximity of $[Z_n]$ and $\langle Z_n \rangle$. Actually, this conclusion requires even less conditions than in Corollary 5.2. They are listed in Corollary 5.3.

The main results of the paper are, in a sense, converse to Corollaries 5.2 and 5.5. They deal with the problem: what conditions should be adjoined to $\langle Z_n \rangle \xrightarrow{\text{law}} H$ in order to ensure $(**)$, where Z is a continuous local martingale with quadratic characteristic H? If the assumptions about the prelimit processes do not guarantee that H performing as $\langle Z \rangle$ determines the distribution of Z, then results of this kind assert existence of convergent subsequences but not convergence of the whole sequence (Theorems 5.1 and 5.4). Combining Theorems 5.4 with 4.5, we obtain Theorem 5.6 asserting that the whole sequence converges to a continuous local martingale whose finite-dimensional distributions are explicitly expressed via those of its initial value and quadratic characteristic. The expression shows that the limiting process has conditionally independent increments—but this conclusion is nothing more than a comment to the theorem.

The proving of the main results needs a lot of preparation. Those technical results which do not deal with the notion of martingale are gathered in Section 2 (excluding Section 2.1), and the more specialized ones are placed in Section 3. The rationale in Sections 3 and 4 would be essentially simpler if we confined ourselves to quasicontinuous processes (for a locally square integrable martingale, this property is tantamount to continuity of its quadratic characteristic). To dispense with this restriction, we use a special technique sketched in Section 2.1.

All vectors are thought of, unless otherwise stated, as columns. The tensor square $xx^\top$ of $x \in \mathbb{R}^d$ will be otherwise written as $x^{\otimes 2}$. We use the Euclidean norm $|\cdot|$ of vectors and the operator norm $\|\cdot\|$ of matrices. The symbols $\mathbb{R}^{d*}$, $\mathfrak{S}$, and $\mathfrak{S}_+$ signify: the space of d-dimensional row vectors, the class of all symmetric square matrices of a fixed size (in our case—d) with real entries, and its subclass of nonnegative (in the spectral sense) matrices, respectively.

By $\mathrm{C_b}(X)$, we denote the space of complex-valued bounded continuous functions on a topological space X. If $X = \mathbb{R}^k$ and the dimension k is determined by the context or does not matter, then we write simply $\mathrm{C_b}$.

Our notation of classes of random processes follows [3]. In particular, $\overline{\mathcal{M}}(\mathbb{F})$ and $\mathcal{M}(\mathbb{F})$ signify the class of all martingales with respect to a filtration (= flow of σ-algebras) $\mathbb{F} \equiv (\mathcal{F}(t),\ t \in \mathbb{R}_+)$ and its subclass of uniformly integrable martingales. An $\mathbb{F}$-martingale U will be called: *square integrable* if $\mathsf{E}|U(t)|^2 < \infty$ for all t and *uniformly square integrable* if $\sup_t \mathsf{E}|U(t)|^2 < \infty$. The classes of such processes will be denoted $\overline{\mathcal{M}}_2(\mathbb{F})$ and $\mathcal{M}_2(\mathbb{F})$, respectively. The symbol $\mathbb{F}$ will be suppressed if the filtration either is determined by the context or does not matter. If $\mathcal{U}$ is a class of $\mathbb{F}$-adapted process, then by $\ell\mathcal{U}$ we denote the respective local class (see [2, Definition I.1.33], where the notation $\mathcal{U}^{\mathrm{loc}}$ is used). Members of $\ell\mathcal{M}$, and $\ell\mathcal{M}_2$ are called *local martingales* and *locally* (better *local*) *square integrable martingales*, respectively. All processes, except quadratic variations and quadratic characteristics, are implied $\mathbb{R}^d$-valued, where d is chosen arbitrarily and fixed.

The integral $\int_0^t \varphi(s)\mathrm{d}X(s)$ will be written shortly (following [1, 2]) as $\varphi \circ X(t)$ if this integral is pathwise (i.e., X is a process of locally bounded variation) or $\varphi \cdot X(t)$ if it is stochastic. We use properties of stochastic integral and other basic facts of stochastic analysis without explanations, relegating the reader to [1–4]. The quadratic variation (see the definition in Section 2.3 [3] or Definition I.4.45 together with Theorem I.4.47 in [2]) of a random process ξ and the quadratic characteristic of $Z \in \ell\mathcal{M}_2$ will be (and already were) denoted $[\xi]$ and $\langle Z\rangle$, respectively. They take values in $\mathfrak{S}_+$, which, of course, does not preclude to regard them as $\mathbb{R}^{d^2}$-valued random processes.

2. Some Technical Results

The Stone-Weierstrass theorem (see, e.g., [5]) concerns compact spaces only. In the following two, its minor generalizations (for real-valued and complex-valued functions, resp.) both the compactness assumption and the conclusion (that the approximation is uniform on the whole space) are weakened. They are proved likewise their celebrated prototype if one argues for the restrictions of continuous functions to some compact subset fixed beforehand.

Lemma 2.1. *Let $\mathfrak{A}$ be an algebra of real-valued bounded continuous functions on a topological space T. Suppose that $\mathfrak{A}$ separates points of T and contains the module of each its member and the unity function. Then, for any real-valued bounded continuous function F, compact set $B \subset T$, and positive number ε, there exists a function $G \in \mathfrak{A}$ such that $\|G\|_\infty \le \|F\|_\infty$ and $\max_{x\in B}|F(x) - G(x)| < \varepsilon$.*

Lemma 2.2. *Let $\mathfrak{A}$ be an algebra of complex-valued bounded continuous functions on a topological space T. Suppose that $\mathfrak{A}$ separates points of T, and contains the conjugate of each its member and the unity function. Then for any complex-valued bounded continuous function F, compact set $B \subset T$, and positive number ε there exists a function $G \in \mathfrak{A}$ such that $\|G\|_\infty \le \|F\|_\infty$ and $\max_{x\in B}|F(x) - G(x)| < \varepsilon$.*

We consider henceforth sequences of random processes or random variables given, maybe, on different probability spaces. So, for the nth member of a sequence, P and E should be understood as P_n and E_n. In what follows, "u.i." means "uniformly integrable".

Lemma 2.3. *In order that a sequence of random variables be u.i., it is necessary and sufficient that each its subsequence contain a u.i. subsequence.*

Proof. Necessity is obvious; let us prove sufficiency.

Suppose that a sequence (α_n) is not u.i. Then, there exists $a > 0$ such that for all $N > 0$

$$\overline{\lim_{n\to\infty}} \mathsf{E}|\alpha_n| I\{|\alpha_n| > N\} \geq 2a. \tag{2.1}$$

Consequently, there exists an increasing sequence (n_k) of natural numbers such that $\mathsf{E}\beta_k I\{\beta_k > k\} \geq a$, where $\beta_k = |\alpha_{n_k}|$. Then, for any infinite set $J \subset \mathbb{N}$ and $N > 0$, we have

$$\underline{\lim}_{k\to\infty, k\in J} \mathsf{E}\,\beta_k I\{\beta_k > N\} \geq a, \tag{2.2}$$

which means that the subsequence (α_{n_k}) does not contain u.i. subsequences. □

Lemma 2.4. *Let for each n $\xi_{n1}, \ldots, \xi_{np}$ be random variables given on a probability space $(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$, and $\mathcal{H}_n$ a sub-σ-algebra of $\mathcal{F}_n$. Suppose that for each $j \in \{1, \ldots, p\}$,*

$$\mathsf{E}(\xi_{nj} \mid \mathcal{H}_n) - \xi_{nj} \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \longrightarrow \infty, \tag{2.3}$$

and for any $J \subset \{1, \ldots, p\}$ the sequence $(\prod_{j\in J} \xi_{nj}, n \in \mathbb{N})$ is u.i. Then,

$$\mathsf{E}\left(\prod_{j=1}^{p} \xi_{nj} \mid \mathcal{H}_n\right) - \prod_{j=1}^{p} \xi_{nj} \xrightarrow{\mathsf{P}} 0. \tag{2.4}$$

Proof. Denote $\eta_{nj} = \mathsf{E}(\xi_{nj} \mid \mathcal{H}_n)$. By the second assumption, the sequences $(\xi_{nj}, n \in \mathbb{N}), (\eta_{nj}, n \in \mathbb{N})$, $j = 1, \ldots, p$, are stochastically bounded, which together with the first assumption yields

$$\prod_{j\in J} \eta_{nj} - \prod_{j\in J} \xi_{nj} \xrightarrow{\mathsf{P}} 0, \tag{2.5}$$

for any $J \subset \{1, \ldots, p\}$. Hence, writing the identity

$$\mathsf{E}(\xi_{n1}\xi_{n2} \mid \mathcal{H}_n) = \mathsf{E}((\xi_{n1} - \eta_{n1})(\xi_{n2} - \eta_{n2}) \mid \mathcal{H}_n) + \eta_{n1}\eta_{n2}, \tag{2.6}$$

and using both assumptions, we get (2.4) for $p = 2$. For arbitrary p, this relation is proved by induction whose step coincides, up to notation, with the above argument. □

The proof of the next statement is similar.

Lemma 2.5. *Let α_n and β_n be random variables given on a probability space $(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$ and $\mathcal{H}_n$ a sub-σ-algebra of $\mathcal{F}_n$. Suppose that*

$$\alpha_n - \mathsf{E}(\alpha_n \mid \mathcal{H}_n) \xrightarrow{\mathsf{P}} 0, \tag{2.7}$$

and the sequences (α_n), (β_n) *and* $(\alpha_n\beta_n)$ *are u.i. Then,*

$$\mathsf{E}\big(\alpha_n\beta_n \mid \mathcal{H}_n\big) - \alpha_n \mathsf{E}\big(\beta_n \mid \mathcal{H}_n\big) \xrightarrow{\mathsf{P}} 0. \tag{2.8}$$

Lemma 2.6. *Let* $n \in \mathbb{N}$ Ξ_n *be an* $\mathbb{R}^k$*-valued random variable given on a probability space* $(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$*, and* $\mathcal{H}_n$ *a sub-*σ*-algebra of* $\mathcal{F}_n$*. Suppose that*

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\{|\Xi_n| > N\} = 0, \tag{2.9}$$

and the relation

$$\mathsf{E}(F(\Xi_n) \mid \mathcal{H}_n) \ - F(\Xi_n) \xrightarrow{\mathsf{P}} 0 \tag{2.10}$$

holds for all F from some class of complex-valued bounded continuous functions on $\mathbb{R}^k$ *which separates points of the latter. Then, it holds for all* $F \in \mathrm{C_b}$.

Proof. Let $\mathfrak{A}$ denote the class of all complex-valued bounded continuous functions on $\mathbb{R}^k$ satisfying (2.10). Obviously, it is linear. By Lemma 2.4, it contains the product of any two its members. So, $\mathfrak{A}$ is an algebra. By assumption, it separates points of $\mathbb{R}^k$. The other two conditions of Lemma 2.2 are satisfied trivially. Thus, that lemma asserts that for any $F \in \mathrm{C_b}$, $N > 0$ and $\varepsilon > 0$, there exists a function $G \in \mathfrak{A}$ such that $\|G\|_\infty \le \|F\|_\infty$ and $\max_{|x|\le N}|F(x) - G(x)| < \varepsilon$. Then,

$$\begin{gathered} |F(\Xi_n) - G(\Xi_n)| I\{|\Xi_n| \le N\} < \varepsilon, \\ |F(\Xi_n) - G(\Xi_n)| \ I\{|\Xi_n| > N\} \le 2\|F\|_\infty \, I\{|\Xi_n| > N\}. \end{gathered} \tag{2.11}$$

By the choice of G

$$\mathsf{E}(G(\Xi_n) \mid \mathcal{H}_n) - G(\Xi_n) \xrightarrow{\mathsf{P}} 0, \tag{2.12}$$

whence by the dominated convergence theorem

$$\mathsf{E}|\mathsf{E}(G(\Xi_n) \mid \mathcal{H}_n) - G(\Xi_n)| \longrightarrow 0. \tag{2.13}$$

Writing the identity

$$\begin{aligned} \mathsf{E}(F(\Xi_n) \mid \mathcal{H}_n) - F(\Xi_n) = {} & \mathsf{E}((F(\Xi_n) - G(\Xi_n))I\{|\Xi_n| \le N\} \mid \mathcal{H}_n) \\ & + \mathsf{E}((F(\Xi_n) - G(\Xi_n))I\{|\Xi_n| > N\} \mid \mathcal{H}_n) + \mathsf{E}(G(\Xi_n) \mid \mathcal{H}_n) - G(\Xi_n) \\ & + (G(\Xi_n) - F(\Xi_n))I\{|\Xi_n| \le N\} + (G(\Xi_n) - F(\Xi_n))I\{|\Xi_n| > N\}, \end{aligned} \tag{2.14}$$

we get from (2.11)–(2.13)

$$\overline{\lim_{n\to\infty}} \mathsf{E}|\mathsf{E}(F(\Xi_n) \mid \mathcal{H}_n) - F(\Xi_n)| \le 2\varepsilon + 4\|F\|_\infty \overline{\lim_{n\to\infty}} \mathsf{P}\{|\Xi_n| > N\}, \tag{2.15}$$

which together with (2.9) and due to arbitrariness of ε proves (2.10). □

Corollary 2.7. *Let for each n $\zeta_{n1},\dots,\zeta_{np}$ be $\mathbb{R}^d$-valued random variables given on a probability space $(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$ and $\mathcal{H}_n$ a sub-σ-algebra of $\mathcal{F}_n$. Suppose that the relations*

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\{|\zeta_{nj}| > N\} = 0, \tag{2.16}$$

$$\mathsf{E}(g(\zeta_{nj}) \mid \mathcal{H}_n) - g(\zeta_{nj}) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \longrightarrow \infty \tag{2.17}$$

hold for all $j \in \{1,\dots,p\}$ and g from some class $\mathfrak{F}$ of complex-valued bounded continuous functions on $\mathbb{R}^d$ which separates points of the latter. Then,

$$\mathsf{E}(F(\zeta_{n1},\dots,\zeta_{np}) \mid \mathcal{H}_n) - F(\zeta_{n1},\dots,\zeta_{np}) \xrightarrow{\mathsf{P}} 0, \tag{2.18}$$

for all $F \in C_b(\mathbb{R}^{pd})$.

