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Research Article

Periodic and Solitary Wave Solutions to the Fornberg-Whitham Equation

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New travelling wave solutions to the Fornberg-Whitham equation $u_t - u_{xxt} + u_x + uu_x = uu_{xxx} + 3u_x u_{xx}$ are investigated. They are characterized by two parameters. The expresssions for the periodic and solitary wave solutions are obtained.

1. Introduction

Recently, Ivanov [1] investigated the integrability of a class of nonlinear dispersive wave equations:

$$u_t - u_{xxt} + \partial_x\left(\kappa u + \alpha u^2 + \beta u^3\right) = \nu u_x u_{xx} + \gamma u u_{xxx}, \tag{1.1}$$

where and $\alpha, \beta, \gamma, \kappa, \nu$ are real constants.

The important cases of (1.1) are as follows. The hyperelastic-rod wave equation

$$u_t - u_{xxt} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}) \tag{1.2}$$

has been recently studied as a model, describing nonlinear dispersive waves in cylindrical compressible hyperelastic rods [2–7]. The physical parameters of various compressible materials put γ in the range from –29.4760 to 3.4174 [2, 4].

The Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.3}$$

describes the unidirectional propagation of shallow water waves over a flat bottom [8, 9]. It is completely integrable [1] and admits, in addition to smooth waves, a multitude of travelling wave solutions with singularities: peakons, cuspons, stumpons, and composite waves [9–12]. The solitary waves of (1.2) are smooth if $\kappa > 0$ and peaked if $\kappa = 0$ [9, 10]. Its solitary waves are stable solitons [13, 14], retaining their shape and form after interactions [15]. It models wave breaking [16–18].

The Degasperis-Procesi equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{1.4}$$

models nonlinear shallow water dynamics. It is completely integrable [1] and has a variety of travelling wave solutions including solitary wave solutions, peakon solutions and shock waves solutions [19–26].

The Fornberg-Whitham equation

$$u_t - u_{xxt} + u_x + uu_x = uu_{xxx} + 3u_x u_{xx} \tag{1.5}$$

appeared in the study qualitative behaviors of wave-breaking [27]. It admits a wave of greatest height, as a peaked limiting form of the travelling wave solution [28], $u(x,t) = A\exp(-1/2|x - 4/3t|)$, where A is an arbitrary constant. It is not completely integrable [1].

The regularized long-wave or BBM equation

$$u_t - u_{xxt} + u_x + uu_x = 0 \tag{1.6}$$

and the modified BBM equation

$$u_t - u_{xxt} + u_x + 3u^2 u_x = 0 \tag{1.7}$$

have also been investigated by many authors [29–37].

Many efforts have been devoted to study (1.2)–(1.4), (1.6), and (1.7), however, little attention was paid to study (1.5). In [38], we constructed two types of bounded travelling wave solutions $u(\xi)(\xi = x - ct)$ to (1.5), which are defined on semifinal bounded domains and called kink-like and antikink-like wave solutions. In this paper, we continue to study the travelling wave solutions to (1.5). Following Vakhnenko and Parkes's strategy in [39], we obtain some periodic and solitary wave solutions $u(\xi)$ to (1.5) which are defined on $(-\infty, +\infty)$. The travelling wave solutions obtained in this paper are obviously different from those obtained in our previous work [38]. To the best of our knowledge, these solutions are new for (1.5). Our work may help people to know deeply the described physical process and possible applications of the Fornberg-Whitham equation.

The remainder of the paper is organized as follows. In Section 2, for completeness and readability, we repeat Appendix A in [39], which discusses the solutions to a first-order ordinary differential equaion. In Section 3, we show that, for travelling wave solutions, (1.5) may be reduced to a first-order ordinary differential equation involving two arbitrary integration constants a and b. We show that there are four distinct periodic solutions corresponding to four different ranges of values of a and restricted ranges of values of b. A short conclusion is given in Section 4.

