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Research Article

Compact Operators Defined on 2-Normed and 2-Probabilistic Normed Spaces

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The compact operators defined on 2-normed spaces are investigated, and then the main ideas are generalized to operators defined on 2-probabilistic normed spaces.

1. Introduction

In 1963, Gähler [1] introduced the notion of a 2-metric, real-valued function of pointtriples on a set X, whose abstract properties were suggested by the area function for a triangle determined by a triple in Euclidean space. A related concept in the category of linear spaces, the theory of 2-norm on a linear space, was also investigated by Gähler in [2]. Since these were studied in many papers, we mention [3–5].

Also, due to vagueness about the distance between points in a metric space, probabilistic metric spaces were introduced by Menger [6] as a generalization of metric spaces. From the vantage point of a sixty-year history, it is safe to say that the probabilistic approach on deterministic results of linear normed spaces is playing an important role in applied mathematics.

In this paper, we first investigate compact operators between 2-normed spaces. Then, according to Menger's probabilistic approach, we discuss on 2-probabilistic normed spaces and extend the main ideas given in first section to operators defined between 2-probabilistic normed spaces.

2. 2-Normed Spaces

In this section, after providing the required preliminaries, we discuss on compact operators between 2-normed spaces.

In the sequel of this paper, it is always assumed that all vector spaces are real with the dimension greater than one.

Definition 2.1 ([7]). Let X be a real linear space. A function $\|\cdot,\cdot\| : X^2 \to \mathbb{R}$ is called a 2-*norm* on X if it satisfies the following conditions, for every $\alpha \in \mathbb{R}$ and $x, y, z \in X$:

(a) $\|x, y\| = 0 \Leftrightarrow x$ and y are linearly dependent,

(b) $\|x, y\| = \|y, x\|$,

(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$,

(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

Then the pair $(X, \|\cdot,\cdot\|)$ is said to be a *linear* 2-*normed space*.

A most standard example of a 2-normed space is $X = \mathbb{R}^2$ equipped with the following 2-norm (the absolute value of the determinant):

$$\|x_1, x_2\|_E = \text{abs}\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \tag{2.1}$$

where $x_i = (x_{i1}, x_{i2})$ for $i = 1, 2$.

Definition 2.2. Let X and Y be two 2-normed spaces, and let $T : X \to Y$ be a linear operator. For any $e \in X$, we say that the operator T is *e-bounded* if there exists $M_e > 0$ such that $\|T(x), T(e)\| \leq M_e\|x, e\|$ for all $x \in X$. An e-bounded operator T, for every e, will be called *bounded*.

For example, the operator $T(x) = cx$, where $c \in \mathbb{R}$ defined on any 2-normed space X is a bounded operator. More examples are the followings.

Example 2.3. The operator $T : (\mathbb{R}^2, \|\cdot,\cdot\|_E) \to (\mathbb{R}^2, \|\cdot,\cdot\|_E)$ defined by

$$T(x_1, x_2) = (x_1, -x_1 - x_2) \tag{2.2}$$

is a bounded linear operator. In fact, for each $e, x \in X$, we have

$$\|T(x), T(e)\|_E = \text{abs}\begin{vmatrix} x_1 & -x_1 & -x_2 \\ e_1 & -e_1 & -e_2 \end{vmatrix} = \text{abs}\begin{vmatrix} x_1 & x_2 \\ e_1 & e_2 \end{vmatrix} = \|x, e\|_E. \tag{2.3}$$

Example 2.4. Consider the real vector space P of all real polynomials on the interval $[0, 1]$. Define

$$\|f, g\| = \sup_{0 \leq t \leq 1}|fg' - f'g| + \sqrt{\int_0^1 |fg' - f'g|^2 dt}, \tag{2.4}$$

for all $f, g \in P$, where the prime denotes differentiation with respect to t. The operator $T : P \to P$ defined by

$$Tf(t) = tf(t) \tag{2.5}$$

is a bounded operator. Indeed,

$$\begin{aligned}\|Tf, Tg\| &= \sup_{0\le t\le 1}\left|(tf)(tg)' - (tf)'(tg)\right| + \sqrt{\int_0^1 \left|(tf)(tg)' - (tf)'(tg)\right|^2 dt} \\ &= \sup_{0\le t\le 1} t^2\left|fg' - f'g\right| + \sqrt{\int_0^1 t^2\left|fg' - f'g\right|^2 dt} \\ &\le \sup_{0\le t\le 1}\left|fg' - f'g\right| + \sqrt{\int_0^1 \left|fg' - f'g\right|^2 dt} \\ &= \|f, g\|,\end{aligned} \tag{2.6}$$

for all $f, g \in P$.

Example 2.5. Let $(X, \|\cdot\|)$ be a normed space. Whereas any normed space may be realized as a function space on the closed unit ball of the dual space X^*, one can define a 2-norm on X by

$$\|x, y\| = \sup\{\left|f(x)g(y) - g(x)f(y)\right| : f, g \in \text{Ball }(X^*)\}, \quad (x, y \in X). \tag{2.7}$$

Now suppose that T is a bounded linear operator on $(X, \|\cdot\|)$ in the usual sense. It can be easily seen that T is bounded on $(X, \|\cdot,\cdot\|)$.

We are interested in calling the 2-norm given in Example 2.5 the 2-norm induced by (ordinary) norm.

Definition 2.6. A sequence $\{x_n\}$ of X is said to be *convergent* if there exists an element $a \in X$ such that $\lim_{n\to\infty}\|x_n - a, x\| = 0$, for all $x \in X$.

Evidently the limit of any convergent sequence is unique.

Definition 2.7. Let X and Y be two 2-normed spaces, and let $T : X \to Y$ be a linear operator. The operator T is said to be *sequentially continuous* at $x \in X$ if for any sequence $\{x_n\}$ of X converging to x we have $T(x_n) \to T(x)$.

Definition 2.8. The *closure* of a subset E of a 2-normed space X is denoted by $\overline{E}$ and defined by the set of all $x \in X$ such that there is a sequence $\{x_n\}$ of E converging to x. We say that E is *closed* if $E = \overline{E}$.

For a 2-normed space X, consider the subsets,

$$\begin{aligned}B_e(a, r) &= \{x : \|x - a, e\| < r\}, \\ B_e[a, r] &= \{x : \|x - a, e\| \le r\},\end{aligned} \tag{2.8}$$

of X. It is clear that $B_e[a, r]$ is closed.

Definition 2.9. A subset A of a 2-normed space X is said to be *locally bounded* if there exist $e \in X \setminus \{0\}$, and $r > 0$ such that $A \subseteq B_e(0, r)$.