Proof. Denote $\Xi_n = (\zeta_{n1},\dots,\zeta_{np})$. Condition (2.17) implies by Lemma 2.4 that relation (2.10) is valid for all F of the kind $F(x_1,\dots,x_p) = \prod_{i=1}^p g_i(x_i)$, where $g_i \in \mathfrak{F}$. Obviously, such functions separate points of $\mathbb{R}^{pd}$. Furthermore, condition (2.16) where j runs over $\{1,\dots,p\}$ is tantamount to (2.9). It remains to refer to Lemma 2.6. □

Corollary 2.8. *Let for each n K_n be an $\mathfrak{S}$-valued random process given on a probability space $(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$, $\mathcal{H}_n$ a sub-σ-algebra of $\mathcal{F}_n$, and ζ_{n0} an $\mathcal{H}_n$-measurable $\mathbb{R}^m$-valued random variable. Suppose that the relations*

$$\begin{gathered}\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\{\|K_n(t)\| > N\} = 0, \\ \mathsf{E}\Big(f\Big(zK_n(t)z^\top\Big) \mid \mathcal{H}_n\Big) - f\Big(zK_n(t)z^\top\Big) \xrightarrow{\mathsf{P}} 0,\end{gathered} \tag{2.19}$$

and (2.16) *hold for $j = 0$, all $t > 0$ and any bounded uniformly continuous function f on $\mathbb{R}$. Then, for any $l \in \mathbb{N}$, $s_l > \cdots > s_1 > 0$ and $F \in C_b(\mathbb{R}^m \times \mathfrak{S}^l)$ the relation*

$$\mathsf{E}(F(\zeta_{n0}, K_n(s_1),\dots,K_n(s_l)) \mid \mathcal{H}_n) - F(\zeta_{n0}, K_n(s_1),\dots,K_n(s_l)) \xrightarrow{\mathsf{P}} 0 \tag{2.20}$$

is valid.

Recall that for any $B \in \mathfrak{S}$

$$\|B\| = \max_{x\in S^{d-1}} \left|x^\top B x\right|, \tag{2.21}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$.

Lemma 2.9. *For any symmetric matrices B_1 and B_2,*

$$\max_{x \in S^{d-1}} \left| x^\top B_1 x x^\top B_2 x \right| \le \|B_1\| \|B_2\|. \tag{2.22}$$

Proof. It suffices to note that the left-hand side of the equality does not exceed $\max_{x,y \in S^{d-1}} |x^\top B_1 x y^\top B_2 y|$. □

Let $X, X_1, X_2 \ldots$ be $\mathbb{R}^d$-valued random processes with trajectories in the Skorokhod space D (= càdlàg processes on $\mathbb{R}_+$). We write $X_n \xrightarrow{\mathrm{D}} X$ if the induced by the processes X_n measures on the Borel σ-algebra in D weakly converge to the measure induced by X. If herein X is continuous, then we write $X_n \xrightarrow{\mathrm{C}} X$. We say that a sequence (X_n) is *relatively compact* (r.c.) *in* D (*in* C) if each its subsequence contains, in turn, a subsequence converging in the respective sense. The weak convergence of finite-dimensional distributions of random processes, in particular the convergence in distribution of random variables, will be denoted $\xrightarrow{\mathrm{d}}$. Likewise $\stackrel{\mathrm{d}}{=}$ means equality of distributions.

Denote $\Pi(t, r) = \{(u, v) \in \mathbb{R}^2 : (v - r)_+ \le u \le v \le t\}$,

$$\Delta_{\mathcal{U}}(f; t, r) = \sup_{(u,v) \in \Pi(t,r)} \left| f(v) - f(u) \right| \quad (f \in \mathrm{D},\ t > 0, r > 0). \tag{2.23}$$

Proposition VI.3.26 (items (i), (ii)) [2] together with VI.3.9 [2] asserts that a sequence (ξ_n) of càdlàg random processes is r.c. in C if and only if for all positive t and ε

$$\lim_{N \to \infty} \varlimsup_{n \to \infty} \mathrm{P}\left\{ \sup_{s \le t} |\xi_n(s)| > N \right\} = 0, \qquad \lim_{r \to 0} \varlimsup_{n \to \infty} \mathrm{P}\{ \Delta_{\mathcal{U}}(\xi_n; t, r) > \varepsilon \} = 0. \tag{2.24}$$

Hence, two consequences are immediate.

Corollary 2.10. *Let (ξ_n) and (Ξ_n) be sequences of $\mathbb{R}^d$-valued and $\mathbb{R}^m$-valued, respectively, càdlàg processes such that (Ξ_n) is r.c. in* C*, $|\xi_n(0)| \le |\Xi_n(0)|$ and for any $v > u \ge 0$*

$$|\xi_n(v) - \xi_n(u)| \le |\Xi_n(v) - \Xi_n(u)|. \tag{2.25}$$

Then, the sequence (ξ_n) is also r.c. in C.

Corollary 2.11. *Let (ξ_n) and (ζ_n) be r.c. in* C *sequences of càdlàg processes taking values in $\mathbb{R}^d$ and $\mathbb{R}^p$, respectively. Suppose also that for each n ξ_n and ζ_n are given on a common probability space. Then the sequence of $\mathbb{R}^{d+p}$-valued processes (ξ_n, ζ_n) is also r.c. in* C.

Lemma 2.12. *Let $(\eta_n^l, l, n \in \mathbb{N})$, (η^l), and (η_n) be sequences of càdlàg random processes such that for any positive t and ε*

$$\lim_{l \to \infty} \varlimsup_{n \to \infty} \mathrm{P}\left\{ \sup_{s \le t} \left| \eta_n^l(s) - \eta_n(s) \right| > \varepsilon \right\} = 0, \tag{2.26}$$

for each l

$$\eta_n^l \xrightarrow{\mathrm{D}} \eta^l \quad \text{as } n \longrightarrow \infty, \tag{2.27}$$

the sequence (η^l) *is r.c. in* D. *Then, there exists a random process* η *such that* $\eta^l \xrightarrow{\mathrm{D}} \eta$.

Proof. Let ρ be a bounded metric in D metrizing Skorokhod's $\mathcal{J}$-convergence (see, e.g., [2, VI.1.26]). Then, condition (2.26) with arbitrary t and ε implies that

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{E}\rho\left(\eta_n^l, \eta_n\right) = 0. \tag{2.28}$$

Hence, by the triangle inequality, we have

$$\lim_{\substack{m\to\infty \\ k\to\infty}} \overline{\lim_{n\to\infty}} \mathsf{E}\rho\left(\eta_n^m, \eta_n^k\right) = 0. \tag{2.29}$$

Let F be a uniformly continuous with respect to ρ bounded functional on D. Denote $A = \sup_{x\in\mathrm{D}}|F(x)|$, $\vartheta(r) = \sup_{x,y\in\mathrm{D}:\ \rho(x,y)<r}|F(x) - F(y)|$. Then, $\vartheta(0+) = 0$ and for any $r > 0$

$$\mathsf{E}\left|F(\eta_n^m) - F\left(\eta_n^k\right)\right| \le A\mathsf{P}\left\{\rho\left(\eta_n^m, \eta_n^k\right) > r\right\} + \vartheta(r), \tag{2.30}$$

which together with (2.29) yields

$$\lim_{\substack{m\to\infty \\ k\to\infty}} \overline{\lim_{n\to\infty}} \left|\mathsf{E}F(\eta_n^m) - \mathsf{E}F\left(\eta_n^k\right)\right| = 0. \tag{2.31}$$

By condition (2.27),

$$\lim_{n\to\infty}\left|\mathsf{E}F(\eta_n^m) - \mathsf{E}F\left(\eta_n^k\right)\right| = \left|\mathsf{E}F(\eta^m) - \mathsf{E}F\left(\eta^k\right)\right|, \tag{2.32}$$

which jointly with (2.31) proves fundamentality and, therefore, convergence of the sequence $(\mathsf{E}F(\eta^l), l \in \mathbb{N})$. Now, the desired conclusion emerges from relative compactness of (η^l) in D. □

Corollary 2.13. *Let the conditions of Lemma 2.12 be fulfilled. Then,* $\eta_n \xrightarrow{\mathrm{D}} \eta$, *where* η *is the existing by Lemma 2.12 random process such that* $\eta^l \xrightarrow{\mathrm{D}} \eta$.

Proof. Repeating the derivation of (2.31) from (2.29), we derive from (2.28) the relation

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \left|\mathsf{E}F\left(\eta_n^l\right) - \mathsf{E}F(\eta_n)\right| = 0. \tag{2.33}$$

It remains to write $|\mathsf{E}F(\eta_n) - \mathsf{E}F(\eta)| \le |\mathsf{E}F(\eta_n) - \mathsf{E}F(\eta_n^l)| + |\mathsf{E}F(\eta_n^l) - \mathsf{E}F(\eta^l)| + |\mathsf{E}F(\eta^l) - \mathsf{E}F(\eta)|$. □

Corollary 2.14. *Let* (η_n^l), (η^l), *and* (η_n) *be sequences of càdlàg random processes such that for any* $t \in \mathbb{R}_+$ *and* $\varepsilon > 0$ (2.26) *holds; for each* $l \in \mathbb{N}$ *relation* (2.27) *is valid; the sequence* (η^l) *is r.c. in* C. *Then, there exists a random process* η *such that* $\eta^l \xrightarrow{\mathrm{C}} \eta$ *and* $\eta_n \xrightarrow{\mathrm{C}} \eta$.

Below, $\mathcal{U}$ is the symbol of the locally uniform (i.e., uniform in every interval) convergence.

Lemma 2.15. *Let* $X, X_1, X_2 \ldots$ *be càdlàg random processes such that* $X_n \xrightarrow{\mathrm{C}} X$. *Then,* $F(X_n) \xrightarrow{\mathrm{d}} F(X)$ *for any* $\mathcal{U}$*-continuous functional* F *on* D.

Proof. Lemma VI.1.33 and Corollary VI.1.43 in [2] assert completeness and separability of the metric space (D, ρ), where ρ is the metric used in the proof of Lemma 2.12. Then, it follows from the assumptions of the lemma by Skorokhod's theorem [6] that there exist given on a common probability space càdlàg random processes $X', X'_1, X'_2 \ldots$ such that $X' \stackrel{\mathrm{d}}{=} X$ (so that X' is continuous), $X'_n \stackrel{\mathrm{d}}{=} X_n$ and $\rho(X'_n, X') \to 0$ a.s. By the choice of ρ, the last relation is tantamount to $X'_n \xrightarrow{\mathcal{J}} X'$ a.s. Hence, and from continuity of X', we get by Proposition VI.1.17 [2] $X'_n \xrightarrow{\mathcal{U}} X'$ a.s. and, therefore, by the choice of F, $F(X'_n) \to F(X')$ a.s. It remains to note that $F(X'_n) \stackrel{\mathrm{d}}{=} F(X_n)$, $F(X') \stackrel{\mathrm{d}}{=} F(X)$. □

2.1. Forestopping of Random Processes

Let $\mathbb{F}$ be a filtration on some probability space, S an $\mathbb{F}$-adapted random process, and τ a stopping time with respect to $\mathbb{F}$. We put $S(0-) = S(0)$ and denote $S^\tau(t) = S(t \wedge \tau)$,

$$ {}^\tau S(t) = S(t) I_{[0,\tau[}(t) + S(\tau-) I_{[\tau,\infty[}(t), \tag{2.34} $$

${}^\tau \mathcal{F}(t) = \mathcal{F}(t) \cap \mathcal{F}(\tau-)$, ${}^\tau \mathbb{F} = ({}^\tau \mathcal{F}(t), t \in \mathbb{R}_+)$. Obviously,

$$ {}^\tau (S^\tau) = {}^\tau S, \tag{2.35} $$

$$ [{}^\tau S] = {}^\tau [S], \tag{2.36} $$

provided $[S]$ exists. In case τ is $\mathbb{F}$-predictable, the operation $S \mapsto {}^\tau S$ was called in [7] the *forestopping*. The following three statements were proved in [7].