2. Solutions to a First-Order Ordinary Differential Equaion

This section is due to Vakhnenko and Parkes (see Appendix A in [39]). For completeness and readability, we repeat it in the following.

Consider solutions to the following ordinary differential equation

$$(\varphi\varphi_{\xi})^2 = \varepsilon^2 f(\varphi), \tag{2.1}$$

where

$$f(\varphi) = (\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi_3 - \varphi)(\varphi_4 - \varphi), \tag{2.2}$$

and φ_1, φ_2, φ_3, φ_4 are chosen to be real constants with $\varphi_1 \leq \varphi_2 \leq \varphi \leq \varphi_3 \leq \varphi_4$.

Following [40] we introduce ζ defined by

$$\frac{d\xi}{d\zeta} = \frac{\varphi}{\varepsilon}, \tag{2.3}$$

so that (2.1) becomes

$$(\varphi_{\zeta})^2 = f(\varphi). \tag{2.4}$$

Equation(2.4) has two possible forms of solution. The first form is found using result 254.00 in [41]. Its parametric form is

$$\begin{aligned} \varphi &= \frac{\varphi_2 - \varphi_1 n \operatorname{sn}^2(w \mid m)}{1 - n \operatorname{sn}^2(w \mid m)}, \\ \xi &= \frac{1}{\varepsilon p}\left(w\varphi_1 + (\varphi_2 - \varphi_1)\Pi(n; w \mid m)\right), \end{aligned} \tag{2.5}$$

with w as the parameter, where

$$m = \frac{(\varphi_3 - \varphi_2)(\varphi_4 - \varphi_1)}{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}, \qquad p = \frac{1}{2}\sqrt{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}, \qquad w = p\zeta, \tag{2.6}$$

$$n = \frac{\varphi_3 - \varphi_2}{\varphi_3 - \varphi_1}. \tag{2.7}$$

In (2.5) $\operatorname{sn}(w \mid m)$ is a Jacobian elliptic function, where the notation is as used in [42, Chapter 16], and the notation is as used in [42, Section 17.2.15].

The solution to (2.1) is given in parametric form by (2.5) with w as the parameter. With respect to w, φ in (2.5) is periodic with period $2K(m)$, where $K(m)$ is the complete elliptic integral of the first kind. It follows from (2.5) that the wavelength λ of the solution to (2.1) is

$$\lambda = \frac{2}{\varepsilon p}\left|\varphi_1 K(m) + (\varphi_2 - \varphi_1)\Pi(n \mid m)\right|, \tag{2.8}$$

where $\Pi(n \mid m)$ is the complete elliptic integral of the third kind.

When $\varphi_3 = \varphi_4$, $m = 1$, (2.5) becomes

$$\begin{gathered} \varphi = \frac{\varphi_2 - \varphi_1 n \tanh^2 w}{1 - n \tanh^2 w}, \\ \xi = \frac{1}{\varepsilon}\left(\frac{w\varphi_3}{p} - 2\tanh^{-1}\left(\sqrt{n}\tanh w\right)\right). \end{gathered} \tag{2.9}$$

The second form of solution of (2.5) is found using result 255.00 in [41]. Its parametric form is

$$\begin{gathered} \varphi = \frac{\varphi_3 - \varphi_4 n \operatorname{sn}^2(w \mid m)}{1 - n\operatorname{sn}^2(w \mid m)}, \\ \xi = \frac{1}{\varepsilon p}\left(w\varphi_4 - (\varphi_4 - \varphi_3)\Pi(n; w \mid m)\right), \end{gathered} \tag{2.10}$$

where m, p, w are as in (2.6), and

$$n = \frac{\varphi_3 - \varphi_2}{\varphi_4 - \varphi_2}. \tag{2.11}$$

The solution to (2.1) is given in parametric form by (2.10) with w as the parameter. The wavelength λ of the solution to (2.1) is

$$\lambda = \frac{2}{\varepsilon p}\left|\varphi_4 K(m) - (\varphi_4 - \varphi_3)\Pi(n \mid m)\right|. \tag{2.12}$$