Example 2.10. Every bounded set in $\mathbb{R}^2$ is a locally bounded set in $(\mathbb{R}^2, \|\cdot,\cdot\|_E)$. In fact, assume that A is a bounded set in $\mathbb{R}^2$. There exists an $M > 0$ such that for every $(x, y) \in A$, $\|(x, y)\| < M$. Putting $e = (1, 0)$, we obtain $A \subseteq B_e(0, M)$.

Example 2.11. For a normed space $(X, \|\cdot\|)$ consider the 2-norm induced by its norm as given in Example 2.5. Suppose that A is a bounded set in $(X, \|\cdot\|)$ and $e \in X \setminus \{0\}$. It can be easily seen that A lies in $B_e(0, r)$, for some $r > 0$.

Definition 2.12. A subset B of a 2-normed space X is said to be *compact* if every sequence $\{x_n\}$ of B has a convergent subsequence in B.

It is clear that every compact set of a normed space X is also compact in its 2-norm induced by norm.

Lemma 2.13. *Every compact subset M of a 2-normed space is closed and locally bounded.*

Proof. The proof of closedness is trivial. If M were not locally bounded, it would contain a sequence $\{y_n\}$ such that $\|y_n, e\| > n$, for any nonzero fixed element e. Now this sequence could not have a convergent subsequence because if $\{y_{n_k}\}$ were a convergent subsequence to y_0, then $\|y_{n_k} - y_0, e\| \to 0$. And for ϵ there would exist a positive integer N such that $\|y_{n_k}, e\| - \|y_0, e\| \le \|y_{n_k} - y_0, e\| < \epsilon$, for each $k > N$ which is a contradiction. □

The following example shows that the converse of Lemma 2.13 is false in general.

Example 2.14. The subset $B_{(1,0)}[0, 1]$ of $(\mathbb{R}^2, \|\cdot,\cdot\|_E)$ is not a compact set. Because the sequence $\{(n, 0)\}$ of $B_{(1,0)}[0, 1]$ has no convergent subsequence. Suppose on the contrary that $(n_k, 0) \to (a, b)$. Hence, for $e = (0, 1)$, we have $\|(n_k, 0) - (a, b), (0, 1)\|_E \to 0$. That is, $|n_k - a| \to 0$ which is impossible.

Lemma 2.15. *Let X be a 2-normed space. Then X is of finite dimension if $B_e[a, r]$ is a compact set in X, for some $a, e \in X$, and $r > 0$.*

Proof. Suppose that $B_e[a, r]$ is compact. Consider the normed space $X/\langle e\rangle$ equipped with the norm:

$$\|x + \langle e\rangle\| = \frac{\|x, e\|}{\|e, e'\|}, \tag{2.9}$$

where $\{e, e'\}$ is a linearly independent set. The subset,

$$A = \left\{x + \langle e\rangle : \|x - a + \langle e\rangle\| \le \frac{r}{\|e, e'\|}\right\}, \tag{2.10}$$

is a closed ball in the usual sense of the normed space $X/\langle e\rangle$. We aim to show that A is a compact set in the normed space $X/\langle e\rangle$. Choose the sequence $\{x_n + \langle e\rangle\}$ of A. Since, for every n,

$$\|x_n + \langle e\rangle - (a + \langle e\rangle)\| = \frac{\|x_n - a, e\|}{\|e, e'\|} \le \frac{r}{\|e, e'\|}, \tag{2.11}$$

and therefore $\{x_n\}$ is a sequence in $B_e[a, r]$. Hence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ to a point x_0. We have

$$\lim_{n\to\infty} \|x_{n_k} + \langle e\rangle - (x_0 + \langle e\rangle)\| = \lim_{n\to\infty} \frac{\|x_{n_k} - x_0, e\|}{\|e, e'\|} = 0. \tag{2.12}$$

Hence $\{x_{n_k} + \langle e\rangle\}_{k=1}^{\infty}$ is a convergent subsequence of $\{x_n + \langle e\rangle\}$. This implies that A is compact. Therefore X is of finite dimension. □

In the rest of this section, the space $X/\langle e\rangle$ will denote the normed space given in the proof of Lemma 2.15.

It is well known that if $\{x_1, \ldots, x_n\}$ is a linear independent set of vectors in a normed space X (of any dimension), then there is a number $c > 0$ such that for all scalars $\alpha_1, \ldots, \alpha_n$ we have

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|). \tag{2.13}$$

The next lemma gives a similar assertion in 2-normed spaces.

Lemma 2.16. *Let $\{x_1, \ldots, x_n, e\}$ be a linearly independent set of vectors in a 2-normed space X (of any dimension). Then, there is a positive number c such that for any choice of scalars $\alpha_1, \ldots, \alpha_n$ we have*

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n, e\| \geq c(|\alpha_1| + \cdots + |\alpha_n|). \tag{2.14}$$

Proof. Consider the normed space $X/\langle e\rangle$ and put $\lambda = \|e, e'\| > 0$. Since $\{x_1, \ldots, x_n, e\}$ is linearly independent in X, so is $\{\lambda x_1 + \langle e\rangle, \ldots, \lambda x_n + \langle e\rangle\}$ in $X/\langle e\rangle$. Thus, there exists $c > 0$ such that for every choice of scalar $\alpha_1, \ldots, \alpha_n$ we have

$$\|\alpha_1(\lambda x_1 + \langle e\rangle) + \cdots + \alpha_n(\lambda x_n + \langle e\rangle)\| \geq c(|\alpha_1| + \cdots + |\alpha_n|). \tag{2.15}$$

Therefore $\|\alpha_1 x_1 + \ldots + \alpha_n x_n, e\| \geq c(|\alpha_1| + \ldots + |\alpha_n|)$. This completes the proof. □

Definition 2.17. Let X and Y be two 2-normed spaces. A linear operator $T : X \to Y$ is called a *compact operator* if it maps every locally bounded sequence $\{x_n\}$ in X onto a sequence $\{T(x_n)\}$ in Y which has a convergent subsequence.