Lemma 2.16. *Let a random process* U *and a stopping time* τ *be* $\mathbb{F}$*-predictable. Then, the process* ${}^\tau U$ *is* ${}^\tau \mathbb{F}$*-predictable.*

Theorem 2.17. *Let* X *be an* $\mathbb{F}$*-martingale and* τ *an* $\mathbb{F}$*-predictable stopping time. Then,* ${}^\tau X$ *is a* ${}^\tau \mathbb{F}$*-martingale. If* X *is uniformly integrable, then so is* ${}^\tau X$.

Lemma 2.18. *Let* V *be an* $\mathbb{R}^d$*-valued right-continuous* $\mathbb{F}$*-predictable random process and* A *a closed set in* $\mathbb{R}^d$. *Then, the stopping time* $\inf\{t : V(t) \in A\}$ *is* $\mathbb{F}$*-predictable.*

The operation of forestopping was used prior to [7] by Barlow [8] who took the assertion of Theorem 2.17 (which he did not even formulate) for granted.

We will need some subtler properties of this operation.

Lemma 2.19. *Let U be a starting from zero locally square integrable martingale with respect to $\mathbb{F}$, N a positive number, and σ an $\mathbb{F}$-predictable stopping time such that*

$$\sigma \leq \inf\{t : \operatorname{tr}\langle U\rangle(t) \geq N\}. \tag{2.37}$$

Then, $\mathsf{E}\sup_{s\in\mathbb{R}_+}|{}^{\sigma}U(s)|^2 \leq 4N$.

Proof. Predictability of σ implies by Theorem 2.1.13 [3] that there exists a sequence (σ_n) of stopping times such that

$$\{\sigma > 0\} \subset \{\sigma_n < \sigma\}, \tag{2.38}$$

$$\sigma_n \nearrow \sigma \quad \text{a.s.} \tag{2.39}$$

By the choice of U there exists a sequence (τ_k) of stopping times such that

$$\tau_k \nearrow \infty \quad \text{a.s,} \tag{2.40}$$

$$U_k \equiv U^{\tau_k} \in \mathcal{M}_2(\mathbb{F}). \tag{2.41}$$

Then,

$$\sup_{s\leq\sigma_n}|U(s)| = \lim_{k\to\infty}\sup_{s\leq\sigma_n\wedge\tau_k}|U(s)|. \tag{2.42}$$

Herein, obviously,

$$\sup_{s\leq\sigma_n\wedge\tau_k}|U(s)| = \sup_{s\leq\sigma_n}|U_k(s)|. \tag{2.43}$$

From (2.41) we have by Doob's inequality

$$\mathsf{E}\sup_{s\leq\sigma_n}|U_k(s)|^2 \leq 4\mathsf{E}|U_k(\sigma_n)|^2. \tag{2.44}$$

Noting that: (1) for any $x \in \mathbb{R}^d$ $|x|^2 = \operatorname{tr} xx^\top$, (2) $\mathsf{E}U_k(\sigma_n)U_k(\sigma_n)^\top = \mathsf{E}\langle U_k\rangle(\sigma_n)$, we may re-write the last inequality in the form

$$\mathsf{E}\sup_{s\leq\sigma_n}|U_k(s)|^2 \leq 4\mathsf{E}\operatorname{tr}\langle U_k\rangle(\sigma_n). \tag{2.45}$$

Writing

$$\langle U_k\rangle(\sigma_n) = \langle U^{\tau_k}\rangle(\sigma_n) = \langle U\rangle^{\tau_k}(\sigma_n) = \langle U\rangle(\tau_k\wedge\sigma_n), \tag{2.46}$$

we get from (2.37) and (2.38) $\operatorname{tr}\langle U_k\rangle(\sigma_n) < N$, which together with (2.45) results in $\mathsf{E}\sup_{s\le\sigma_n} |U_k(s)|^2 < 4N$. Then, from (2.42) and (2.43), we have by Fatou's theorem

$$\mathsf{E}\sup_{s\le\sigma_n}|U(s)|^2 \le 4N. \tag{2.47}$$

The assumption $U(0) = 0$ yields

$$\mathsf{E}\sup_{s<\sigma}|U(s)|^2 = \mathsf{E}\sup_{s<\sigma}|U(s)|^2 I\{\sigma > 0\}. \tag{2.48}$$

Relations (2.38) and (2.39) imply that

$$\sup_{s<\sigma}|U(s)|^2 I\{\sigma > 0\} = \lim_{n\to\infty}\sup_{s\le\sigma_n}|U(s)|^2 I\{\sigma > 0\}, \tag{2.49}$$

which together with (2.47) yields by Fatou's theorem $\mathsf{E}\sup_{s<\sigma}|U(s)|^2 I\{\sigma > 0\} \le 4N$. It remains to note that $\sup_{s\in\mathbb{R}_+}|{}^{\sigma}U(s)| = \sup_{s<\sigma}|U(s)|I\{\sigma > 0\}$ in view of (2.34). □

Lemma 2.20. *Let U be a locally square integrable martingale with respect to $\mathbb{F}$ such that*

$$\mathsf{E}|U(0)|^2 < \infty, \tag{2.50}$$

and for any t

$$\mathsf{E}\max_{s\le t}|\Delta U(s)|^2 < \infty. \tag{2.51}$$

Let, further, N be a positive number and σ a predictable time satisfying condition (2.37). *Then, $U^{\sigma} \in \overline{\mathcal{M}}_2(\mathbb{F})$.*

Proof. In view of (2.50) it suffices to show that $(U - U(0))^{\sigma} \in \overline{\mathcal{M}}_2(\mathbb{F})$. In other words, we may consider that $U(0) = 0$. Then condition (2.51) and the evident inequality

$$\sup_{s\le t}|U^{\sigma}(s)| \le \sup_{s\le t}|{}^{\sigma}U(s)| + \max_{s\le t}|\Delta U(s)| \tag{2.52}$$

imply by Lemma 2.19 that for any t

$$\mathsf{E}\sup_{s\le t}|U^{\sigma}(s)|^2 < \infty. \tag{2.53}$$

It remains to prove that for all $t_2 > t_1 \ge 0$,

$$\mathsf{E}(U^{\sigma}(t_2) \mid \mathcal{F}(t_1)) = U^{\sigma}(t_1). \tag{2.54}$$

Taking a sequence (τ_n) with properties (2.40) and (2.41), we write

$$\mathsf{E}(U(t_2 \wedge \sigma \wedge \tau_k) \mid \mathcal{F}(t_1)) = U^{\sigma}(t_1 \wedge \sigma \wedge \tau_k). \tag{2.55}$$

To deduce (2.54) from this inequality and (2.40), it suffices to note that

$$|U(t \wedge \sigma \wedge \tau_k)| \leq \sup_{s \leq t} |U^{\sigma}(s)|, \tag{2.56}$$

so that (2.53) provides uniform integrability of the sequence $(U(t_2 \wedge \sigma \wedge \tau_k),\, k \in \mathbb{N})$. □

Corollary 2.21. *Under the conditions of Lemma 2.20* ${}^{\sigma}U \in \mathcal{M}_2({}^{\sigma}\mathbb{F})$.

Theorem 2.22. *Let U be a locally square integrable martingale with respect to $\mathbb{F}$ satisfying conditions* (2.50) *and* (2.51), *N a positive number, and σ a predictable time satisfying condition* (2.37). *Then,* $\langle {}^{\sigma}U \rangle = {}^{\sigma}\langle U \rangle$.

Proof. Denote $X = (U^{\otimes 2} - \langle U \rangle)^{\sigma} \equiv (U^{\sigma})^{\otimes 2} - \langle U^{\sigma} \rangle$, $Y = {}^{\sigma}U^{\otimes 2} - {}^{\sigma}\langle U \rangle \equiv {}^{\sigma}(U^{\otimes 2} - \langle U \rangle)$. It suffices to show that Y is a ${}^{\sigma}\mathbb{F}$-martingale. To deduce this fact from Theorem 2.17, we note that, firstly, $X \in \overline{\mathcal{M}}(\mathbb{F})$ by construction and Lemma 2.20, and, secondly, $Y = {}^{\sigma}X$ by construction of both processes and because of (2.35). □

3. Martingale Preliminaries

The next statement is obvious.

Lemma 3.1. *Let (M^l) be a sequence of martingales such that*

$$M^l \xrightarrow{\mathrm{d}} M, \tag{3.1}$$

and for any t the sequence $(|M^l(t)|)$ is uniformly integrable. Then, M is a martingale.

Lemma 3.2. *Let (M^l) be a sequence of martingales such that* (3.1) *holds and*

$$\sup_{l \in \mathbb{N},\, t \in \mathbb{R}_+} \mathsf{E}\, \mathrm{tr} \left\langle M^l \right\rangle (t) < \infty. \tag{3.2}$$

Then, $\sup_t \mathsf{E}|M(t)|^2 < \infty$.

Proof. By condition (3.2) and the definition of quadratic characteristic, there exists a constant C such that $\mathsf{E}|M^l(t)|^2 \leq C$ for all t and l. Hence, and from (3.1), we have by Fatou's theorem (applicable due to the above-mentioned Skorokhod's principle of common probability space) $\mathsf{E}|M(t)|^2 \leq C$. □

Corollary 3.3. *Let a sequence (M^l) of square integrable martingales satisfy conditions* (3.1) *and* (3.2). *Then, M is a uniformly integrable martingale.*

Lemma 3.4. *Let Y be a local martingale and K be an $\mathfrak{S}$-valued random process. Suppose that they are given on a common probability space and $(Y, K) \stackrel{\mathrm{d}}{=} (Y, [Y])$. Then for any t $K(t) = [Y](t)$ a.s.*

Proof. By assumption,

$$\sum_{i=1}^{n} (Y(t_i) - Y(t_{i-1}))^{\otimes 2} - K(t) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{n} (Y(t_i) - Y(t_{i-1}))^{\otimes 2} - [Y](t), \tag{3.3}$$

for all n, t and $t_0 < t_1 < \cdots t_n$. Hence, recalling the definition of quadratic variation, we get $[Y](t) - K(t) \stackrel{\mathrm{d}}{=} 0$. □

We shall identify indistinguishable processes, writing simply $\xi = \eta$ if $\mathsf{P}\{\forall t \in \mathbb{R}_+, \xi(t) = \eta(t)\} = 1$. Theorem 2.3.5 [3] asserts that

$$[Y] = \langle Y \rangle, \tag{3.4}$$

for a continuous local martingale Y. Hence, and from Lemma 3.4, we have

Corollary 3.5. *Let Y be a continuous local martingale and K an $\mathfrak{S}$-valued random process. Suppose that they are given on a common probability space and $(Y, K) \stackrel{\mathrm{d}}{=} (Y, \langle Y \rangle)$. Then, $K = \langle Y \rangle$.*

Proof. Lemma 3.4 and formula (3.4) yield $\mathsf{P}\{\forall t \in \mathbb{Q}_+, K(t) = \langle Y \rangle(t)\} = 1$. Continuity of both processes enables us to substitute $\mathbb{Q}_+$ by $\mathbb{R}_+$. □

Lemma 3.6. *Let U be a locally square integrable martingale. Then, $\|\langle U \rangle\|$ is an increasing process.*

Proof. For any $x \in \mathbb{R}^d$, the process $x^\top U$ is a numeral locally square integrable martingale and, therefore, the process $\langle x^\top U \rangle$ increases. It remains to note that $\langle x^\top U \rangle = x^\top \langle U \rangle x$ and to recall formula (2.21). □

Lemma 3.7. *Let Z_1 and Z_2 be locally square integrable martingales with respect to a common filtration. Then,*

$$\|\langle Z_1, Z_2 \rangle + \langle Z_2, Z_1 \rangle\| \leq 2\sqrt{\|\langle Z_1 \rangle\| \|\langle Z_2 \rangle\|}. \tag{3.5}$$

Proof. For $d = 1$ (then $\langle Z_2, Z_1 \rangle = \langle Z_1, Z_2 \rangle$), this is the Kunita-Watanabe inequality [3, page 118]. In the general case, we take an arbitrary vector $x \in S^{d-1}$ and write

$$x^\top (\langle Z_1, Z_2 \rangle + \langle Z_2, Z_1 \rangle) x = 2\left\langle x^\top Z_1, x^\top Z_2 \right\rangle \leq 2\sqrt{\langle x^\top Z_1 \rangle \langle x^\top Z_2 \rangle} = 2\sqrt{x^\top \langle Z_1 \rangle x x^\top \langle Z_2 \rangle x}, \tag{3.6}$$

hereupon the required conclusion ensues from (2.21) and Lemma 2.9. □

Lemma 3.8. *Let U_1 and U_2 be locally square integrable martingales with respect to some common filtration. Then, for any $t > 0$*

$$\sup_{s\le t}\|\langle U_1\rangle(s) - \langle U_2\rangle(s)\| \le \|\langle U_1 - U_2\rangle(t)\| + 2\sqrt{\|\langle U_1 - U_2\rangle(t)\|}\sqrt{\|\langle U_2\rangle(t)\|}. \tag{3.7}$$

Proof. Writing the identities

$$\langle U_1\rangle = \langle U_1 - U_2 + U_2\rangle = \langle U_1 - U_2\rangle + \langle U_1 - U_2,\ U_2\rangle + \langle U_2,\ U_1 - U_2\rangle + \langle U_2\rangle, \tag{3.8}$$

we deduce from Lemma 3.7 that

$$\|\langle U_1\rangle(s) - \langle U_2\rangle(s)\| \le \|\langle U_1 - U_2\rangle(s)\| + 2\sqrt{\|\langle U_2 - U_1\rangle(s)\|}\sqrt{\|\langle U_2\rangle(s)\|}. \tag{3.9}$$

It remains to note that the right-hand side increases in s by Lemma 3.6. □

For a function $f \in \mathrm{D}$ we denote $\Delta f(t) = f(t) - f(t-)$.
Let us introduce the conditions:

(RC) The sequence $(\operatorname{tr}\langle Y_n\rangle)$ is r.c. in C.