When $\varphi_1 = \varphi_2$, $m = 1$, (2.10) becomes

$$\begin{gathered} \varphi = \frac{\varphi_3 - \varphi_4 n \tanh^2 w}{1 - n \tanh^2 w}, \\ \xi = \frac{1}{\varepsilon}\left(\frac{w\varphi_2}{p} + 2\tanh^{-1}\left(\sqrt{n}\tanh w\right)\right). \end{gathered} \tag{2.13}$$

3. Periodic and Solitary Wave Solutions to Equation (1.5)

Equation (1.5) can also be written in the form

$$(u_t + uu_x)_{xx} = u_t + uu_x + u_x. \tag{3.1}$$

Let $u = \varphi(\xi) + c$ with $\xi = x - ct$ be a travelling wave solution to (3.1), then it follows that

$$(\varphi\varphi_\xi)_{\xi\xi} = \varphi\varphi_\xi + \varphi_\xi. \tag{3.2}$$

Integrating (3.2) twice with respect to ξ, we have

$$(\varphi\varphi_\xi)^2 = \frac{1}{4}\left(\varphi^4 + \frac{8}{3}\varphi^3 + a\varphi^2 + b\right), \tag{3.3}$$

where a and b are two arbitrary integration constants.

Equation (3.3) is in the form of (2.1) with $\varepsilon = 1/2$ and $f(\varphi) = (\varphi^4 + 8/3\varphi^3 + a\varphi^2 + b)$. For convenience we define $g(\varphi)$ and $h(\varphi)$ by

$$\begin{aligned} f(\varphi) &= \varphi^2 g(\varphi) + b, \quad \text{where } g(\varphi) = \varphi^2 + \frac{8}{3}\varphi + a, \\ f\prime(\varphi) &= 2\varphi h(\varphi), \quad \text{where } h(\varphi) = 2\varphi^2 + 4\varphi + a, \end{aligned} \tag{3.4}$$

and define φ_L, φ_R, b_L, and b_R by

$$\begin{aligned} \varphi_L &= -\frac{1}{2}\left(2 + \sqrt{4-2a}\right), \qquad \varphi_R = -\frac{1}{2}\left(2 - \sqrt{4-2a}\right), \\ b_L &= -\varphi_L^2 g(\varphi_L) = \frac{a^2}{4} - 2a + \frac{8}{3} + \frac{2}{3}(2-a)\sqrt{4-2a}, \\ b_R &= -\varphi_R^2 g(\varphi_R) = \frac{a^2}{4} - 2a + \frac{8}{3} - \frac{2}{3}(2-a)\sqrt{4-2a}. \end{aligned} \tag{3.5}$$

Obviously, φ_L, φ_R are the roots of $h(\varphi) = 0$.

In the following, suppose that $a < 2$ and $a \neq 0$ such that $f(\varphi)$ has three distinct stationary points: φ_L, φ_R, 0 and comprise two minimums separated by a maximum. Under this assumption, (3.3) has periodic and solitary wave solutions that have different analytical forms depending on the values of a and b as follows.

(1) $a < 0$

In this case $\varphi_L < 0 < \varphi_R$ and $f(\varphi_L) < f(\varphi_R)$. For each value $a < 0$ and $0 < b < b_R$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(a)), there are periodic inverted loop-like solutions to (3.3) given by (2.5) so that $0 < m < 1$, and with wavelength given by (2.8); see Figure 2(a), for an example.