Lemma 2.18. *Let X and Y be two 2-normed spaces, and let $T : X \to Y$ be a compact operator. Then for every $e \in X$, T induces the ordinary compact operator $T' : X/\langle e\rangle \to Y/\langle T(e)\rangle$ defined by $T'(x + \langle e\rangle) = T(x) + \langle T(e)\rangle$, for all $x \in X$.*

Proof. Suppose $e \in X$, and $\{x_n + \langle e\rangle\}$ is a bounded sequence in the normed space $X/\langle e\rangle$. There exists $M > 0$ such that every $\|x_n + \langle e\rangle\| < M$ and so $\|x_n, e\| < M\|e, e'\|$, for all n. Since T is compact, the sequence $\{T(x_n)\}$ has a convergent subsequence $\{T(x_{n_k})\}$ to a point y_0. Thus, $\lim_{n\to\infty}\|T(x_{n_k}) - y_0, y\| = 0$, for all $y \in Y$or $\lim_{n\to\infty}\|T(x_{n_k}) - y_0 + \langle T(e)\rangle\| = 0$. This shows that T' is a compact operator. □

Lemma 2.19. *Let X and Y be two 2-normed spaces. If $T : X \to Y$ is a surjective bounded linear operator, then it is sequentially continuous.*

Proof. If $x_n \to a$, then $\|x_n - a, e\| \to 0$, for each $e \in X$. Since T is bounded for every $e \in X$, there exists M_e such that $\|T(x_n) - T(a), T(e)\| \leq M_e \|x_n - a, e\|$ for all n. Thus $T(x_n) \to T(a)$. □

Corollary 2.20. *Let X and Y be two 2-normed spaces. Then*

(a) *every compact operator $T : X \to Y$ is bounded;*

(b) *if $\dim X = \infty$, then the identity operator $I : X \to X$ is not a compact operator.*

Proof. (a) Choose $e \in X$. Let T' be the compact operator induced by T(as explained in Lemma 2.18). Since T' is a compact operator, there exists $M_e > 0$ such that

$$\|T(x) + \langle T(e) \rangle\| \leq M_e \|x + \langle e \rangle\|, \tag{2.16}$$

for all $x \in X$. That is, for all $x \in X$

$$\frac{\|T(x), T(e)\|}{\|T(e), y_0\|} \leq M_e \frac{\|x, e\|}{\|e, e'\|}, \tag{2.17}$$

where $\{y_0, T(e)\}$ and $\{e, e'\}$ are linearly independent sets. This implies that T is bounded.

(b) Choose $e \in X$. The identity operator I maps $B_e[0,1]$ to itself. Suppose on the contrary that I is a compact operator. Let $\{x_n\}$ be a sequence of $B_e[0,1]$. Because $\{x_n\}$ is a locally bounded sequence, it has a convergent subsequence. Hence $B_e[0,1]$ is compact and therefore X is of finite dimension by Lemma 2.15, which is a contradiction. □

Remark 2.21. Suppose X and $Y$are two 2-normed spaces, T_1 and T_2 are compact operators from X into Y, and $c \in \mathbb{R}$. Then $cT_1 + T_2$ is a compact operator. To see this, let $\{x_n\}$ be any locally bounded sequence in X. The sequence $\{T_1(x_n)\}$ has a convergent subsequence $\{T_1(x_{n_k})\}$. The sequence $\{T_2(x_{n_k})\}$ has a convergent subsequence $T_2(z_n)$. Let $T_1(z_n) \to u$, and let $T_2(z_n) \to v$. If $y \in Y$, $c \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \left\|(cT_1 + T_2)(z_n) - cu - v, y\right\| \leq \lim_{n\to\infty} |c| \left\|T_1(z_n) - u, y\right\| + \lim_{n\to\infty} \left\|T_2(z_n) - v, y\right\|. \tag{2.18}$$

Thus $\lim_{n\to\infty} \|cT_1 + T_2(z_n) - cu - v, y\| = 0$, for all $y \in Y$. This implies that $cT_1 + T_2$ is a compact operator.

Theorem 2.22. *Let X be a 2-normed space, let $T : X \to X$ be a compact operator, and let $S : X \to X$ be a bijective bounded operator. Then ST and TS are compact operators.*

Proof. Let $\{x_n\}$ be any locally bounded sequence in X. Then $\{T(x_n)\}$ has a convergent subsequence $\{T(x_{n_k})\}$. Put $\lim_{n\to\infty} T(x_{n_k}) = y_0$. Since S is bijective and bounded, by Lemma 2.19, we have $S(T(x_{n_k})) \to S(y_0)$. Hence $S(T(x_n))$ has a convergent subsequence. This proves ST is compact. Now, to show that TS is compact, for any locally bounded sequence $\{x_n\}$, there exist $e \in X$ and $M > 0$ such that $x_n \in B_e(0, M)$ for all n, that is,

$\|x_n, e\| < M$, for all $n \geq 1$. Since S is bounded, the sequence $\{S(x_n)\}$ is a locally bounded sequence in X. Because T is compact, $\{T(S(x_n))\}$ has a convergent subsequence. This completes the proof. □

Theorem 2.23. *Let X and Y be two 2-normed spaces. If $T : X \to Y$ is a linear operator where $\dim X < \infty$, then T is bounded.*

Proof. Choose $e \in X$. Since $\dim X < \infty$, so $\dim X/\langle e\rangle < \infty$. Therefore the operator $T' : X/\langle e\rangle \to Y/\langle T(e)\rangle$ definedby $T'(x + \langle e\rangle) = T(x) + \langle T_e\rangle$, for all $x \in X$, is a bounded operator. Thus, for every $e \in X$ there exists $M_e > 0$ such that $\|T(x) + \langle T(e)\rangle\| \leq M_e\|x + \langle e\rangle\|$, for all $x \in X$. Therefore

$$\frac{\|T(x), T(e)\|}{\|T(e), y_0\|} \leq M_e \frac{\|x, e\|}{\|e, e'\|}, \tag{2.19}$$

where $\{y_0, T(e)\}$ and $\{e, e'\}$ are linearly independent subsets. Thus T is bounded. □

Theorem 2.24. *Let $T : X \to X$ be a compact operator on a 2-normed space X. Then for every $\lambda \neq 0$, the null space $N(T_\lambda)$ of $T_\lambda = T - \lambda I$ is of finite dimension.*

Proof. Consider the subset $M = B_e[a, r]$ of $N(T_\lambda)$. We show that M is compact, then apply Lemma 2.15. If $\{x_n\}$ is a sequence in M, then $\{x_n\}$ is locally bounded and $\{T(x_n)\}$ has a convergent subsequence $\{T(x_{n_k})\}$. Now $x_n \in M \subset N(T_\lambda)$ implies $T_\lambda(x_n) = T(x_n) - \lambda x_n = 0$, so that $x_n = \lambda^{-1}T(x_n)$ because $\lambda \neq 0$. Consequently, $\{\lambda^{-1}T(x_{n_k})\}$ will be a convergent subsequence of $\{x_n\}$ in M. Hence M is compact, because $\{x_n\}$ was arbitrary in M. This shows that $\dim N(T_\lambda) < \infty$. □

Definition 2.25. A sequence $\{x_n\}$ of 2-normed space X is called a *Cauchy sequence* if $\lim_{m,n\to\infty}\|x_n - x_m, x\| = 0$, for all $x \in X$.

We will say that the 2-normed space X is a 2-*Banach space* if every Cauchy sequence in X is a convergent sequence in X.