(UI1) The sequence $(|Y_n(t) - Y_n(0)|^2)$ is u.i.

(UI2) For any $z \in \mathbb{R}^{d*}$ the sequence $(\operatorname{tr}\langle zY_n\rangle(t))$ is u.i.

(UI3) The sequence $(\sup_{s\le t}|Y_n(s) - Y_n(0)|^2)$ is u.i.

Lemma 3.9. *Let (Y_n) be a sequence of local square integrable martingales satisfying the conditions: (RC),*

$$\lim_{L\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\{|Y_n(0)| > L\} = 0, \tag{3.10}$$

and, for each $t > 0$, the condition

$$\max_{s\le t}|\Delta Y_n(s)| \xrightarrow{\mathsf{P}} 0. \tag{3.11}$$

Then, (Y_n) is r.c. in C.

Proof. It follows from (RC) and (3.10) by Rebolledo's theorem [2, VI.4.13] that (Y_n) is r.c. in D. Hereon, the desired conclusion follows from Proposition VI.3.26 (items (i) and (iii)) [2] with account of VI.3.9 [2]. □

Combining Lemma 3.9 with Corollary 2.11, we get

Corollary 3.10. *Under the assumptions of Lemma 3.9, the sequence of compound processes $(Y_n, \langle Y_n\rangle)$ is r.c. in* C.

Some statements below deal with random processes on $[0,t]$, not on $\mathbb{R}_+$. In this case, the time variable is denoted by s and C means $\mathrm{C}[0,t]$ instead of $\mathrm{C}(\mathbb{R}_+)$.

Lemma 3.11. *Let* (Y_n) *be a r.c. in* C *and satisfying condition (UI1) sequence of martingales on* $[0,t]$. *Then, for any* $z \in \mathbb{R}^{d*}$, *the sequence* $([zY_n](t))$ *is u.i.*

Proof. The obvious equality $[V - V(0)] = [V]$ allows us to consider that $Y_n(0) = 0$. Then, condition (UI1) together with Doob's inequality yields

$$\sup_n \mathsf{E} \sup_{s \le t} |Y_n(s)|^2 < \infty, \tag{3.12}$$

whence

$$\sup_n \mathsf{E} \max_{s \le t} |\Delta Y_n(s)| < \infty. \tag{3.13}$$

By assumption, for any infinite set $J_0 \subset \mathbb{N}$, there exist an infinite subset $J \subset J_0$ and a random process Y such that

$$Y_n \xrightarrow{\mathrm{C}} Y \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{3.14}$$

Condition (UI1) implies that Y is a square integrable martingale and for any $z \in \mathbb{R}^{d*}$,

$$\mathsf{E}(zY_n(t))^2 \longrightarrow \mathsf{E}(zY(t))^2 \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{3.15}$$

From (3.14) and (3.13), we have by Corollary VI.6.7 [2]

$$[zY_n] \xrightarrow{\mathrm{C}} [zY] \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{3.16}$$

Hence, and from (3.15), recalling that for any $\mathbb{R}$-valued $M \in \overline{\mathcal{M}}_2$ one has $\mathsf{E}(M - M(0))^2 = \mathsf{E}[M]$ [3, Theorem 2.2.4], we get

$$\mathsf{E}[zY_n](t) \longrightarrow \mathsf{E}[zY](t) \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{3.17}$$

Comparing this relation with (3.16), we conclude that the sequence $([zY_n](t), n \in J)$ is uniformly integrable. Hence, in view of arbitrariness of J_0, we deduce by Lemma 2.3 uniform integrability of $([zY_n](t), n \in \mathbb{N})$. □

Lemma 3.12. *Let* (Y_n) *be a sequence of martingales on* $[0,t]$ *satisfying condition (UI1) and (UI2). Suppose that there exists an* $\mathbb{R}^d \times \mathfrak{S}_+$*-valued random process* (Y, K) *such that*

$$(Y_n, \langle Y_n \rangle) \xrightarrow{\mathrm{C}} (Y, K). \tag{3.18}$$

Then, firstly,

$$[Y_n] - \langle Y_n \rangle \xrightarrow{\mathrm{C}} O, \tag{3.19}$$

where O is the null matrix, Y is a continuous martingale, and, secondly,

$$(Y, K) \stackrel{\mathrm{d}}{=} (Y, \langle Y \rangle). \tag{3.20}$$

Proof. For the same reason as in the proof of Lemma 3.11, we may consider that $Y_n(0) = 0$. Then, as was shown above, condition (UI1) implies (3.13). Combining the latter with

$$Y_n \xrightarrow{\mathrm{C}} Y, \tag{3.21}$$

(a part of (3.18)), we get by Corollary VI.6.7 [2] that

$$(Y_n, [Y_n]) \xrightarrow{\mathrm{C}} (Y, [Y]). \tag{3.22}$$

From (3.18) and (3.22), we get by Corollary 2.11 that for any infinite set $J_0 \subset \mathbb{N}$, there exist an infinite subset $J \subset J_0$ and an $\mathfrak{S}_+ \times \mathfrak{S}_+$-valued random process (Q^J, R^J) such that

$$([Y_n], \langle Y_n \rangle) \longrightarrow \left(Q^J, R^J\right) \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{3.23}$$

(Of course $Q^J \stackrel{\mathrm{d}}{=} K,\ R^J \stackrel{\mathrm{d}}{=} [Y]$.)

Denote $Z_n = [Y_n] - \langle Y_n \rangle$. This is a martingale by Lemma 10.4 in [4]. Relation (3.23) implies that

$$Z_n \xrightarrow{\mathrm{C}} Q^J - R^J \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{3.24}$$

For any $z \in \mathbb{R}^{d*}$, the sequence $(zZ_n(t)z^\top)$ is, by Lemma 3.11 and condition (UI2), u.i. So, relation (3.24) implies by Lemma 3.1 that $z(Q^J - R^J)z^\top$ is a martingale. Also, it implies its continuity. Relation (3.23) shows that the processes $zQ^Jz^\top$ and $zR^Jz^\top$ increase and start from zero. So, $z(Q^J - R^J)z^\top$ starts from zero and has finite variation in $[0,t]$. These four properties of $Q^J - R^J$ imply together that $Q^J(s) - R^J(s) = O$ for all $s \in [0,t]$. Thus, any subsequence $(Z_n, n \in J_0)$ contains, in turn, a subsequence $(Z_n, n \in J)$ such that $Z_n \to O$ as $n \to \infty,\ n \in J$. This proves (3.19).

From (3.22) and (3.19), we have $(Y_n, \langle Y_n \rangle) \xrightarrow{\mathrm{C}} (Y, [Y])$. And this is, in view of (3.4), tantamount to

$$(Y_n, \langle Y_n \rangle) \xrightarrow{\mathrm{C}} (Y, \langle Y \rangle). \tag{3.25}$$

Comparing this relation with (3.18), we arrive at (3.20). □

Remark 3.13. The second conclusion of Lemma 3.12 implies by Corollary 3.5 that $K = \langle Y \rangle$.

Corollary 3.14. *Let a sequence (Y_n) of martingales on $[0,t]$ satisfy conditions (RC), (3.11), (UI1), and (UI2). Then, relation (3.19) holds.*

Proof. By Corollary 3.10 for any infinite set $J_0 \subset \mathbb{N}$, there exist an infinite set $J \subset J_0$ and an $\mathbb{R}^d \times \mathfrak{S}_+$-valued random process (Y, K) such that relation (3.18) holds as $n \to \infty$, $n \in J$. Then, by Lemma 3.12, so does (as $n \to \infty$, $n \in J$) (3.19). Due to arbitrariness of J_0 this relation holds when n ranges over $\mathbb{N}$, too. □

Corollary 3.15. *Let (Y_n) be a sequence of martingales on $\mathbb{R}_+$ satisfying conditions (RC) and for all $t > 0$, conditions* (3.11)*, (UI1), (UI2). Then, relation* (3.19) *holds.*

Lemma 3.16. *Let a sequence (Y_n) of martingales on $\mathbb{R}_+$ satisfy conditions (RC) and, for any $t > 0$,* (3.11) *and (UI3). Then, relation* (3.19) *holds.*

Proof. Denote $\sigma_n^N = \inf\{s : \operatorname{tr}\langle Y_n\rangle(s) \geq N\}$, $Y_n^N = {}^{\sigma_n^N} Y_n$. Obviously,

$$\left\{\sigma_n^N < t\right\} \subset \{\operatorname{tr}\langle Y_n\rangle(t) \geq N\}. \tag{3.26}$$

By Corollary 2.21 Y_n^N is a square integrable martingale with respect to ${}^{\sigma_n^N}\mathbb{F}_n$. By Theorem 2.22,

$$\left\langle Y_n^N \right\rangle = {}^{\sigma_n^N} \langle Y_n \rangle . \tag{3.27}$$

Condition (RC) implies relative compactness of the sequence $(\langle Y_n^N \rangle, n \in \mathbb{N})$. By construction $|Y_n^N(t) - Y_n^N(0)| \leq \sup_{s \leq t}|Y_n(t) - Y_n(0)|$. So, condition (UI3) implies that for any t and N the sequence $(|Y_n^N(t) - Y_n^N(0)|^2, n \in \mathbb{N})$ is u.i. Thus, Corollary 3.15 asserts that for any N

$$\left[Y_n^N\right] - \left\langle Y_n^N \right\rangle \xrightarrow{\mathrm{C}} O \quad \text{as } n \longrightarrow \infty. \tag{3.28}$$

Equalities (3.27) and (2.36) yield the relation

$$\left\{\sup_{s \leq t}\left(\left\| \left[Y_n^N\right](s) - [Y_n](s) \right\| + \left\| \left\langle Y_n^N \right\rangle(s) - \langle Y_n\rangle(s) \right\|\right) > 0\right\} \subset \left\{\sigma_n^N < t\right\}, \tag{3.29}$$

which together with (3.26), (3.28) and (RC) entails (3.19). □

Corollary 3.15 and Lemma 3.16 are only the steps towards the final result about asymptotic proximity of quadratic variations and quadratic characteristics—Corollary 5.3.

Corollary 3.17. *Let a sequence (Y_n) of martingales on $\mathbb{R}_+$ satisfy conditions (RC) and for any $t > 0$,* (3.11) *and (UI3). Suppose also that there exists an $\mathbb{R}^d \times \mathfrak{S}_+$-valued random process (Y, K) such that relation* (3.18) *is valid. Then, Y is a continuous martingale, and* (3.20) *holds.*

Proof. Lemma 3.16 asserts (3.19). The implications ((3.21) and (UI1) ⇒ (3.22)); ((3.22) and (3.19) ⇒ (3.25)), were established in the proof of Lemma 3.12. □

4. Sequences of Martingales with Asymptotically Conditionally Independent Increments

Lemma 4.1. *Let for each* n M_n *be an* $\mathbb{R}$*-valued square integrable martingale on* $[0,t]$ *with respect to a flow* $(\mathcal{G}_n(s), s \in [0,t])$ *and* $\mathcal{H}_n$ *a sub-*σ*-algebra of* $\mathcal{G}_n(0)$*. Suppose that conditions (UI1) and (RC) are fulfilled for* $Y_n = M_n$*,*

$$M_n(0) = 0, \tag{4.1}$$

$$\max_{s\le t}|\Delta M_n(s)| \xrightarrow{\mathsf{P}} 0 \tag{4.2}$$

and there exists a nonrandom number N *such that for all* n

$$\langle M_n\rangle(t) \le N. \tag{4.3}$$

Then,

$$\mathsf{E}\left(e^{iM_n(t)+\langle M_n\rangle(t)/2} \mid \mathcal{H}_n\right) \xrightarrow{\mathsf{P}} 1. \tag{4.4}$$

Proof. Conditions (RC) ($Y_n = M_n$), (4.1), and (4.2) entail, by Lemma 3.9, relative compactness of (M_n) in C.