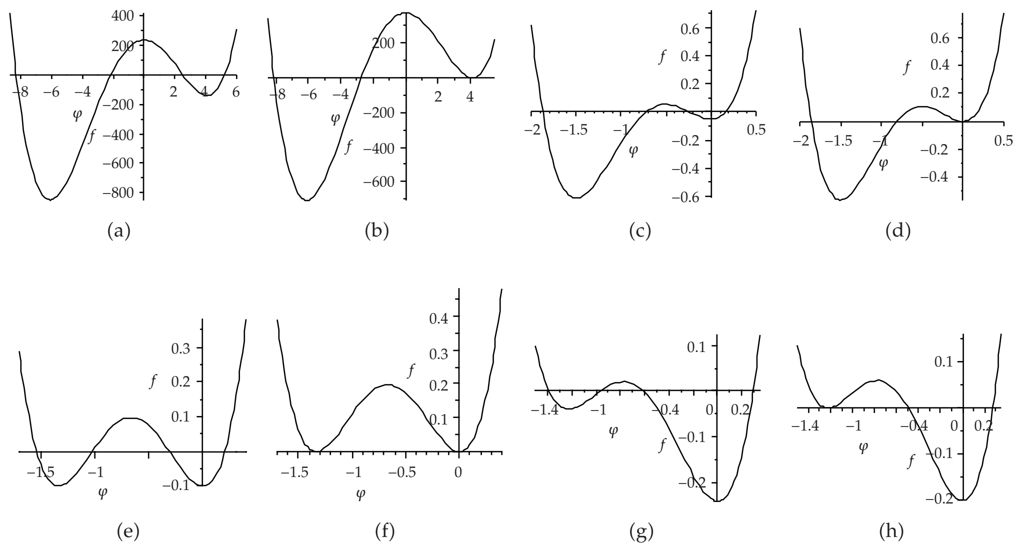

Figure 1: The curve of $f(\varphi)$. (a) $a = -50$, $b = 233$; (b) $a = -50$, $b = 374.1346$; (c) $a = 1.5$, $b = -0.05$; (d) $a = 1.5$, $b = 0$; (e) $a = 16/9$, $b = -0.1$; (f) $a = 16/9$, $b = 0$; (g) $a = 1.9$, $b = -0.24$; (h) $a = 1.9$, $b = -0.2010$.

The case $a < 0$ and $b = b_R$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(b)) corresponds to the limit $\varphi_3 = \varphi_4 = \varphi_R$ so that $m = 1$, and then the solution is an inverted loop-like solitary wave given by (2.9) with $\varphi_2 \le \varphi < \varphi_R$ and

$$\begin{aligned} \varphi_1 &= -\frac{1}{6}\left(2 + 3\sqrt{4-2a} + 2\sqrt{4 + 6\sqrt{4-2a}}\right), \\ \varphi_2 &= -\frac{1}{6}\left(2 + 3\sqrt{4-2a} - 2\sqrt{4 + 6\sqrt{4-2a}}\right); \end{aligned} \tag{3.6}$$

see Figure 3(a), for an example.

(2) $0 < a < 16/9$

In this case $\varphi_L < \varphi_R < 0$ and $f(\varphi_L) < f(0)$. For each value $0 < a < 16/9$ and $b_R < b < 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(c)), there are periodic hump-like solutions to (3.3) given by (2.5) so that $0 < m < 1$, and with wavelength given by (2.8); see Figure 2(b), for an example.

The case $0 < a < 16/9$ and $b = 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(d)) corresponds to the limit $\varphi_3 = \varphi_4 = 0$ so that $m = 1$, and then the solution can be given by (2.9) with φ_1 and φ_2 given by the roots of $g(\varphi) = 0$, namely

$$\varphi_1 = -\frac{4}{3} - \frac{1}{3}\sqrt{16-9a}, \qquad \varphi_2 = -\frac{4}{3} + \frac{1}{3}\sqrt{16-9a}. \tag{3.7}$$

In this case we obtain a weak solution, namely, the periodic upward-cusp wave

$$\varphi = \varphi(\xi - 2j\xi_m), \quad (2j-1)\xi_m < \xi < (2j+1)\xi_m, \quad j = 0, \pm 1, \pm 2, \ldots, \tag{3.8}$$