Theorem 2.26. *Let X, Y, and Z be 2-normed spaces, let $T : Z \subset X \to Y$ be a surjective bounded operator, and let Y is a 2-Banach space. Then T has an extension $\overline{T} : \overline{Z} \to Y$, where $\overline{T}$ is an e-bounded operator for each $e \in Z$.*

Proof. We consider any $x \in \overline{Z}$. There is a sequence $\{x_n\}$ in Z such that $x_n \to x$. Since T is linear and bounded for every $e \in Z$, there exists $M_e > 0$ such that

$$\|T(x_n) - T(x_m), T(e)\| \leq M_e\|x_n - x_m, e\|, \tag{2.20}$$

for all n, m. This shows that $\{T(x_n)\}$ is Cauchy in Y, because $\{x_n\}$ is convergent. By assumption, Y is a 2-Banach space, so that $\{T(x_n)\}$ converges in Y say $T(x_n) \to y$. We define $\overline{T}$ by $\overline{T}(x) = y$. This definition is independent of the particular choice of a sequence in Z converging to x. Because suppose that $x_n \to x$ and $z_n \to x$. Then $v_m \to x$, where $\{v_m\}$ is the sequence $\{x_1, z_1, x_2, z_2, \ldots\}$. Hence $\{T(v_m)\}$ is convergent and the two subsequences $\{T(x_n)\}$ and $\{T(z_n)\}$ of $\{T(v_m)\}$ must have the same limit. This proves that $\overline{T}$ is uniquely defined at every $x \in \overline{Z}$. Clearly $\overline{T}$ is linear and $\overline{T}(x) = T(x)$ for every $x \in Z$, so that $\overline{T}$ is an extension of T. On the other hand,

$$\|T(x), T(e)\| \leq M_e \|x, e\| \tag{2.21}$$

for all x. Thus,

$$\left\|\overline{T}(x), T(e)\right\| \leq \left\|\overline{T}(x) - T(x_n), T(e)\right\| + \|T(x_n), T(e)\|. \tag{2.22}$$

When $n \to \infty$, $\|\overline{T}(x), T(e)\| \leq M_e\|x, e\|$. Therefore $\overline{T}$ is an e-bounded linear operator for each $e \in Z$. □

3. 2-Probabilistic Normed Spaces

In this section, we aim to consider compact operators between 2-probabilistic normed spaces. We need some preliminaries which are given first.

Definition 3.1. A function $f : \mathbb{R} \to [0, \infty)$ is called a *distribution function* if it is nondecreasing and right-continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$, and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We will denote the set of all distribution functions by $\mathfrak{D}$.

Definition 3.2. A pair (X, N) is called a 2-*probabilistic normed space* (briefly, a 2*PN-space*) if X is a real vector space with $\dim X > 1$, N is a mapping from $X \times X$ into $\mathfrak{D}$ (for $x \in X$, the distribution function $N(x, y)$ is denoted by $N_{x,y}$, and $N_{x,y}(t)$ is the value $N_{x,y}$ at $t \in \mathbb{R}$) satisfyingthe following conditions:

(2PN-I) $N_{x,y}(0) = 0$, for all $x, y \in X$,

(2PN-II) $N_{x,y}(t) = 1$ for all $t > 0$ if and only if x and y are linearly dependent,

(2PN-III) $N_{x,y}(t) = N_{y,x}(t)$, for all $x, y \in X$,

(2PN-IV) $N_{\alpha x,y}(t) = N_{x,y}(t/|\alpha|)$, for all $\alpha \in \mathbb{R} \setminus \{0\}$, and for all $x, y \in X$,

(2PN-V) $N_{x+y,z}(s + t) \geq \min\{N_{x,z}(s), N_{y,z}(t)\}$ for all $x, y, z \in X$, and $s, t \in \mathbb{R}$.

We call the mapping $(x, y) \to N_{x,y}$ a 2-*probabilistic norm* (2*P-norm*) on X.

Example 3.3. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. Every 2-norm induces a 2P-norm on X as follows:

$$N_{x,y}(t) = \begin{cases} \dfrac{t}{t + \|x, y\|}, & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{3.1}$$

This 2-probabilistic norm is called the *standard* 2*P-norm*.

Theorem 3.4 ([8]). *Let* (X, N) *be a* 2*PN-space. Assume that the condition* $(2PN\text{-}VI)$ $N_{x,y}(t) > 0$, *for all* $t \in (0, \infty)$ *implies that* $\{x, y\}$ *is linearly dependent. For* $\alpha \in (0, 1)$, *define*

$$\|x, y\|_\alpha = \inf\{t : N_{x,y}(t) \geq \alpha\}. \tag{3.2}$$

Then $\{\|\cdot,\cdot\|_\alpha : \alpha \in (0,1)\}$ *is an ascending family of* 2*-norms on* X*. These* 2*-norms are called* α*-*2*-norms on* X *corresponding to (or induced by) the* 2*-probabilistic norm* N *on* X*.*

The following example gives us a $2PN$-space satisfying condition ($2PN$-VI).

Example 3.5. Suppose that $(X, \|\cdot,\cdot\|)$ is a 2-normed space. Define

$$N_{x,y}(t) = \begin{cases} 0, & t \le \|x,y\|, \\ 1, & t > \|x,y\|, \end{cases} \tag{3.3}$$

where $x, y \in X$, and $t \in \mathbb{R}$. Then the $2PN$-space (X, N) satisfies ($2PN$-VI).

Definition 3.6. Let (X, N) be a $2PN$-space, and let $\{x_n\}$ be a sequence of X. Then the sequence $\{x_n\}$ is said to be *convergent* to $x_0 \in X$ and denoted by $x_n \to x_0$ if $\lim_{n\to\infty} N_{x_n - x_0, x}(t) = 1$ for all $x \in X$ and $t > 0$.

Definition 3.7. Let $T : (X, N) \to (Y, N')$ be a linear operator, where (X, N) and (Y, N') are $2PN$-spaces. For an element $e \in X$,

(1) the operator T is called e*-*2*-probabilistic continuous* (e-2-PC) at $z \in X$ if for any $\epsilon > 0$, and $\alpha \in (0,1)$ there exists $\delta > 0$ such that

$$N_{x-z,e}(\delta) \ge \alpha \Longrightarrow N'_{T(x)-T(z),T(e)}(\epsilon) \ge \alpha, \tag{3.4}$$

for all $x \in X$.

If T is e-2-PC at each point of X, then T is said to be e-2-PC on X. If T is e-2-PC on X for each $e \in X$, then T is said to be 2*-probabilistic continuous* (2-PC) on X.