Denote $T_n = \langle M_n\rangle/2$, $\xi_n = e^{iM_n+T_n}$, $X_n = ([M_n] - \langle M_n\rangle)/2$, $\gamma_n = \xi_n^- \circ X_n$,

$$\eta_n = \sum_{s\le t}\xi_n(s-)\left(e^{\Delta T_n(s)+i\Delta M_n(s)} - 1 - \Delta T_n(s) - i\Delta M_n(s) + \frac{1}{2}(\Delta M_n(s))^2\right). \tag{4.5}$$

Condition (RC) ($Y_n = M_n$) implies that

$$\max_{s\le t}\Delta T_n(s) \xrightarrow{\mathsf{P}} 0. \tag{4.6}$$

In view of (4.1) $\xi_n(0) = 1$. Then, by Itô's formula

$$\begin{aligned}\xi_n(t) &= 1 + i\xi_n^- \cdot M_n(t) + \xi_n^- \circ T_n(t) - \frac{1}{2}\xi_n^- \circ \langle M_n^c\rangle(t)\\ &\quad + \sum_{s\le t}\xi_n(s-)\left(e^{i\Delta M_n(s)+\Delta T_n(s)} - 1 - i\Delta M_n(s) - \Delta T_n(s)\right).\end{aligned} \tag{4.7}$$

Hence, recalling that $\langle M_n^c\rangle(t) = [M_n](t) - \sum_{s\le t}(\Delta M_n(s))^2$, we get

$$\xi_n(t) = 1 + i\xi_n^- \cdot M_n(t) - \xi_n^- \circ X_n(t) + \eta_n. \tag{4.8}$$

By the definition of ξ_n and by condition (4.3),

$$\sup_{s\le t}|\xi_n(s)| \le e^{N/2}. \tag{4.9}$$

Consequently,

$$\mathsf{E}\left(\xi_n^- \cdot M_n(t) \mid \mathscr{H}_n\right) = 0, \tag{4.10}$$

and $\mathsf{E}|\xi_n^- \cdot M_n(t)|^2 = \mathsf{E}(|\xi_n^-|^2 \circ \langle M_n\rangle(t))$. The right-hand side of the last equality being less than $e^N\ N$, the sequence $(\xi_n^- \cdot M_n(t))$ is u.i., and so is $([M_n](t))$ by Lemma 3.11 whose conditions (those not postulated) we have verified. This together with (4.9) and (4.3) implies uniform integrability of $(\xi_n^- \circ X_n(t))$. Now, (4.8) and inequality (4.9) show that (η_n) has this property, too.

By construction and Lemma 10.4 in [4], X_n is a martingale. Then, it follows from (4.9) that $\mathsf{E}(\xi_n^- \circ X_n(t) \mid \mathscr{H}_n) = 0$, which together with (4.10) and (4.8) yields

$$\mathsf{E}(\xi_n(t) \mid \mathscr{H}_n) = 1 + \mathsf{E}\left(\eta_n \mid \mathscr{H}_n\right). \tag{4.11}$$

So, it suffices to show that

$$\eta_n \xrightarrow{\mathsf{P}} 0. \tag{4.12}$$

Obviously, for any real a and b

$$\begin{gathered} e^{a+bi} - a = (e^a - 1 - a)e^{bi} + a\left(e^{bi} - 1\right) + e^{bi}, \\ |e^a - 1 - a| \le |a|^2 e^{|a|}, \qquad \left|e^{bi} - 1\right| \le |b|, \qquad \left|e^{bi} - 1 - bi + \frac{b^2}{2}\right| \le |b|^3. \end{gathered} \tag{4.13}$$

Hence, from (4.5), (4.9), we get

$$\left|\eta_n\right| \le e^N\left(e^{e^N}\sum_{s\le t}(\Delta T_n(s))^2 + \max_{s\le t}|\Delta M_n(s)|\left(\sum_{s\le t}\Delta T_n(s) + \sum_{s\le t}(\Delta M_n(s))^2\right)\right). \tag{4.14}$$

Now, (4.12) ensues from (4.6), (4.3), (4.2), and stochastic boundedness of the sequence $([M_n](t))$. □

Lemma 4.2. *Let for each n, M_n be an $\mathbb{R}$-valued starting from zero locally square integrable martingale with respect to some flow $(\mathcal{G}_n(t), t \in \mathbb{R}_+)$ and $\mathscr{H}_n$ a sub-σ-algebra of $\mathcal{G}_n(0)$. Suppose that condition (RC) is fulfilled for $Y_n = M_n$;*

$$\mathsf{E}\max_{s\le t}(\Delta M_n(s))^2 \longrightarrow 0, \tag{4.15}$$

for any $t > 0$*; there exists a nonrandom function* $q : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that*

$$\langle M_n \rangle(t) \leq q(t), \tag{4.16}$$

for all n *and* t*. Then, for any* t*, relation* (4.4) *holds.*

Proof. Let us denote, only in this proof, $\tau_{Nn} = \inf\{t : |M_n(t)| \geq N\}$, $M_{Nn}(t) = M_n(t \wedge \tau_{Nn})$, $T_{Nn}(t) = T_n(t \wedge \tau_{Nn})$ (so that $M_{Nn} \in \ell\mathcal{M}_2$ and $\langle M_{Nn} \rangle = 2T_{Nn}$), $\xi_{Nn} = e^{iM_{Nn}+T_{Nn}}$. The evident inequality

$$\sup_{s \leq t} |M_{Nn}(s)| \leq N + \max_{s \leq t} |\Delta M_n(s)|, \tag{4.17}$$

and condition (4.15) show us that for any positive t and N, the sequence $(M_{Nn}(t)^2,\ n \in \mathbb{N})$ is u.i.

By assumption, there exists a sequence (σ_k) of stopping times such that $\sigma_k \nearrow \infty$ a.s. and for each k, $M^{\sigma_k} \in \mathcal{M}_2$. Then, for any $t > s \geq 0$, $N > 0$ and $n, k \in \mathbb{N}$,

$$\mathsf{E}(M_{Nn}(t \wedge \sigma_k) \mid \mathcal{G}_n(s)) = \mathsf{E}(M^{\sigma_k}(t \wedge \tau_{Nn}) \mid \mathcal{G}_n(s)) = M^{\sigma_k}(s \wedge \tau_{Nn}) = M_{Nn}(s \wedge \sigma_k). \tag{4.18}$$

Writing $|M_{Nn}(t \wedge \sigma_k)| \leq \sup_{s \leq t} |M_{Nn}(s)|$, we deduce from (4.17) and (4.15) uniform integrability of the sequence $(M_{Nn}(t \wedge \sigma_k),\ k \in \mathbb{N})$. So, letting $k \to \infty$ in (4.18), we get

$$\mathsf{E}(M_{Nn}(t) \mid \mathcal{G}_n(s)) = M_{Nn}(s), \tag{4.19}$$

that is, M_{Nn} is a martingale. It is square integrable because of (4.17) and (4.15). Thus, for any N and t the sequence $(M_{Nn}, n \in \mathbb{N})$ satisfies all the conditions of Lemma 4.1 which, therefore, asserts that

$$\mathsf{E}(\xi_{Nn}(t) \mid \mathcal{H}_n) \xrightarrow{\mathsf{P}} 1 \quad \text{as } n \longrightarrow \infty. \tag{4.20}$$

Here, in $|\xi_n(t)| \vee |\xi_{Nn}(t)| \leq e^{q(t)}$ because of (4.16), so

$$|\xi_n(t) - \xi_{Nn}(t)| \leq 2e^{q(t)} I\{\tau_{Nn} \leq t\}. \tag{4.21}$$

Obviously, $\{\tau_{Nn} \leq t\} \subset \{\sup_{s \leq t} |M_n(s)| \geq N\}$. From (4.16), we have by the Lenglart-Rebolledo inequality

$$\lim_{N \to \infty} \overline{\lim_{n \to \infty}} \mathsf{P}\left\{ \sup_{s \leq t} |M_n(s)| \geq N \right\} = 0. \tag{4.22}$$

The last three relations imply that

$$\lim_{N \to \infty} \overline{\lim_{n \to \infty}} \mathsf{E}|\xi_n(t) - \xi_{Nn}(t)| = 0, \tag{4.23}$$

which together with (4.20) yields (4.4). □

Lemma 4.3. *Let for each n, $M_n, M_n^1, M_n^2 \dots$ be $\mathbb{R}$-valued starting from zero locally square integrable martingales with respect to a flow $(\mathcal{G}_n(t),\ t \in \mathbb{R}_+)$ and $\mathscr{H}_n$ a sub-σ-algebra of $\mathcal{G}_n(0)$. Suppose that for all $m \in \mathbb{N}$, $t > 0$, $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathsf{E} \max_{u\le t} (\Delta M_n^m(u))^2 = 0, \tag{4.24}$$

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \left\langle M_n^l - M_n \right\rangle(t) > \varepsilon \right\} = 0, \tag{4.25}$$

there exists a nonrandom function $q : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\langle M_n \rangle(t) \vee \langle M_n^m \rangle(t) \le q(t), \tag{4.26}$$

for all m, n, and t; for each m, the sequence $(\langle M_n^m \rangle,\ n \in \mathbb{N})$ is r.c. in C. *Then, for any t relation* (4.4) *holds.*

Proof. Denote $T_n^m = \langle M_n^m \rangle / 2$. By Lemma 4.2 $\mathsf{E}(e^{iM_n^m(t)+T_n^m(t)} \mid \mathscr{H}_n) \xrightarrow{\mathsf{P}} 1$ as $n \to \infty$. So, in view of (4.26), it suffices to prove that for any $\varepsilon > 0$,

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{E}\left| e^{iM_n^l(t)+T_n^l(t)} - e^{iM_n(t)+T_n(t)} \right| = 0. \tag{4.27}$$

Conditions (4.25) and (4.26) imply by Lemma 3.8 that for all positive t and ε

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \left| T_n^l(t) - T_n(t) \right| > \varepsilon \right\} = 0. \tag{4.28}$$

Furthermore, (4.25) together with the Lenglart-Rebolledo inequality and the assumed equalities $M_n(0) = 0 = M_n^l(0)$ yields

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \left| M_n^l(t) - M_n(t) \right| > \varepsilon \right\} = 0, \tag{4.29}$$

which jointly with the previous relation and condition (4.25) entails (4.27). □

Lemma 4.4. *Let for each n, M_n be an $\mathbb{R}$-valued starting from zero locally square integrable martingale with respect to a flow $(\mathcal{G}_n(t),\ t \in \mathbb{R}_+)$ and $\mathscr{H}_n$ a sub-σ-algebra of $\mathcal{G}_n(0)$. Suppose that conditions (RC) $(Y_n = M_n)$,* (4.24) *(for all m and t) and* (4.25) *(for all t and ε) are fulfilled; for all $m \in \mathbb{N}$ and $u_2 > u_1 \ge 0$,*

$$\langle M_n^m \rangle(u_2) - \langle M_n^m \rangle(u_1) \le \langle M_n \rangle(u_2) - \langle M_n \rangle(u_1); \tag{4.30}$$

for any $t > 0$ and bounded uniformly continuous function $f : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathsf{E}\big(f(\langle M_n \rangle(t)) \mid \mathscr{H}_n\big) - f(\langle M_n \rangle(t)) \xrightarrow{\mathsf{P}} 0. \tag{4.31}$$

Then, for any t,

$$\mathsf{E}\left(e^{iM_n(t)} \mid \mathscr{H}_n\right) - e^{-\langle M_n\rangle(t)/2} \xrightarrow{\mathsf{P}} 0. \tag{4.32}$$

Proof. (1°) Let us fix t and denote $\alpha_n = e^{T_n(t)}$, $\beta_n = e^{iM_n(t)}$ so that $\xi_n(t) = \alpha_n\beta_n$. If there exists a nonrandom constant N such that $\langle M_n\rangle(s) \le N$ for all n and s, then all the conditions of Lemma 4.3 are fulfilled, and therefore,

$$\mathsf{E}(\alpha_n\beta_n \mid \mathscr{H}_n) \xrightarrow{\mathsf{P}} 1. \tag{4.33}$$

Also, under this assumption $\alpha_n = g^N(T_n(t))$, where $g^N(x) = e^{x^{[N]}}$,

$$x^{[N]} = \frac{Nx}{N \vee |x|}. \tag{4.34}$$

So, substituting $f(x) = g^N(2x)$ to (4.31), we obtain (2.7), whence by Lemma 2.5, relation (2.8) follows. Juxtaposing it with (4.33), we get $\alpha_n\mathsf{E}(\beta_n \mid \mathscr{H}_n) - 1 \xrightarrow{\mathsf{P}} 0$. Dividing both sides of this relation by α_n (≥ 1), we arrive at (4.32).

(2°) Let us waive the extra assumption.