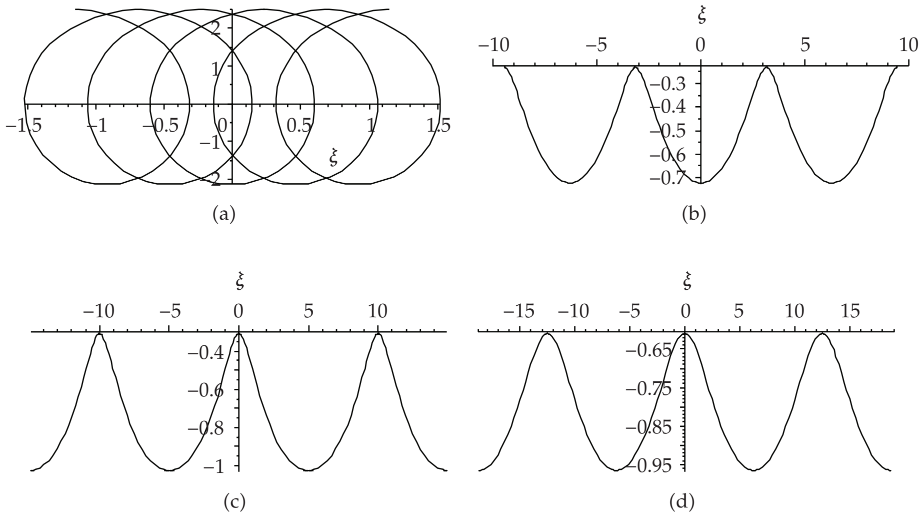

Figure 2: Periodic solutions to (3.3) with $0 < m < 1$. (a) $a = -50$, $b = 233$ so $m = 0.7885$; (b) $a = 1.5$, $b = -0.05$ so $m = 0.6893$; (c) $a = 16/9$, $b = -0.1$ so $m = 0.8254$; (d) $a = 1.9$, $b = -0.24$ so $m = 0.6121$.

where

$$\varphi(\xi) = \left(\varphi_2 - \varphi_1 \tanh^2\left(\frac{\xi}{4}\right)\right)\cosh^2\left(\frac{\xi}{4}\right), \tag{3.9}$$

$$\xi_m = 4\tanh^{-1}\sqrt{\frac{\varphi_2}{\varphi_1}}, \tag{3.10}$$

see Figure 3(b), for an example.

(3) $a = 16/9$

In this case $\varphi_L < \varphi_R < 0$ and $f(\varphi_L) = f(0)$. For $a = 16/9$ and each value $b_R < b < 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(e)), there are periodic hump-like solutions to (3.3) given by (2.10) so that $0 < m < 1$, and with wavelength given by (2.12); see Figure 2(c), for an example.

The case $a = 16/9$ and $b = 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(f)) corresponds to the limit $\varphi_1 = \varphi_2 = \varphi_L = -4/3$ and $\varphi_3 = \varphi_4 = 0$ so that $m = 1$. In this case neither (2.9) nor (2.13) is appropriate. Instead we consider (3.3) with $\mathrm{f}(\varphi) = 1/4\varphi^2(\varphi + 4/3)^2$ and note that the bound solution has $-4/3 < \varphi \le 0$. On integrating (3.3) and setting $\varphi = 0$ at $\xi = 0$ we obtain a weak solution

$$\varphi = \frac{4}{3}\exp\left(-\frac{1}{2}|\xi|\right) - \frac{4}{3}, \tag{3.11}$$

that is, a single peakon solution with amplitude $4/3$, see Figure 3(c).