(2) the linear operator T is called e*-*2*-probabilistic bounded* (e-2-PB) for $e \in X$ on X if for every $\alpha \in (0,1)$ there exists $m_{e,\alpha} > 0$ such that

$$N_{x,e}\left(\frac{t}{m_{e,\alpha}}\right) \ge \alpha \Longrightarrow N'_{T(x),T(e)}(t) \ge \alpha, \tag{3.5}$$

for all $x \in X$ and $t \in \mathbb{R}$.

If T is e-2-PB on X for each $e \in X$, then T is said to be 2*-probabilistic bounded* (2-PB) on X.

Example 3.8. Suppose that X is a $2PN$-space and that $T : X \to X$ is a linear operator defined by $T(x) = cx$, $c \in \mathbb{R}$. Then T is a 2-PC operator. Because, for any $e \in X$, $\epsilon > 0$, and $\alpha \in (0,1)$ it suffices to choose $\delta = \epsilon/c^2$. Now, for $x \in X$ if $N_{x-z,e}(\delta) \ge \alpha$, then $N'_{T(x)-T(z),T(e)}(\epsilon) \ge \alpha$.

Theorem 3.9. *Let* (X, N) *and* (Y, N') *be two* $2PN$*-spaces, and let* $T : (X, N) \to (Y, N')$ *be a linear operator.*

(a) *If* T *is* e*-*2*-*PC *for* $e \in X$ *at* $x_0 \in X$*, then* T *is* e*-*2*-*PC *on* X*.*

(b) T *is* 2*-*PC *on* X *if and only if* T *is* 2*-*PB *on* X*.*

Proof. (a) Since T is e-2-PC at x_0, for each $\epsilon > 0$, and $\alpha \in (0,1)$, there exists $\delta > 0$ such that

$$N_{x-x_0,e}(\delta) \geq \alpha \Longrightarrow N'_{T(x)-T(x_0),T(e)}(\epsilon) \geq \alpha, \tag{3.6}$$

for all $x \in X$. Taking $y \in X$, and $x \in X$ such that $N_{x-y,e}(\delta) \geq \alpha$ we get

$$N'_{T\left(x+x_0-y\right)-T(x_0),T(e)}(\epsilon) \geq \alpha, \tag{3.7}$$

or

$$N'_{T(x)-T\left(y\right),T(e)}(\epsilon) \geq \alpha. \tag{3.8}$$

Since y is arbitrary, it follows that T is e-2-PC on X.

(b) First we suppose that T is 2-PB on X. Choose $e \in X$, $\alpha \in (0,1)$, and $\epsilon > 0$ arbitrarily. There exists $m_{e,\alpha} > 0$ such that

$$N_{x,e}\left(\frac{\epsilon}{m_{e,\alpha}}\right) \geq \alpha \Longrightarrow N'_{T(x),T(e)}(\epsilon) \geq \alpha, \tag{3.9}$$

for all $x \in X$. This shows that T is e-2-PC at zero and by part (a) it is e-2-PC on X.

Conversely, suppose that T is 2-PC at 0. Using e-2P continuity of T at 0 and taking $\epsilon = 1$, and $\alpha \in (0,1)$, there exists $\delta > 0$ such that

$$N_{x-0,e}(\delta) \geq \alpha \Longrightarrow N'_{T(x)-T(0),T(e)}(1) \geq \alpha, \tag{3.10}$$

or

$$N_{x,e}(\delta) \geq \alpha \Longrightarrow N'_{T(x),T(e)}(1) \geq \alpha, \tag{3.11}$$

for all $x \in X$. Choose $m_{e,\alpha} = 1/\delta$. Then

$$N_{x,e}\left(\frac{t}{m_{e,\alpha}}\right) = N_{x/t,e}(\delta) \geq \alpha \Longrightarrow N'_{T(x/t),T(e)}(1) = N'_{T(x),T(e)}(t) \geq \alpha, \tag{3.12}$$

for all $x \neq 0$ and $t > 0$. This implies that T is e-2-PB on X. Because e was arbitrary, T is 2-PB. □

Theorem 3.10. *Let (X, N) and (Y, N') be two 2PN-spaces satisfying (2PN-VI). If the linear operator $T : (X, \|\cdot,\cdot\|_\alpha) \to (Y, \|\cdot,\cdot\|'_\alpha)$ is bounded with respect to α-2-norms corresponding to N and N' for each $\alpha \in (0,1)$, then $T : (X, N) \to (Y, N')$ is 2-PB.*

Proof. Fix $e \in X$. For any $\alpha \in (0,1)$, there exists $m_{e,\alpha}$ such that for all $x \in X$,

$$\|T(x), T(e)\|_\alpha \leq m_{e,\alpha}\|x, e\|_\alpha. \tag{3.13}$$

Then for $x \neq 0$, and $t > 0$,

$$\|m_{e,\alpha}x, e\|_\alpha \leq t \Longrightarrow \|T(x), T(e)\|_\alpha \leq t. \tag{3.14}$$

On the other hand,

$$\inf\{s : N_{m_{e,\alpha}x,e}(s) \geq \alpha\} \leq t \Longrightarrow \inf\left\{s : N'_{T(x),T(e)}(s) \geq \alpha\right\} \leq t. \tag{3.15}$$

It is clear that

$$\begin{gathered}\inf\{s : N_{m_{e,\alpha}x,e}(s) \geq \alpha\} \leq t \Longleftrightarrow N_{m_{e,\alpha}x,e}(t) \geq \alpha, \\ \inf\left\{s : N'_{T(x),T(e)}(s) \geq \alpha\right\} \leq t \Longleftrightarrow N'_{T(x),T(e)}(t) \geq \alpha.\end{gathered} \tag{3.16}$$

Thus, for any $\alpha \in (0,1)$, there exists $m_{e,\alpha} > 0$ such that for all $t \in \mathbb{R}$, $x \in X$,

$$N_{x,e}\left(\frac{t}{m_{e,\alpha}}\right) \geq \alpha \Longrightarrow N'_{T(x),T(e)}(t) \geq \alpha, \tag{3.17}$$

that is, T is 2-PB. □

Theorem 3.11. *Let $T : (X, N) \to (Y, N')$ be a linear surjective operator, where (X, N) and (Y, N') are $2PN$-spaces. If T is 2-PC on X, then T is sequentially continuous, that is, for any sequence $\{x_n\}$ converging to x, $T(x_n) \to T(x)$.*

Proof. If $x_n \to x$, then $\lim_{n\to\infty} N_{x_n-x,e}(t) = 1$, for each $e \in X$, and $t > 0$. Since T is 2-PC, it is 2-PB by Theorem 3.9. Thus, for each $\alpha \in (0,1)$ there exists $m_{e,\alpha}$ such that if $N_{x_n-x,e}(t/m_{e,\alpha}) \geq \alpha$, then $N'_{T(x_n)-T(x),T(e)}(t) \geq \alpha$, for all $n \in \mathbb{N}$, and $t \in \mathbb{R}$. Hence, $T(x_n) \to T(x)$. □

Definition 3.12. A subset B in a $2PN$-space (X, N) is called *compact* if each sequence of B has a convergent subsequence in B.