Denote $\sigma_{kn} = \inf\{s : \langle M_n\rangle(s) \ge k\}$,

$$T_{kn}(s) = T_n(s)I_{[0,\sigma_{kn}[}(s) + T_n(\sigma_{kn}-)I_{[\sigma_{kn},\infty[}(s), \tag{4.35}$$

$$T^m_{kn}(s) = T^m_n(s)I_{[0,\sigma_{kn}[}(s) + T^m_n(\sigma_{kn}-)I_{[\sigma_{kn},\infty[}(s), \tag{4.36}$$

and likewise with M instead of T. Lemma 2.18 asserts predictability of σ_{kn}. By construction and condition (4.33), $\sigma_{kn} \le \inf\{s : \langle M^m_n\rangle(s) \ge k\}$. Thus, Theorem 2.22 asserts that M_{kn} and M^m_{kn} are square integrable martingales and $\langle M_{kn}\rangle = 2T_{kn}$, $\langle M^m_{kn}\rangle = 2T^m_{kn}$. Consequently, for any $t_2 > t_1 > 0$

$$\langle M^m_{kn}\rangle(t_2) - \langle M^m_{kn}\rangle(t_1) \le \langle M_{kn}\rangle(t_2) - \langle M_{kn}\rangle(t_1). \tag{4.37}$$

In view of (4.35) and (4.34),

$$(T_n(t))^{[k]} - T_{kn}(t) = \left(\frac{kT_n(\sigma_{kn})}{k \vee T_n(\sigma_{kn})} - T_n(\sigma_{kn}-)\right)I_{[\sigma_{kn},\infty[}(t), \tag{4.38}$$

whence

$$\left|(T_n(t))^{[k]} - T_{kn}(t)\right| \le \Delta T_n(\sigma_{kn} \wedge t) \le \max_{s\le t}\Delta T_n(s). \tag{4.39}$$

Here, in condition (4.6) is fulfilled because of (RC).

Let f be a bounded uniformly continuous function. Then,

$$\mathsf{E}\Big(f\Big((T_n(t))^{[k]}\Big) \mid \mathscr{H}_n\Big) - f\Big((T_n(t))^{[k]}\Big) \xrightarrow{\mathsf{P}} 0, \tag{4.40}$$

by condition (4.30);

$$f(T_{kn}(t)) - f\Big((T_n(t))^{[k]}\Big) \xrightarrow{\mathsf{P}} 0, \tag{4.41}$$

on the strength of (4.39), (4.6) and uniform continuity of f. From the second relation, we get, since f is bounded,

$$\mathsf{E}\Big(f(T_{kn}(t)) - f\Big((T_n(t))^{[k]}\Big) \mid \mathscr{H}_n\Big) \xrightarrow{\mathsf{P}} 0. \tag{4.42}$$

These three relations together yield

$$\mathsf{E}\big(f(T_{kn}(t)) \mid \mathscr{H}_n\big) - f(T_{kn}(t)) \xrightarrow{\mathsf{P}} 0. \tag{4.43}$$

Thus, the sequences $(M_{kn},\ n \in \mathbb{N})$ and $(M_{kn}^m,\ n \in \mathbb{N})$ satisfy all the conditions of the lemma plus the above extra assumption. Then, according to item (1°) $\mathsf{E}(e^{iM_{kn}(t)} \mid \mathscr{H}_n) - e^{-\langle M_{kn}\rangle(t)/2} \xrightarrow{\mathsf{P}} 0$ as $n \to \infty$. Hence, and from (RC), relation (4.32) emerges by the same argument as (4.4) was derived from (4.20) and (4.22). □

Theorem 4.5. *Let for each $n \in \mathbb{N}$ $X_n, X_n^1, X_n^2 \ldots$ be locally square integrable martingales with respect to a flow $\mathbb{F}_n$. Suppose that the sequence $(\mathrm{tr}\langle X_n\rangle)$ is r.c. in C and for all $m \in \mathbb{N}$, $t > s > 0$, $\varepsilon > 0$, $t_2 > t_1 \geq 0$, $z \in \mathbb{R}^{d*}$ and bounded uniformly continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$*

$$\lim_{n\to\infty} \mathsf{E} \max_{u\le t} |\Delta X_n^m(u)|^2 = 0, \tag{4.44}$$

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\Big\{\mathrm{tr}\Big\langle X_n^l - X_n\Big\rangle(t) > \varepsilon\Big\} = 0, \tag{4.45}$$

$$\langle zX_n^m\rangle(t_2) - \langle zX_n^m\rangle(t_1) \leq \langle zX_n\rangle(t_2) - \langle zX_n\rangle(t_1), \tag{4.46}$$

$$\mathsf{E}\big(f(\langle zX_n\rangle(t)) \mid \mathcal{F}_n(s)\big) - f(\langle zX_n\rangle(t)) \xrightarrow{\mathsf{P}} 0. \tag{4.47}$$

Then, (1) *for any $p, l \in \mathbb{N}$, $t_p > \cdots > t_1 > t_0 \geq s > 0$, $z_1, \ldots, z_p \in \mathbb{R}^{d*}$, and $s_l > \cdots > s_1 > 0$*

$$\begin{aligned}\mathsf{E}&\left(\exp\left\{i\sum_{j=1}^{p} z_j\big(X_n(t_j) - X_n(t_{j-1})\big)\right\} \mid \mathcal{F}_n(s)\right)\\ &- \exp\left\{-\frac{1}{2}\sum_{j=1}^{p} z_j\big(\langle X_n\rangle(t_j) - \langle X_n\rangle(t_{j-1})\big)z_j^\top\right\} \xrightarrow{\mathsf{P}} 0\end{aligned} \tag{4.48}$$

(2) *under the extra assumption that the sequence* $(X_n(0))$ *is stochastically bounded,*

$$\mathsf{E}\left(\exp\left\{i\sum_{j=1}^{p} z_j\left(X_n(t_j)-X_n(t_{j-1})\right)\right\}F(X_n(0),\langle X_n\rangle(s_1),\ldots,\langle X_n\rangle(s_l))\mid \mathcal{F}_n(s)\right)$$
$$-\exp\left\{-\frac{1}{2}\sum_{j=1}^{p} z_j\left(\langle X_n\rangle(t_j)-\langle X_n\rangle(t_{j-1})\right)z_j^{\top}\right\}F(X_n(0),\langle X_n\rangle(s_1),\ldots,\langle X_n\rangle(s_l))\xrightarrow{\mathsf{P}}0, \tag{4.49}$$

for all $p,l\in\mathbb{N}$, $t_p>\cdots>t_1>t_0\geq s>0$, $z_1,\ldots,z_p\in\mathbb{R}^{d*}$, $s_l>\cdots>s_1>0$, *and* $F\in C_b(\mathbb{R}^d\times\mathfrak{S}^l)$.

Proof. The relative compactness condition implies that for any t, the sequence $(\|\langle X_n\rangle\|(t))$ is stochastically bounded. Then, it follows from (4.47) by Corollary 2.8 that for any $t>t'\geq s>0$, $\varepsilon>0$, $z\in\mathbb{R}^{d*}$ and $\varphi\in C_b(\mathbb{R}_+^2)$

$$\mathsf{E}\left|\mathsf{E}\left(\varphi\left(\langle zX_n\rangle(t),\langle zX_n\rangle(t')\right)\mid\mathcal{F}_n(s)\right)-\varphi\left(\langle zX_n\rangle(t),\langle zX_n\rangle(t')\right)\right|\longrightarrow 0. \tag{4.50}$$

If, moreover, the sequence $(X_n(0))$ is stochastically bounded, then the same corollary asserts that, in the notation of formula (4.49),

$$\mathsf{E}(F(X_n(0),\langle X_n\rangle(s_1),\ldots,\langle X_n\rangle(s_l))\mid\mathcal{F}_n(s))$$
$$-F(X_n(0),\langle X_n\rangle(s_1),\ldots,\langle X_n\rangle(s_l))\xrightarrow{\mathsf{P}}0. \tag{4.51}$$

Let us fix j and denote $M_n(u)=z_jX_n(t_{j-1}+u)-z_jX_n(t_{j-1})$ (likewise with a superscript), $\mathcal{G}_n(u)=\mathcal{F}_n(t_{j-1}+u)$. Then,

$$\langle M_n\rangle(u)=\langle z_jX_n\rangle(t_{j-1}+u)-\langle z_jX_n\rangle(t_{j-1}), \tag{4.52}$$

$$M_n(t_j-t_{j-1})=z_jX_n(t_j)-z_jX_n(t_{j-1}),$$
$$\langle M_n\rangle(t_j-t_{j-1})=\langle z_jX_n\rangle(t_j)-\langle z_jX_n\rangle(t_{j-1}),$$
$$\max_{u\leq t}|\Delta M_n^m(u)|=\max_{u\leq t_{j-1}+t}\left|z_j\Delta X_n^m(u)\right|, \tag{4.53}$$
$$\left\langle M_n^l-M_n\right\rangle(u)=\left\langle z_j\left(X_n^l-X_n\right)\right\rangle(t_{j-1}+u)-\left\langle z_j\left(X_n^l-X_n\right)\right\rangle(t_{j-1}).$$

So we have the implications: (4.44) $\Rightarrow$ (4.24); (4.45) $\Rightarrow$ (4.25). Setting in (4.50) $t=t_{j-1}+u$, $t'=t_{j-1}$, $z=z_j$, $\varphi(x,y)=f(x-y)$ ($f\in C_b(\mathbb{R}_+)$), and taking to account (4.52), we get (4.31) with $\mathcal{H}_n=\mathcal{G}_n(0)$. Equality (4.52) shows that the sequence $(\langle M_n\rangle)$ is r.c. in C, since $(\langle X_n\rangle)$ has this property. The similar equality for M_n^m and condition (4.46) imply (4.30). Thus, Lemma 4.4 asserts that for any t relation (4.32) with $\mathscr{H}_n = \mathcal{G}_n(0)$ holds. Putting $t = t_j - t_{j-1}$, we convert it to

$$\mathsf{E}\Big(e^{iz_j(X_n(t_j)-X_n(t_{j-1}))} \mid \mathcal{F}_n(t_{j-1})\Big) - e^{-z_j(\langle X_n\rangle(t_j)-\langle X_n\rangle(t_{j-1}))z_j^\top/2} \xrightarrow{\mathsf{P}} 0. \tag{4.54}$$

Denote the left-hand side of this relation by $\varkappa_n$. Inequality $|\varkappa_n| \le 2$ allows to rewrite it in the form $\mathsf{E}|\varkappa_n| \to 0$. Consequently, for any $s \in [0, t_{j-1}]$

$$\mathsf{E}\Big(e^{iz_j(X_n(t_j)-X_n(t_{j-1}))} \mid \mathcal{F}_n(s)\Big) - \mathsf{E}\Big(e^{-z_j(\langle X_n\rangle(t_j)-\langle X_n\rangle(t_{j-1}))z_j^\top/2} \mid \mathcal{F}_n(s)\Big) \xrightarrow{\mathsf{P}} 0. \tag{4.55}$$

Hence, and from (4.50) ($\varphi(x, y) = f(x - y)$), we have for $j = 1, \ldots, p$

$$\mathsf{E}\Big(e^{iz_j(X_n(t_j)-X_n(t_{j-1}))} \mid \mathcal{F}_n(s)\Big) - e^{-z_j(\langle X_n\rangle(t_j)-\langle X_n\rangle(t_{j-1}))z_j^\top/2} \xrightarrow{\mathsf{P}} 0. \tag{4.56}$$

Now, (4.48) emerges from Lemma 2.4.

Relation (4.49) follows from (4.48) and (4.51) by Lemma 2.5. □

Remark 4.6. Relation (4.49) *implies that every partial limit (with respect to the weak convergence in law) of a sequence* (X_n) *is a process with conditionally independent increments.*

The following result can facilitate the verification of condition (4.47).