(4) $16/9 < a < 2$

In this case $\varphi_L < \varphi_R < 0$ and $f(\varphi_L) > f(0)$. For each value $16/9 < a < 2$ and $b_R < b < b_L$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(g)), there are periodic hump-like

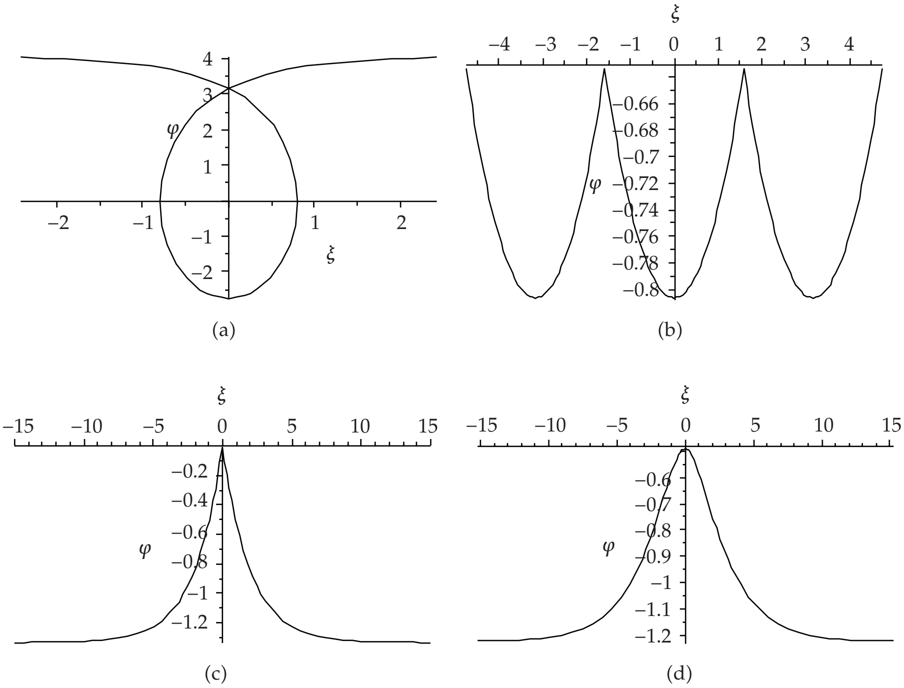

Figure 3: Solutions to (3.3) with $m = 1$. (a) $a = -50$, $b = 374.1346$; (b) $a = 1.5$, $b = 0$; (c) $a = 16/9$, $b = 0$; (d) $a = 1.9$, $b = -0.2010$.

solutions to (3.3) given by (2.10) so that $0 < m < 1$, and with wavelength given by (2.12); see Figure 2(d), for an example.

The case $16/9 < a < 2$ and $b = b_L$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(h)) corresponds to the limit $\varphi_1 = \varphi_2 = \varphi_L$ so that $m = 1$, and then the solution is a hump-like solitary wave given by (2.13) with $\varphi_L < \varphi \le \varphi_3$ and

$$\varphi_3 = \frac{1}{6}\left(-2 + 3\sqrt{4-2a} - 2\sqrt{4 - 6\sqrt{4-2a}}\right),$$
$$\varphi_4 = \frac{1}{6}\left(-2 + 3\sqrt{4-2a} + 2\sqrt{4 - 6\sqrt{4-2a}}\right), \tag{3.12}$$

see Figure 3(d), for an example.

On the above, we have obtained expressions of parametric form for periodic and solitary wave solutions $\varphi(\xi)$ to (3.3). So in terms of $u = \varphi(\xi) + c$, we can get expressions for the periodic and solitary wave solutions $u(\xi)$ to (1.5).

4. Conclusion

In this paper, we have found expressions for new travelling wave solutions to the Fornberg-Whitham equation. These solutions depend, in effect, on two parameters a and m. For $m = 1$, there are inverted loop-like ($a < 0$), single peaked ($a = 16/9$), and hump-like ($16/9 < a < 2$)

solitary wave solutions. For $m = 1, 0 < a < 16/9$ or $0 < m < 1, a < 2$, and $a \neq 0$, there are periodic hump-like wave solutions.

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References

[1] R. Ivanov, "On the integrability of a class of nonlinear dispersive wave equations," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 4, pp. 462–468, 2005.