Definition 3.13. Let (X, N) be a $2PN$-space. For $e, x \in X$, $\alpha \in (0,1)$, and $r > 0$ we define the *locally ball* $B_{e,\alpha}[x, r]$ by $\{y \in X : N_{x-y,e}(r) \geq \alpha\}$.

It is clear that every locally ball is a closed set.

Definition 3.14. A subset B of a $2PN$-space (X, N) is said to be 2-*probabilistic locally bounded* (2-PLB), if there are $t > 0$, $e \in X \setminus \{0\}$, and $0 < r < 1$ such that $N_{x,e}(t) > 1 - r$, for all $x \in B$.

Example 3.15. The subset $C = \{(x, y) : y = \arcsin x\}$ is a 2-PLB set in $(\mathbb{R}^2, N)$, where N is the standard 2-probabilistic norm. In fact, $C \subseteq B_{(1,0),1/2}((0,0), 1)$. Since, if $(x, y) \in C$, then $\|(x, y), (1,0)\|_E = |y| < 1$. That is, $(x, y) \in B_{(1,0)}((0,0), 1)$.

Definition 3.16. The *closure* of a subset E of a $2PN$-space (X, N) is denoted by $\overline{E}$ and defined by the set of all $x \in X$ such that there is a sequence $\{x_n\}$ of E converging to x. We say that E is closed if $E = \overline{E}$.

Definition 3.17. Let (X, N) and (Y, N') be $2PN$-spaces. A linear operator $T : (X, N) \to (Y, N')$ is called a *compact operator* if it maps every 2-PLB sequence $\{x_n\}$ in X onto a sequence $\{T(x_n)\}$ in Y which has a convergent subsequence.

Example 3.18. Let $\|\cdot,\cdot\|_1$ and $\|\cdot,\cdot\|_2$ be two 2-norms, and let $T : (X, \|\cdot,\cdot\|_1) \to (Y, \|\cdot,\cdot\|_2)$ be a compact operator. Then $T : (X, N_1) \to (Y, N_2)$ is a compact operator, where N_1 and N_2 are 2-probabilistic norms defined by

$$N_{i_{x,y}}(t) = \begin{cases} \dfrac{t}{t + \|x, y\|_i}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{3.18}$$

for $i = 1, 2$. To see this, let $\{x_n\}$ be a 2-PLB-sequence in (X, N_1). There exist $t_0 > 0$, $e \in X$, and $\alpha \in (0, 1)$ such that for all $n \in \mathbb{N}$,

$$N_{1_{x_n,e}}(t_0) > \alpha. \tag{3.19}$$

Therefore $t_0/(t_0 + \|x_n, e\|_1) > \alpha$, and this implies that $\{x_n\}$ is a locally bounded sequence in $(X, \|\cdot,\cdot\|_1)$. Now, the compactness of $T : (X, \|\cdot,\cdot\|_1) \to (Y, \|\cdot,\cdot\|_2)$ implies that the sequence $\{T(x_n)\}$ has a convergent subsequence $\{T(x_{n_k})\}$; that is, there exists a $b \in Y$ such that

$$\lim_{n_k \to \infty} \|T(x_{n_k}) - b, v\|_2 = 0, \tag{3.20}$$

for all $v \in Y$. Hence,

$$\lim_{n_k \to \infty} N_{2_{T(x_{n_k})-b,v}}(t) = 1, \tag{3.21}$$

for all $v \in Y$ and $t > 0$. Thus $T : (X, N_1) \to (Y, N_2)$ is a compact operator.

Lemma 3.19. *Let (X, N) be a $2PN$-space satisfying $(2PN\text{-}VI)$, and let $\{x_n\}$ be a sequence in X. Then $x_n \to x_0$ in (X, N) if and only if $x_n \to x_0$ in $(X, \|\cdot,\cdot\|_\alpha)$ for each $\alpha \in (0, 1)$.*

Proof. Suppose that $x_n \to x_0$ in (X, N). Choose $\alpha \in (0, 1)$, $x \in X$, and $t > 0$. There exists $k \in \mathbb{N}$ such that $N_{x_n - x_0, x}(t) > 1 - \alpha$, for all $n \geq k$. It follows that $\|x_n - x_0, x\|_{1-\alpha} \leq t$, for all $n \geq k$. Thus $\|x_n - x_0, x\|_{1-\alpha} \to 0$. Conversely, choose $x \in X$. Let $\|x_n - x_0, x\|_\alpha \to 0$, for every $\alpha \in (0, 1)$. Fix $\alpha \in (0, 1)$, and $t > 0$. There exists $k \in \mathbb{N}$ such that

$$\wedge\{r > 0 : N_{x_n - x_0, x}(r) \geq 1 - \alpha\} < t, \tag{3.22}$$

for all $n \geq k$. Hence, for every $n \geq k$ there is $0 < t_n < t$ such that

$$N_{x_n - x_0, x}(t_n) \geq 1 - \alpha. \tag{3.23}$$

It implies that

$$N_{x_n - x_0, x}(t) \geq 1 - \alpha, \tag{3.24}$$

for all $n \geq k$, that is, $x_n \to x_0$ in (X, N). □

Lemma 3.20. *Let (X, N) be a $2PN$-space satisfying $(2PN\text{-}VI)$. Then X is of finite dimension if the locally ball $B_{e,\alpha}[x, r]$ is a compact set in (X, N).*

Proof. Let $\|\cdot,\cdot\|_\alpha$ be the α-2-norm induced by N. To show that X is of finite dimension, it suffices to prove that by Lemma 2.15, the subset

$$B_e[x, r] = \{y \in X : \|x - y, e\|_\alpha \leq r\} \tag{3.25}$$

is a compact set in $(X, \|\cdot,\cdot\|_\alpha)$. It is clear that $B_e[x, r] = B_{e,\alpha}[x, r]$. Choose a sequence $\{x_n\}$ of $B_e[x, r]$. Since $B_{e,\alpha}[x, r]$ is compact, it has a convergent subsequence $\{x_{n_k}\}$. Lemma 3.19 implies that $\{x_{n_k}\}$ is convergent in $\|\cdot,\cdot\|_\alpha$. Thus $B_e[x, r]$ is compact in $(X, \|\cdot,\cdot\|_\alpha)$, and consequently X is of finite dimension. □