Lemma 4.7. *Let for each $n \in \mathbb{N}$ Q_n be an $\mathfrak{S}$-valued or $\mathbb{R}^k$-valued random process adapted to a flow $\mathbb{F}_n$ on a probability space $(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$. Suppose that there exists a sequence (Λ_n) of scalar random processes such that, for any $n \in \mathbb{N}$ and $u > 0$, $\Lambda_n(u)$ is an $\mathcal{F}_n(0)$-measurable positive random variable; for all $t > s > 0$,*

$$Q_n(t) - \frac{\Lambda_n(t)}{\Lambda_n(s)} Q_n(s) \xrightarrow{\mathsf{P}} 0. \tag{4.57}$$

Then, for all for $t > s > 0$ and bounded uniformly continuous functions g on $\mathfrak{S}$ (or on $\mathbb{R}^k$),

$$\mathsf{E}(g(Q_n(t)) \mid \mathcal{F}_n(s)) - g(Q_n(t)) \xrightarrow{\mathsf{P}} 0. \tag{4.58}$$

Proof. Denote $\lambda_n(t, s) = \Lambda_n(t)/\Lambda_n(s)$. Condition (4.57) implies that

$$\mathsf{P}\{|Q_n(t) - \lambda_n(t, s)Q_n(s)| > \varepsilon \mid \mathscr{H}_n\} \xrightarrow{\mathsf{P}} 0, \tag{4.59}$$

for any $t > s > 0$, $\varepsilon > 0$ and sequence $(\mathcal{H}_n)$ whose nth member is a sub-σ-algebra of $\mathcal{F}_n$. Hence, and from the evident inequality

$$\begin{aligned} &\mathsf{E}\left(\left|g(Q_n(t)) - g(\lambda_n(t,s)Q_n(s))\right| \mid \mathcal{H}_n\right) \\ &\quad \le 2\|g\|_\infty \mathsf{P}\{\|Q_n(t) - \lambda_n(t,s)Q_n(s)\| > \varepsilon \mid \mathcal{H}_n\} + \sup_{\|A-B\|\le\varepsilon} \left|g(A) - g(B)\right|, \end{aligned} \tag{4.60}$$

we get by the choice of g

$$\mathsf{E}\left(g(Q_n(t)) - g(\lambda_n(t,s)Q_n(s)) \mid \mathcal{H}_n\right) \xrightarrow{\mathsf{P}} 0. \tag{4.61}$$

Setting here at first $\mathcal{H}_n = \mathcal{F}_n(s)$ and then $\mathcal{H}_n = \mathcal{F}_n(t)$, subtracting the second relation from the first and recalling that the random variable $\lambda_n(t,s)Q_n(s)$ is, by the assumptions about Q_n and Λ_n, $\mathcal{F}_n(s)$-measurable, we arrive at (4.58). □

Example 4.8. Let $\mathcal{F}_n(t) = \mathcal{F}(nt)$ and $Q_n(t) = n^{-1}R(nt)$, where R is an $\mathbb{F}$-adapted random process (so that Q_n is $\mathbb{F}_n$-adapted). Writing

$$Q_n(t) - \frac{t}{s}Q_n(s) = t\left(\frac{R(nt)}{nt} - \frac{R(nt)}{ns}\right), \tag{4.62}$$

we see that condition (4.57) will be fulfilled with $\Lambda_n(t) = t$ if we demand that $t^{-1}R(t)$ tend in probability to some limit as $t \to \infty$.

5. The Convergence Theorems

Theorem 5.1. *Let (Y_n) be a sequence of local square integrable martingales satisfying conditions (RC),* (3.10)*, and, for each t, the condition*

$$\mathsf{E}\max_{s\le t}|\Delta Y_n(s)|^2 \longrightarrow 0. \tag{5.1}$$

Then, for any infinite set $J_0 \subset \mathbb{N}$ there exist an infinite set $J \subset J_0$ and a continuous local martingale Y^J such that

$$(Y_n, \langle Y_n\rangle) \xrightarrow{\mathsf{C}} \left(Y^J, \left\langle Y^J\right\rangle\right) \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{5.2}$$

Proof. (1°) Denote $\tau_n^l = \inf\{t : |Y_n(t) - Y_n(0)| \ge l\}$, $Y_n^l(t) = Y_n(t\wedge\tau_n^l)$, $K_n = \langle Y_n\rangle$, $K_n^l = \langle Y_n^l\rangle$ (so that $K_n^l(t) = K_n(t\wedge\tau_n^l)$),

$$\eta_n = (Y_n, K_n), \qquad \eta_n^l = \left(Y_n^l, K_n^l\right), \tag{5.3}$$

regarding η_n and η_n^l as $\mathbb{R}^{d+d^2}$-valued processes.

Conditions (RC), (3.10) and (5.1) imply by Corollary 3.10 that the sequence (η_n) is r.c. in C. Then, by Corollaries 2.10 and 2.11, for any $l \in \mathbb{N}$ the sequence of compound processes

$(\eta_n^1,\ldots,\eta_n^l,K_n)$ is r.c. in C, too. Hence, using the diagonal method, we deduce that for any infinite set $J_0\subset\mathbb{N}$, there exist an infinite set $J\subset J_0$ and random processes $Y^1,K^1,Y^2,K^2\ldots$ such that for all $l\in\mathbb{N}$

$$\left(\eta_n^1,\ldots,\eta_n^l,K_n\right)\xrightarrow{\mathrm{C}}\left(\eta^1,\ldots,\eta^l,K\right)\quad\text{as } n\longrightarrow\infty,\ n\in J, \tag{5.4}$$

where

$$\eta^i=\left(Y^i,K^i\right). \tag{5.5}$$

The distribution of the right-hand side of (5.4) may depend on J, so the minute notation would be something like $(\eta^{J,1},\ldots,\eta^{J,l},K^J)$. We suppress, "for technical reasons", the superscript J, keeping, however, it in mind.

(2°) By the definition of Y_n^l,

$$\sup_{s\le t}\left|Y_n^l(s)-Y_n^l(0)\right|\le l+\max_{s\le t}|\Delta Y_n(s)|, \tag{5.6}$$

which together with (5.1) shows that for any l and t, the sequence $(\sup_{s\le t}|Y_n^l(s)-Y_n^l(0)|^2$, $n\in\mathbb{N})$ is uniformly integrable. Then, it follows from (5.3)–(5.5) by Corollary 3.17 and Remark 3.13 that Y^l is a continuous martingale and

$$K^l=\left\langle Y^l\right\rangle. \tag{5.7}$$

(3°) Writing

$$\left\{\sup_{s\le t}\left|\eta_n^l(s)-\eta_n(s)\right|>0\right\}\subset\left\{\tau_n^l<t\right\}\subset\left\{\sup_{s\le t}|Y_n(s)-Y_n(0)|\ge l\right\}, \tag{5.8}$$

and recalling that (Y_n) is r.c. in C, we arrive at (2.26).

(4°) Note that the processes $\eta^1,\eta^2\ldots$ are given, in view of (5.4), on a common probability space. Let us show that

$$\lim_{l\to\infty}\sup_{i>l}\mathsf{E}\varrho\left(\eta^i,\eta^l\right)=0, \tag{5.9}$$

where ϱ is the metric in D defined by

$$\varrho(f,q)=\sum_{m=1}^{\infty}2^{-m}\left(1\wedge\sup_{s\le m}\left|f(s)-g(s)\right|\right). \tag{5.10}$$

From (5.4), we have by Lemma 2.15

$$\sup_{s\le m}\left|\eta_n^i(s)-\eta_n^l(s)\right|\xrightarrow{\mathrm{d}}\sup_{s\le m}\left|\eta^i(s)-\eta^l(s)\right|\quad\text{as } n\longrightarrow\infty,\ n\in J, \tag{5.11}$$

for all natural m, i and l. Then, Alexandrov's theorem asserts that for any $\varepsilon > 0$,

$$\mathsf{P}\left\{\sup_{s\le m}\left|\eta^i(s) - \eta^l(s)\right| > \varepsilon\right\} \le \varliminf_{n\to\infty,\, n\in J} \mathsf{P}\left\{\sup_{s\le m}\left|\eta_n^i(s) - \eta_n^l(s)\right| > \varepsilon\right\}, \tag{5.12}$$

which together with the definitions of η_n^k, $\varliminf$ and $\varlimsup$ yields, for $i > l$,

$$\mathsf{P}\left\{\sup_{s\le m}\left|\eta^i(s) - \eta^l(s)\right| > \varepsilon\right\} \le \varlimsup_{n\to\infty,\, n\in J} \mathsf{P}\left\{\sup_{s\le m}|Y_n(s)| > l\right\}. \tag{5.13}$$

Hence, and from the evident inequality

$$\mathsf{E}(1\wedge\gamma) \le \varepsilon + \mathsf{P}\{\gamma > \varepsilon\}, \tag{5.14}$$

where γ is an arbitrary nonnegative random variable, we get for $i > l$,

$$\mathsf{E}\left(1\wedge \sup_{s\le m}\left|\eta^i(s) - \eta^l(s)\right|\right) \le \varepsilon + \varlimsup_{n\to\infty,\, n\in J} \mathsf{P}\left\{\sup_{s\le m}|Y_n(s)| \ge l\right\}. \tag{5.15}$$

By the Lenglart-Rebolledo inequality,

$$\mathsf{P}\left\{\sup_{s\le m}|Y_n(s)| \ge l\right\} \le \frac{a}{l^2} + \mathsf{P}\{\operatorname{tr} K_n(m) \ge a\}, \tag{5.16}$$

for any $a > 0$. Relation (5.4) implies, by Alexandrov's theorem, that

$$\varlimsup_{n\to\infty,\, n\in J} \mathsf{P}\{\operatorname{tr} K_n(m) \ge a\} \le \mathsf{P}\{\operatorname{tr} K(m) \ge a\}, \tag{5.17}$$

which together with (5.10)–(5.16) yields

$$\sup_{i>l} \mathsf{E}\varrho\left(\eta^i, \eta^l\right) \le \varepsilon + \frac{a}{l^2} + \sum_{m=1}^{\infty} 2^{-m}\mathsf{P}\{\operatorname{tr} K(m) \ge a\}. \tag{5.18}$$

Hence, letting $l \to \infty$, then $a \to \infty$ finally $\varepsilon \to 0$, we obtain (5.9).

(5°) Obviously, ϱ metrizes the $\mathcal{U}$-convergence and the metric space (C, ϱ) is complete. Relation (5.9) means that the sequence (η^l) of C-valued random elements is fundamental in probability. Then, by the Riesz theorem, each of its subsequences contains a subsequence converging w.p.1. The limits of every two convergent subsequences coincide w.p.1 because of (5.9). So, there exists a C-valued random element (= continuous random process) η such that

$$\lim_{l\to\infty} \mathsf{E}\varrho\left(\eta^l, \eta\right) = 0. \tag{5.19}$$

And this is a fortified form of the relation

$$\eta^l \xrightarrow{\mathrm{C}} \eta. \tag{5.20}$$

In particular, the sequence (η^l) is r.c. in C (which can be proved directly, but such proof does not guarantee that partial limits are given on the same probability space that the prelimit processes are).

(6°) Relation (5.4) together with the conclusions of items (3°) and (5°) shows that all the conditions of Corollary 2.14 (with the range of n restricted to J) are fulfilled (and even overfulfilled: relation (5.20) proved above without recourse to Corollary 2.14 contains both an assumption and a conclusion of the latter). So, Corollary 2.14 asserts, in addition to (5.20), that

$$\eta_n \xrightarrow{\mathrm{C}} \eta \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{5.21}$$

This pair of relations can be rewritten, in view of (5.3) and (5.5), in the form

$$\left(Y^l, K^l\right) \xrightarrow{\mathrm{C}} (Y, K), \tag{5.22}$$

$$(Y_n, K_n) \xrightarrow{\mathrm{C}} (Y, K) \quad \text{as } n \longrightarrow \infty,\ n \in J, \tag{5.23}$$

where (Y, K) is a synonym of η. We wish to stress again that, firstly, all the processes in (5.22) are given on a common probability space and, secondly, they depend on the choice of J.

(7°) Let us show that Y is a local martingale.

Denote $\sigma_m = \inf\{t : \operatorname{tr} K(t) \geq m\}$, and $M_m(t) = Y(t \wedge \sigma_m)$, $M_m^l(t) = Y^l(t \wedge \sigma_m)$. Equalities (5.19), (5.10), and (5.5) yield

$$\lim_{l \to \infty} \mathsf{E}\varrho\left(M_m^l, M_m\right) = 0, \tag{5.24}$$

whence

$$M_m^l \xrightarrow{\mathrm{d}} M_m \quad \text{as } l \longrightarrow \infty. \tag{5.25}$$

On the strength of (5.7),

$$\left\langle M_m^l \right\rangle(t) = \mathrm{K}^l(t \wedge \sigma_m). \tag{5.26}$$

By the construction of the processes Y_n^l and K_n^l for any $s \in \mathbb{R}_+$ and $n \in \mathbb{N}$, the sequence $(\operatorname{tr} K_n^l(s), l \in \mathbb{N})$ increases. Then, due to (5.4) so does $(\operatorname{tr} K^l(s), l \in \mathbb{N})$. Hence, we have with account of (5.19), (5.10), and (5.5)

$$\operatorname{tr} K^l(s) \leq \operatorname{tr} K(s), \tag{5.27}$$

for all s and l. Comparing this with (5.26), we see that

$$\mathsf{E}\ \mathrm{tr}\left\langle M_m^l \right\rangle(t) \le \mathsf{E}\ \mathrm{tr}\, K(t \wedge \sigma_m). \tag{5.28}$$

But $\mathrm{tr}\, K$ is a continuous increasing process, so $\mathrm{tr}\, K(\sigma_m) = m,\ \mathrm{tr}\, K(t \wedge \sigma_m) \le m$. Now, it follows from (5.25) and (5.28) by Corollary 3.3 that M_m is a uniformly integrable martingale. Thus, the sequence (σ_m) localizes Y.