[2] H.-H. Dai, "Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod," *Acta Mechanica*, vol. 127, no. 1–4, pp. 193–207, 1998.

[3] A. Constantin and W. A. Strauss, "Stability of a class of solitary waves in compressible elastic rods," *Physics Letters A*, vol. 270, no. 3-4, pp. 140–148, 2000.

[4] H.-H. Dai and Y. Huo, "Solitary shock waves and other travelling waves in a general compressible hyperelastic rod," *Proceedings of the Royal Society A*, vol. 456, no. 1994, pp. 331–363, 2000.

[5] Z. Yin, "On the Cauchy problem for a nonlinearly dispersive wave equation," *Journal of Nonlinear Mathematical Physics*, vol. 10, no. 1, pp. 10–15, 2003.

[6] Y. Zhou, "Stability of solitary waves for a rod equation," *Chaos, Solitons & Fractals*, vol. 21, no. 4, pp. 977–981, 2004.

[7] Y. Zhou, "Blow-up of solutions to a nonlinear dispersive rod equation," *Calculus of Variations and Partial Differential Equations*, vol. 25, no. 1, pp. 63–77, 2006.

[8] R. S. Johnson, "Camassa-Holm, Korteweg-de Vries and related models for water waves," *Journal of Fluid Mechanics*, vol. 455, pp. 63–82, 2002.

[9] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.

[10] Y. A. Li, P. J. Olver, and P. Rosenau, "Non-analytic solutions of nonlinear wave models," in *Nonlinear Theory of Generalized Functions*, vol. 401 of *Chapman & Hall/CRC Research Notes in Mathematics*, pp. 129–145, Chapman & Hall/CRC, Boca Raton, Fla, USA, 1999.

[11] J. Lenells, "Traveling wave solutions of the Camassa-Holm equation," *Journal of Differential Equations*, vol. 217, no. 2, pp. 393–430, 2005.

[12] Y. A. Li and P. J. Olver, "Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation," *Journal of Differential Equations*, vol. 162, no. 1, pp. 27–63, 2000.

[13] A. Constantin and W. A. Strauss, "Stability of the Camassa-Holm solitons," *Journal of Nonlinear Science*, vol. 12, no. 4, pp. 415–422, 2002.

[14] A. Constantin and W. A. Strauss, "Stability of peakons," *Communications on Pure and Applied Mathematics*, vol. 53, no. 5, pp. 603–610, 2000.

[15] R. S. Johnson, "On solutions of the Camassa-Holm equation," *Proceedings of the Royal Society A*, vol. 459, no. 2035, pp. 1687–1708, 2003.

[16] A. Constantin, "Existence of permanent and breaking waves for a shallow water equation: a geometric approach," *Annales de l'Institut Fourier*, vol. 50, no. 2, pp. 321–362, 2000.

[17] A. Constantin and J. Escher, "Wave breaking for nonlinear nonlocal shallow water equations," *Acta Mathematica*, vol. 181, no. 2, pp. 229–243, 1998.

[18] A. Constantin and J. Escher, "On the blow-up rate and the blow-up set of breaking waves for a shallow water equation," *Mathematische Zeitschrift*, vol. 233, no. 1, pp. 75–91, 2000.

[19] H. Lundmark, "Formation and dynamics of shock waves in the Degasperis-Procesi equation," *Journal of Nonlinear Science*, vol. 17, no. 3, pp. 169–198, 2007.

[20] J. Escher, Y. Liu, and Z. Yin, "Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation," *Indiana University Mathematics Journal*, vol. 56, no. 1, pp. 87–118, 2007.

[21] G. M. Coclite, K. H. Karlsen, and N. H. Risebro, "Numerical schemes for computing discontinuous solutions of the Degasperis-Procesi equation," *IMA Journal of Numerical Analysis*, vol. 28, no. 1, pp. 80–105, 2008.