Remark 3.21. The converse of Lemma 3.20 generally is not true. For example, consider $B_{(1,0),1/2}[0, 1]$ in $(\mathbb{R}^2, N)$, where N is a standard $2PN$-norm. Clearly,

$$B_{(1,0),1/2}[0, 1] = B_{(1,0)}[0, 1], \tag{3.26}$$

where $B_{(1,0)}[0, 1]$ is the subset of standard 2-normed space $\mathbb{R}^2$. On the contrary, if $B_{(1,0),1/2}[0, 1]$ were a compact set, then for each $\{x_n\} \in B_{(1,0),1/2}[0, 1] = B_{(1,0)}[0, 1]$ there would exist a converging subsequence $\{x_{n_k}\}$. Say $x_{n_k} \to a$, where $a \in B_{(1,0),1/2}[0, 1]$. Thus

$$\lim_{n\to\infty} N_{x_{n_k} - a, e}(t) = \lim_{n\to\infty} \frac{t}{t + \|x_{n_k} - a, e\|} = 1, \tag{3.27}$$

for all $e \in X$, $t > 0$. This implies that

$$\lim_{n\to\infty} \|x_{n_k} - a, e\| = 0, \tag{3.28}$$

for all e. Therefore $B_{(1,0)}[0, 1]$ is a compact set which is a contradiction by Example 2.14.

Lemma 3.22. *Let $T : (X, N) \to (Y, N')$ be a compact operator, where (X, N) and (Y, N') are $2PN$-spaces satisfying $(2PN\text{-}VI)$. If the 2-norms $\|\cdot,\cdot\|_\alpha$, $\|\cdot,\cdot\|'_\alpha$ are α-2-norms induced by N and N', respectively, then $T : (X, \|\cdot,\cdot\|_\alpha) \to (Y, \|\cdot,\cdot\|'_\alpha)$ is a compact operator for all $\alpha \in (0, 1)$.*

Proof. Let $\alpha \in (0, 1)$. We show that for each locally bounded sequence $\{x_n\}$ in $(X, \|\cdot,\cdot\|_\alpha)$, the sequence $\{T(x_n)\}$ has a convergent subsequence in $(Y, \|\cdot,\cdot\|'_\alpha)$. Let $\{x_n\}$ be a locally bounded sequence in $(X, \|\cdot,\cdot\|_\alpha)$. There exist $e \in X$ and $M > 0$ such that

$$\|x_n, e\|_\alpha < M, \tag{3.29}$$

for all $n \in \mathbb{N}$. By the definition of $\|\cdot,\cdot\|_\alpha$, for every $n \geq 1$ there exists $t_n > 0$ such that $t_n < M$ and $N_{x_n,e}(t_n) \geq \alpha$ for all n. Because N is nondecreasing, $\alpha \leq N_{x_n,e}(t_n) \leq N_{x_n,e}(M)$. Hence

$$N_{x_n,e}(M) \geq \alpha, \tag{3.30}$$

for all n. That is, $\{x_n\}$ is 2-PLB in (X, N). Thus $\{T(x_n)\}$ has a convergent subsequence $\{T(x_{n_k})\}$ in (Y, N'). By Lemma 3.19, $\{T(x_{n_k})\}$ is convergent in $\|\cdot,\cdot\|'_\alpha$. □

Theorem 3.23. *Let (X, N) and (Y, N') be two $2PN$-spaces satisfying $(2PN\text{-}VI)$. Then*

(a) *every compact operator $T : (X, N) \to (Y, N')$ is 2-PC;*

(b) *if $\dim X = \infty$, then the identity operator $I : (X, N) \to (X, N)$ is not a compact operator.*

Proof. (a) Choose $\alpha \in (0, 1)$ and $e \in X$. By Lemma 3.22, $T : (X, \|\cdot,\cdot\|_\alpha) \to (Y, \|\cdot,\cdot\|'_\alpha)$ is a compact operator between 2-α-normed spaces $(X, \|\cdot,\cdot\|_\alpha)$ and $(Y, \|\cdot,\cdot\|'_\alpha)$, where $\|\cdot,\cdot\|_\alpha$ and $\|\cdot,\cdot\|'_\alpha$ are induced 2-norms. Therefore T is bounded by Corollary 2.20. There exists $m_{e,\alpha} > 0$ such that

$$\|T(x), T(e)\|'_\alpha \leq m_{e,\alpha}\|x, e\|_\alpha, \tag{3.31}$$

for all $x \in X$. Hence T is 2-PB by Theorem 3.10. Now Theorem 3.9 implies that T is 2-PC. (b) Choose $e \in X$ and $\alpha \in (0, 1)$. The identity operator I maps the locally ball $B_{e,\alpha}[0, 1]$ to itself. Suppose on the contrary that I is a compact operator. Let $\{x_n\}$ be a sequence in $B_{e,\alpha}[0, 1]$. Because I is a compact operator, the 2-PLB sequence $\{x_n\}$ has a convergent subsequence. Hence $B_{e,\alpha}[0, 1]$ is compact. Thus X is of finite dimension by Lemma 3.20, which is a contradiction. □

Remark 3.24. Let (X, N) and (Y, N') be two $2PN$-spaces. If T_1 and T_2 are compact operators from X into Y, and $\alpha \in \mathbb{R}$, then $\alpha T_1 + T_2$ is a compact operator. Because, for each $\{x_n\}$ that is a 2-PLB sequence in X, the sequence $\{T_1(x_n)\}$ has a convergent subsequence $\{T_1(x_{n_k})\}$, and the sequence $\{T_2(x_{n_k})\}$ has a convergent subsequence $\{T_2(z_n)\}$. Hence, $\{T_1(z_n)\}$ and $\{T_2(z_n)\}$ are convergent sequences. Let $T_1(z_n) \to u$ and $T_2(z_n) \to v$, where $u, v \in Y$. We have

$$\lim_{n\to\infty} N'_{(T_1+T_2)(z_n)-u-v,y}(t) \geq \lim_{n\to\infty} \min\left\{N'_{T_1(z_n)-u,y}\left(\frac{t}{2}\right), N'_{T_2(z_n)-v,y}\left(\frac{t}{2}\right)\right\}, \tag{3.32}$$

for all $y \in Y$ and $t > 0$. Thus

$$\lim_{n\to\infty} N'_{T_1+T_2(z_n)-u-v,y}(t) = 1, \tag{3.33}$$

for all $y \in Y$, and $t > 0$. This implies that $T_1 + T_2$ is a compact operator. Now for all $\alpha \in \mathbb{R} \setminus \{0\}$ if $T(x_{n_k}) \to y_0$, then

$$\lim_{n\to\infty} N'_{\alpha T_1(x_{n_k})-\alpha y_0,y}(t) = \lim_{n\to\infty} N'_{T_1(x_{n_k})-y_0,y}\left(\frac{t}{|\alpha|}\right) = 1, \tag{3.34}$$

for all $y \in Y$ and $t > 0$. Hence αT_1 is also a compact operator.