(8°) Relation $Y^l \xrightarrow{\mathrm{C}} Y$ (a part of (5.22)) where the prelimit processes are, according to item (2°), continuous martingales implies by Corollary VI.6.7 [2] that

$$\left(Y^l, \left[Y^l\right]\right) \xrightarrow{\mathrm{C}} (Y, [Y]). \tag{5.29}$$

Comparing this with (5.22), we get with account of (3.4) $(Y, K) \overset{\mathrm{d}}{=} (Y, \langle Y \rangle)$, hereupon Corollary 3.5 asserts that $K = \langle Y \rangle$. □

Corollary 5.2. *Let (Y_n) be a sequence of local square integrable martingales satisfying conditions (RC),* (3.21)*, and, for all $t > 0$,* (5.1)*. Then, Y is a continuous local martingale and relation* (3.25) *holds.*

Proof. Let J_0 be an arbitrary infinite set of natural numbers. Then, Theorem 5.1 whose condition (3.10) is covered by (3.21) asserts existence of an infinite set $J \subset J_0$ and a continuous local martingale Y^J such that (5.2) holds. By assumption, the distribution of Y^J and, consequently, of $(Y^J, \langle Y^J \rangle)$ does not depend on J, which allows to delete the superscript in (5.2). Hence, taking to account arbitrariness of J_0, we conclude that (5.2) holds for $J = \mathbb{N}$. □

Corollary 5.3. *Let a sequence (Y_n) of locally square integrable martingales satisfy conditions (RC) and, for all $t > 0$,* (5.1)*. Then, relation* (3.19) *holds.*

Proof. It was shown in items (1°) and (2°) of the proof of Theorem 5.1 that for each l, the sequence $(Y_n^l, n \in \mathbb{N})$ satisfies all the conditions of Lemma 3.16 which, therefore, asserts that $[Y_n^l] - \langle Y_n^l \rangle \xrightarrow{\mathrm{C}} O$ as $n \to \infty$. Hence, by the same argument as in item (3°), relation (3.19) follows. □

Theorem 5.4. *Let for each $n \in \mathbb{N}$ $X_n, X_n^1, X_n^2 \ldots$ be locally square integrable martingales with respect to a common filtration. Suppose that for all $m \in \mathbb{N}$, $t > 0$ and $\varepsilon > 0$ conditions* (4.44) *and* (4.45) *are fulfilled, and*

$$\lim_{L\to\infty} \sup_l \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \left| X_n^l(0) \right| > L \right\} = 0, \tag{5.30}$$

$$\lim_{L\to\infty} \sup_l \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \mathrm{tr}\left\langle X_n^l \right\rangle(t) > L \right\} = 0, \tag{5.31}$$

$$\lim_{r\to 0} \sup_l \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \sup_{(t_1,t_2)\in\Pi(t,r)} \left(\mathrm{tr}\left\langle X_n^l \right\rangle(t_2) - \mathrm{tr}\left\langle X_n^l \right\rangle(t_1) \right) > \varepsilon \right\} = 0, \tag{5.32}$$

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \left| X_n^l(0) - X_n(0) \right| > \varepsilon \right\} = 0. \tag{5.33}$$

Then, for any infinite set $J_0 \subset \mathbb{N}$, there exist an infinite set $J \subset J_0$ and a continuous local martingale X such that

$$(X_n, \langle X_n \rangle) \xrightarrow{C} (X, \langle X \rangle) \quad \text{as } n \longrightarrow \infty, \ n \in J. \tag{5.34}$$

Note that relation (5.34) is, up to notation, a duplicate of (5.2). So, the superscript J on the right-hand side is tacitly implied (but suppressed because the conditions of the theorem contain another superscript).

Proof. Conditions (5.31) and (5.32) imply that for each m, the sequence $(\langle X_n^m \rangle, n \in \mathbb{N})$ is r.c. in C. Then, it follows from (4.44) and (5.30) by Lemma 3.9 that the sequence $(X_n^m, n \in \mathbb{N})$ is r.c. in C. So, there exist an infinite set $J_m \subset J_{m-1}$ and a random process X^m such that

$$X_n^m \xrightarrow{C} X^m \quad \text{as } n \longrightarrow \infty, \ n \in J_m. \tag{5.35}$$

Consequently, if we denote by J the set whose mth member is that of J_m, then for each m,

$$X_n^m \xrightarrow{C} X^m \quad \text{as } n \longrightarrow \infty, \ n \in J. \tag{5.36}$$

And this together with (4.44) and relative compactness of $(\langle X_n^m \rangle, n \in \mathbb{N})$ implies by Corollary 5.2 that X^m is a continuous local martingale and

$$\eta_n^m \equiv (X_n^m, \langle X_n^m \rangle) \xrightarrow{C} \eta^m \equiv (X^m, \langle X^m \rangle) \quad \text{as } n \longrightarrow \infty, \ n \in J. \tag{5.37}$$

Then, it follows from (5.30)–(5.32) that

$$\begin{gathered}\lim_{L\to\infty} \sup_l \mathsf{P}\left\{ \left| X^l(0) \right| > L \right\} = 0, \qquad \lim_{L\to\infty} \sup_l \mathsf{P}\left\{ \operatorname{tr}\left\langle X^l \right\rangle(t) > L \right\} = 0, \\ \lim_{r\to 0} \sup_l \mathsf{P}\left\{ \sup_{(t_1,t_2)\in\Pi(t,r)} \left(\operatorname{tr}\left\langle X^l \right\rangle(t_2) - \operatorname{tr}\left\langle X^l \right\rangle(t_1) \right) > \varepsilon \right\} = 0,\end{gathered} \tag{5.38}$$

and therefore, the sequences $(\langle X^l \rangle)$, (X^l) and (η^l) are r.c. in C.

Conditions (5.33) and (4.45) imply by the Lenglart-Rebolledo inequality that for all positive t and ε

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \sup_{s\le t} \left| X_n^l(s) - X_n(s) \right| > \varepsilon \right\} = 0. \tag{5.39}$$

Conditions(5.31) and (4.45) imply by Lemma 3.8 that

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \mathsf{P}\left\{ \sup_{s\le t} \left\| \left\langle X_n^l \right\rangle(s) - \langle X_n \rangle(s) \right\| > \varepsilon \right\} = 0, \tag{5.40}$$

which together with the previous relation yields (2.26). Then, Corollary 2.14 asserts existence of a random process $\eta \equiv (X, H)$ such that

$$\left(X^l, \left\langle X^l \right\rangle\right) \xrightarrow{\mathrm{C}} (X, H), \tag{5.41}$$

$$(X_n, \langle X_n \rangle) \xrightarrow{\mathrm{C}} (X, H) \quad \text{as } n \longrightarrow \infty,\ n \in J. \tag{5.42}$$

The ensuing relation $X^l \xrightarrow{\mathrm{C}} X$, continuity (due to (5.37)) of all X^l and relative compactness of $(\langle X^l \rangle)$ imply by Corollary 5.2 that X is a continuous local martingale and $(X^l, \langle X^l \rangle) \xrightarrow{\mathrm{C}} (X, \langle X \rangle)$. Comparing this with (5.41), we get $(X, H) \overset{\mathrm{d}}{=} (X, \langle X \rangle)$, which converts (5.42) to (5.34). □

Repeating the deduction of Corollary 5.2 from Theorem 5.1, we get from Theorem 5.4 the following conclusion.

Corollary 5.5. *Let for each $n \in \mathbb{N}$ $X_n, X_n^1, X_n^2 \dots$ be locally square integrable martingales with respect to a common filtration. Suppose that, they have the same initial value; conditions* (4.44), (4.45), (5.31), *and* (5.32) *are fulfilled for all m and t; there exists a random process X such that $X_n \xrightarrow{\mathrm{C}} X$. Then, X is a continuous local martingale and $(X_n, \langle X_n \rangle) \xrightarrow{\mathrm{C}} (X, \langle X \rangle)$.*

Theorem 5.6. *Let for each $n \in \mathbb{N}$ $X_n, X_n^1, X_n^2 \dots$ be locally square integrable martingales with respect to a flow $\mathbb{F}_n$. Suppose that, conditions* (4.44)–(4.47) *and* (5.30)–(5.33) *are fulfilled for all $m \in \mathbb{N}, t > s \geq 0, > 0$, $t_2 > t_1 \geq 0, z \in \mathbb{R}^{d*}$, and bounded uniformly continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$; there exist an $\mathbb{R}^d$-valued random variable $\mathring{X}$ and an $\mathfrak{S}_+$-valued random process H such that*

$$(X_n(0), \langle X_n \rangle) \xrightarrow{\mathrm{C}} \left(\mathring{X}, H\right). \tag{5.43}$$

Then, (1) *for any $p, l \in \mathbb{N}$, $t_p > \cdots > t_0 \geq 0$, $s_l > \cdots > s_1 > 0$, $z_1, \dots z_p \in \mathbb{R}^{d*}$ and bounded continuous function $F : \mathbb{R}^d \times \mathfrak{S}_+^l \to \mathbb{R}$*

$$\begin{aligned} &\mathsf{E}\left(\exp\left\{i\sum_{j=1}^{p} z_j\left(X_n(t_j) - X_n(t_{j-1})\right)\right\} F\left(X_n(0), \langle X_n\rangle(s_1), \dots, \langle X_n\rangle(s_l)\right)\right) \\ &\longrightarrow \mathsf{E}\left(\exp\left\{-\frac{1}{2}\sum_{j=1}^{p} z_j\left(H(t_j) - H(t_{j-1})\right) z_j^\top\right\} F\left(\mathring{X}, H(s_1), \dots, H(s_l)\right)\right), \end{aligned} \tag{5.44}$$

(2) *there exists a continuous local martingale* X *with initial value* $\overset{\circ}{X}$ *and quadratic characteristic* H *such that* $(X_n, \langle X_n \rangle) \overset{C}{\to} (X, H)$ *and*

$$\begin{aligned}
&\mathsf{E}\left(\exp\left\{ i\sum_{j=1}^{p} z_j (X(t_j) - X(t_{j-1})) \right\} F(X(0), \langle X \rangle(s_1), \dots, \langle X \rangle(s_l)) \right) \\
&\quad = \mathsf{E}\left(\exp\left\{ -\frac{1}{2}\sum_{j=1}^{p} z_j (H(t_j) - H(t_{j-1})) z_j^{\top} \right\} F\left(\overset{\circ}{X}, H(s_1), \dots, H(s_l) \right) \right),
\end{aligned} \tag{5.45}$$

for any $p, l \in \mathbb{N}$, $t_p > \cdots > t_0 \geq 0$, $s_l > \cdots > s_1 > 0$, $z_1, \dots z_p \in \mathbb{R}^{d*}$ *and bounded continuous function* $F : \mathbb{R}^d \times \mathfrak{S}_+^l \to \mathbb{R}$.

Proof. Since the assumptions of this theorem contain those of Theorem 5.4, the conclusion of the latter is valid. It implies, in particular, that the sequence $(\langle X_n \rangle)$ is r.c. in C. So, firstly, the assumptions of Theorem 4.5 are also fulfilled (and therefore the conclusions are valid), and, secondly, the relation

$$\lim_{t \to 0} \overline{\lim_{n \to \infty}} \mathsf{P}\{|X_n(t) - X_n(0)| > \varepsilon\} = 0 \tag{5.46}$$

holds.

If $t_0 > 0$, then Theorem 4.5 asserts relation (4.49) which together with (5.43) yields, by the dominated convergence theorem, (5.44). Relation (5.46) and continuity (due to (5.43)) of H enable us to let $t_0 \to 0$ in (5.44), thus waiving the interim assumption $t_0 > 0$.

Combining (4.49) with the conclusion of Theorem 5.4 and with the dominated convergence theorem, we see that for any infinite set $J_0 \subset \mathbb{N}$, there exist an infinite set $J \subset J_0$ and a continuous local martingale X such that for all $p, l \in \mathbb{N}$, $t_p > \cdots > t_0 > 0$, $s_l > \cdots > s_1 > 0$, $z_1, \dots z_p \in \mathbb{R}^{d*}$ and bounded continuous function $F : \mathbb{R}^d \times \mathfrak{S}_+^l \to \mathbb{R}$

$$\begin{aligned}
&\mathsf{E}\left(\exp\left\{ i\sum_{j=1}^{p} z_j (X_n(t_j) - X_n(t_{j-1})) \right\} F(X_n(0), \langle X_n \rangle(s_1), \dots, \langle X_n \rangle(s_l)) \right) \\
&\longrightarrow \mathsf{E}\left(\exp\left\{ -\frac{1}{2}\sum_{j=1}^{p} z_j (\langle X \rangle(t_j) - \langle X \rangle(t_{j-1})) z_j^{\top} \right\} F(X(0), \langle X \rangle(s_1), \dots, \langle X \rangle(s_l)) \right),
\end{aligned} \tag{5.47}$$

as $n \to \infty$, $n \in J$. The comparison of (5.43) and (5.34) shows that the right-hand side of (5.47) equals

$$\mathsf{E}\left(\exp\left\{ -\frac{1}{2}\sum_{j=1}^{p} z_j (H(t_j) - H(t_{j-1})) z_j^{\top} \right\} F\left(\overset{\circ}{X}, H(s_1), \dots, H(s_l) \right) \right), \tag{5.48}$$

and, therefore, does not depend on the choice of J_0 and J. So, (5.47) holds as n ranges over $\mathbb{N}$, too. This together with (5.44) proves the second statement under the extra assumption $t_0 > 0$ which can be waived exactly as above. □

Corollary 5.7. *Let the conditions of Theorem 5.6 be fulfilled. Then, X has conditionally with respect to* $\mathcal{G} \equiv \sigma(\overset{\circ}{X}, \langle X\rangle(\cdot))$ *independent increments.*

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