[22] G. M. Coclite and K. H. Karlsen, "On the uniqueness of discontinuous solutions to the Degasperis-Procesi equation," *Journal of Differential Equations*, vol. 234, no. 1, pp. 142–160, 2007.

[23] J. Lenells, "Traveling wave solutions of the Degasperis-Procesi equation," *Journal of Mathematical Analysis and Applications*, vol. 306, no. 1, pp. 72–82, 2005.

[24] Y. Matsuno, "Multisoliton solutions of the Degasperis-Procesi equation and their peakon limit," *Inverse Problems*, vol. 21, no. 5, pp. 1553–1570, 2005.

[25] H. Lundmark and J. Szmigielski, "Multi-peakon solutions of the Degasperis-Procesi equation," *Inverse Problem*, vol. 19, no. 6, pp. 1241–1245, 2003.

[26] Y. Matsuno, "Cusp and loop soliton solutions of short-wave models for the Camassa-Holm and Degasperis-Procesi equations," *Physics Letters A*, vol. 359, no. 5, pp. 451–457, 2006.

[27] G. B. Whitham, "Variational methods and applications to water wave," *Proceedings of the Royal Society A*, vol. 299, no. 1456, pp. 6–25, 1967.

[28] B. Fornberg and G. B. Whitham, "A numerical and theoretical study of certain nonlinear wave phenomena," *Philosophical Transactions of the Royal Society of London. Series A*, vol. 289, no. 1361, pp. 373–404, 1978.

[29] A. C. Bryan and A. E. G. Stuart, "Solitons and the regularized long wave equation: a nonexistence theorem," *Chaos, Solitons & Fractals*, vol. 7, no. 11, pp. 1881–1886, 1996.

[30] D. Bhardwaj and R. Shankar, "A computational method for regularized long wave equation," *Computers & Mathematics with Applications*, vol. 40, no. 12, pp. 1397–1404, 2000.

[31] S. I. Zaki, "Solitary waves of the splitted RLW equation," *Computer Physics Communications*, vol. 138, no. 1, pp. 80–91, 2001.

[32] J. I. Ramos, "Solitary waves of the EW and RLW equations," *Chaos, Solitons & Fractals*, vol. 34, no. 5, pp. 1498–1518, 2007.

[33] J. I. Ramos, "Solitary wave interactions of the GRLW equation," *Chaos, Solitons & Fractals*, vol. 33, no. 2, pp. 479–491, 2007.

[34] J. Nickel, "Elliptic solutions to a generalized BBM equation," *Physics Letters A*, vol. 364, no. 3-4, pp. 221–226, 2007.

[35] A. H. A. Ali, A. A. Soliman, and K. R. Raslan, "Soliton solution for nonlinear partial differential equations by cosine-function method," *Physics Letters A*, vol. 368, no. 3-4, pp. 299–304, 2007.

[36] D. Feng, J. Li, J. Lü, and T. He, "New explicit and exact solutions for a system of variant RLW equations," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 715–720, 2008.

[37] Z. Lin, "Instability of nonlinear dispersive solitary waves," *Journal of Functional Analysis*, vol. 255, no. 5, pp. 1191–1224, 2008.

[38] J. Zhou and L. Tian, "A type of bounded traveling wave solutions for the Fornberg-Whitham equation," *Journal of Mathematical Analysis and Applications*, vol. 346, no. 1, pp. 255–261, 2008.

[39] V. O. Vakhnenko and E. J. Parkes, "Periodic and solitary-wave solutions of the Degasperis-Procesi equation," *Chaos, Solitons & Fractals*, vol. 20, no. 5, pp. 1059–1073, 2004.

[40] E. J. Parkes, "The stability of solutions of Vakhnenko's equation," *Journal of Physics A*, vol. 26, no. 22, pp. 6469–6475, 1993.

[41] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Die Grundlehren der Mathematischen Wissenschaften, Band 67, Springer, New York, NY, USA, 1971.

[42] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, NY, USA, 1972.