Theorem 3.25. *Let (X, N) be a $2PN$-space, let $T : (X, N) \to (X, N)$ be a compact operator, let and that $S : (X, N) \to (X, N)$ be a bijective 2-PC operator. Then ST and TS are compact operators.*

Proof. Let $\{x_n\}$ be a 2-PLB sequence in X. Then $\{T(x_n)\}$ has a convergent subsequence $\{T(x_{n_k})\}$. Let $\lim_{n\to\infty} T(x_{n_k}) = y$. Since S is 2-PC, by Theorem 3.11 we have $S(T(x_{n_k})) \to S(y)$. Hence $ST(x_n)$ has a convergent subsequence and this shows that ST is compact. Now, we show that TS is compact. There are $t_0 > 0$, $e \in X$, and $r_0 \in (0, 1)$ such that $N_{x_n,e}(t_0) > 1 - r_0$ for all $n \geq 1$ since $\{x_n\}$ is a 2-PLB sequence. The operator S is 2-PB by Theorem 3.9 and so there is $m_{e,1-r_0} > 0$ such that

$$N_{x_n,e}\left(\frac{t_0}{m_{e,1-r_0}}\right) \geq 1 - r_0 \Longrightarrow N'_{S(x_n),S(e)}(t_0) \geq 1 - r_0, \tag{3.35}$$

for all n. It follows that $\{S(x_n)\}$ is a 2-PLB sequence in X. Because T is a compact operator, $\{T(S(x_n))\}$ has a convergent subsequence. This completes the proof. □

Theorem 3.26. *Let $T : X \to X$ be a compact operator on a $2PN$-space X. Then for every $\lambda \neq 0$, the null space $N(T_\lambda)$ of $T_\lambda = T - \lambda I$ is of finite dimension.*

Proof. We choose a locally ball M in $N(T_\lambda)$ and show that it is compact, then apply Lemma 3.20. Let $\{x_n\}$ be a sequence in M. Then $\{x_n\}$ is locally bounded, and $\{T(x_n)\}$ has a convergent subsequence $\{T(x_{n_k})\}$. Now $x_n \in M \subset N(T_\lambda)$ implies $T_\lambda(x_n) = T(x_n) - \lambda x_n = 0$, so that $x_n = \lambda^{-1}T(x_n)$. Consequently, $x_{n_k} = \lambda^{-1}T(x_{n_k})$ is convergent. Now, the closedness of M implies that the limit of $\{x_{n_k}\}$ belongs to M. This proves that $\dim N(T_\lambda) < \infty$. □

Definition 3.27. Let (X, N) be a $2PN$-space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n\to\infty} N_{x_{n+p}-x_n,x}(t) = 1$ for all $x \in X$, $t > 0$, and $p \in \mathbb{N}$.

We say that a $2PN$-space (X, N) is *complete* if every Cauchy sequence in X is convergent to a point of X.

Theorem 3.28. *Let X, Y, and Z be two $2PN$-spaces, let $T : Z \subset X \to Y$ be a surjective 2-PB operator, and let Y be a complete space. Then T has an extension*

$$\overline{T} : \overline{Z} \longrightarrow Y, \tag{3.36}$$

where $\overline{T}$ is an e-2-PB operator for each $e \in Z$.

Proof. We consider any $x \in \overline{Z}$. There is a sequence $\{x_n\}$ in Z such that $x_n \to x$. Since T is linear and 2-PB, for every $\alpha \in (0, 1)$ and $e \in Z$ there exists $m_{e,\alpha} > 0$ such that

$$N_{x_m-x_n,e}\left(\frac{t}{m_{e,\alpha}}\right) \geq \alpha \Longrightarrow N'_{T(x_m)-T(x_n),T(e)}(t) \geq \alpha, \tag{3.37}$$

for all $t > 0$ and $m, n \in \mathbb{N}$. But the sequence $\{x_n\}$ is Cauchy, thus for all $t > 0$, there exists $k \in \mathbb{N}$ such that for all $m, n > k$ we have

$$N_{x_m-x_n,e}\left(\frac{t}{m_{e,\alpha}}\right) \geq \alpha. \tag{3.38}$$

Therefore, for $m, n > k$,

$$N'_{T(x_m)-T(x_n),T(e)}(t) \ \geq \alpha. \tag{3.39}$$

This shows that $\{T(x_n)\}$ is Cauchy in Y. Thus $\{T(x_n)\}$ is convergent to an element $y \in Y$. Now, we define $\overline{T}$ by $\overline{T}(x) = y$. In exactly the same way as presented in the proof of Theorem 2.26 we see that this definition is independent of the particular choice of a sequence in Z converging to x. Clearly $\overline{T}$ is linear and $\overline{T}(x) = T(x)$ for every $x \in Z$, so that $\overline{T}$ is an extension of T. We now use the 2-probabilistic boundedness of T on Z. Let $\alpha \in (0,1)$ and $e \in Z$. There exists $m_{e,\alpha} > 0$ such that

$$N_{x,e}\left(\frac{t}{m_{e,\alpha}}\right) \geq \alpha \Longrightarrow N'_{T(x),T(e)}(t) \geq \alpha, \tag{$*$}$$

for all $t > 0$ and $x \in Z$. Choose $t > 0$ and $x \in \overline{Z}$ such that

$$N_{x,e}\left(\frac{t}{4m_{e,\alpha}}\right) \geq \alpha. \tag{3.40}$$

Now, we show that $N'_{\overline{T}(x),T(e)}(t) \geq \alpha$. Because $x \in \overline{Z}$, there exists $\{x_n\} \subseteq Z$ such that $x_n \to x$. Therefore, for $n \in \mathbb{N}$ sufficiently large, we have

$$N_{x_n,e}\left(\frac{t}{2m_{e,\alpha}}\right) \geq \min\left\{N_{x_n-x,e}\left(\frac{t}{4m_{e,\alpha}}\right), N_{x,e}\left(\frac{t}{4m_{e,\alpha}}\right)\right\} \geq \alpha. \tag{3.41}$$

By $(*)$, $N'_{T(x_n),T(e)}(t/2) \geq \alpha$, and since $\lim_{n\to\infty} T(x_n) = \overline{T}(x)$, we obtain

$$N'_{T(x_n)-\overline{T}(x),T(e)}\left(\frac{t}{2}\right) \geq \alpha. \tag{3.42}$$

Hence

$$N'_{\overline{T}(x),T(e)}(t) \geq \min\left\{N'_{\overline{T}(x)-T(x_n),T(e)}\left(\frac{t}{2}\right), N'_{T(x_n),T(e)}\left(\frac{t}{2}\right)\right\} \geq \alpha. \tag{3.43}$$

Therefore $\overline{T}$ is a 2-PB linear operator on Z. □

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